Limit Models, Superlimit Models, and Two Cardinal Problems in Abstract Elementary Classes

by

Fred Drueck
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Defense Committee:
John Baldwin, Chair and Advisor
David Marker
Lynn Scow
Ramin Takloo-Bighash
Monica VanDieren, Robert Morris University, Pittsburgh

For my mother and my father.

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[1] This thesis was partially supported by the bowl of candy formerly kept in SEO 335.

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LIST OF ABBREVIATIONS

AEC	Abstract Elementary Class
AP	Amalgamation Property
AMS	American Mathematical Society
JEP	Joint Embedding Property
UIC	University of Illinois at Chicago

SUMMARY

In the introduction and Chapter 2 we introduce various conditions which could potentially define "superstability" for abstract elementary classes. In particular, we examine various notions of "superlimit model" and "superlimit class" which we will apply toward proving gap-transfer theorems for AECs. In Chapter 3 we examine Lessmann's analogue of Vaught's Theorem for abstract elementary classes. We provide a sufficient condition for the construction of an $(\mathrm{LS}(\mathbf{K})^+, \mathrm{LS}(\mathbf{K}))$-model. In Chapter 4 we discuss progress that has been made in proving uniqueness of limit models from various superstability assumptions. In Chapter 5 we give a sufficient condition, or perhaps more accurately, various sufficient conditions for the existence of an $(\aleph_2, \aleph_0)$-model to exist, using the existence of a simplified morass. Our work here is guided by the presentation of Jensen's work in proving the classical gap-2 transfer result for first order logic in (Devlin, 1984).

CHAPTER 1

INTRODUCTION

Abstract elementary classes (AECs) were introduced by Shelah in (Shelah, 1985) to axiomatize certain combinatorial properties of elementary substructure. One of the goals in doing so was to provide a uniform framework for addressing model theoretic questions about non-first-order logics. In the study of abstract elementary classes (AEC) semantic notions tend to be more useful then syntactic notions. The goal is, after all, to offer a uniform framework which applies regardless of the particular logic one works with. This does not always mean that syntactic notions are not useful in the study of AECs, but it does restrict the settings in which we are able to work with syntactic tools.

While some of the venerable model theoretic tools of classical model theory, such as the downward Löwenheim-Skolem theorem are available to us in the abstract context, others (such as the upward Löwenheim-Skolem theorem and compactness) are not available in this more general context. In a certain sense, abstract elementary classes strip away almost all syntactic objects of study, leaving only semantic objects. As such, it is seldom useful to consider first order formulas or syntactic types as objects of study in an abstract elementary class. On the other hand, automorphisms and embeddings remain well-defined and useful notions.

In Chapter 5, we notice in particular that when working in generic AECs, we cannot shift vocabularies with the same ease one can in first order logic. In particular, when working in a non-empty elementary class, one is free to expand the language, expand a particular structure in the class to this larger language, and then consider structures elementarily equivalent to the expanded structure. As long as the size of the structure in the original elementary class is not smaller than the cardinality of the expanded language, this produces a non-trivial elementary class whose members reduct to members of the original elementary class. Shifting vocabularies in arbitrary AECs, by contrast, is non-trivial.

To replace syntactic types, one often considers "galois types"; in general this is a rather complicated notion defined in terms of an equivalence relations on a triple consisting of two models and a distinguished element. However, in the context where one has the joint embedding property (JEP) and the amalgamation property (AP), there is a simpler characterization of galois type. In this context, one can think of a galois type over some parameter model $\mathscr{N}$ as the orbit of some element in a large homogeneous structure (a monster model) under automorphisms which fix $\mathscr{N}$. In this dissertation, we work only with classes which have such a monster mode. In this setting galois types should feel vaguely familiar to anyone used to working in first order syntactic types.

In this dissertation we study sufficient conditions for the existence and or transfer of two cardinal models in abstract elementary classes. In classical first order model theory, a two cardinal model is some $\mathscr{L}$-structure $\mathscr{M}$ with $|M| = \kappa$ for which there was some $\mathscr{L}$-formula ϕ such that $|\phi(\mathscr{M})| = \lambda$. We say such a model is a (κ, λ)-model. A related notion is "Vaughtian pair", that is, a pair of models $\mathscr{M} \prec \mathscr{N}$, $\mathscr{M} \subsetneq \mathscr{N}$ with some formula ϕ such that $\phi(\mathscr{M}) = \phi(\mathscr{N})$. By downward Löwenheim-Skolem, it's easy to see that the existence of a two cardinal model implies the existence of a Vaughtian pair (witnessed by the same ϕ). Under various conditions, one can reverse the implication and build a two cardinal model starting with a Vaughtian pair.

In the AEC context, we like to avoid syntactic notions in general. As such we redefine "Vaughtian pair", "two cardinal model", and "(κ, λ)"-model in terms of semantic notions, e.g galois types or invariant sets. In particular, we define a p-Vaughtian pair, for p a galois type, as a pair of models $\mathscr{M} \prec_{\mathbf{K}} \mathscr{N}$ such that $p(\mathscr{M}) = p(\mathscr{N})$. That is, the orbit p of the monster $\mathfrak{C}$ has the same intersection with $\mathscr{N}$ as it does with $\mathscr{M}$. We define (κ, λ)-model as a structure $\mathscr{M}$ where $|M| = \kappa$ such that for some invariant set $X \subseteq \mathfrak{C}$, $|X \cap M| = \lambda$. We ask then, under what conditions may Vaughtian pairs and two cardinal models exist in a non-elementary AEC?

This is a question that has been well-studied in the context of elementary classes by Vaught, Chang, Jensen, and others. While we avoid in-depth discussion of categoricity transfer, the relationship of non-existence of Vaughtian pairs and two cardinal models with categoricity in elementary classes is one of the motivating factors in exploring this question in the context of abstract elementary classes. In particular, Baldwin and Lachlan show that for countable languages an elementary class is uncountably categorical if and only if the class is ω-stable and has no Vaughtian-pairs. The Baldwin-Lachlan Theorem shows that three areas of model theoretic research, namely the study of two cardinal models, categoricity, and stability theory are all intertwined in the first order case. Lachlan and Shelah showed another tie between stability theory and two-cardinal models, essentially a stronger version of Chang's two cardinal theorem restricted to stable elementary classes:

Theorem 1.0.1 (GCH)(Chang) [1] *Suppose there exists a (κ, λ)-model for the first order theory T then:*

1. *For λ' where $\kappa > \lambda' > \lambda$ there exists a (κ, λ')-model of T.*
2. *If $\mathrm{cf}(\kappa) < \mathrm{cf}(\lambda)$ then there exists a $(2^\kappa, \lambda)$-model of T.*
3. *If $\mathrm{cf}(\kappa) = \mathrm{cf}(\lambda)$ then there exists a $(2^\kappa, 2^\lambda)$-model of T.*

Theorem 1.0.2 (Lachlan, Shelah) [1] *If for a stable theory T there exists a (κ, λ)-model with $\kappa > \lambda$ then there exists a (κ', λ')-model for any $\kappa \geq \lambda \geq |T|$*

The assumption of stability removes the need to assume **GCH** and also provides an even stronger transfer theorem. We investigate some stability-related questions in abstract elementary classes, in hopes that this will be applicable to our study of two-cardinal models and/or upwards categoricity transfer. In particular we investigate conditions that could define an analogue of "superstability" for Abstract Elementary Classes.

For elementary classes, the definition of "superstability" has long been settled. There are a number of equivalent conditions that characterize when a first order theory T is "superstable". We list just three below:

1. T is λ-stable for all $\lambda \geq 2^{|T|}$.
2. There are no infinite forking chains within models of T, that is $\kappa(\mathbf{K}, \mu) = \omega$ for all μ.

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[1] *reproduced from (Chang and Keisler, 1977)*

[1] *See (Baldwin, 1988) Chapter IX section 5 for a more detailed discussion.*

3. The union of an increasing chain of saturated models of T is saturated.

Since these three conditions are known to be equivalent, it matters little which condition is actually defined to be "superstability". For a non-elementary AEC $\mathbf{K}$ it remains an open question whether the following conditions are equivalent:

1. $\mathbf{K}$ is λ-stable for all $\lambda \geq \mu$ for some $\mu \geq \mathrm{LS}(\mathbf{K})$.
2. There is no chain $(\mathscr{M}_i)_{i<\omega}$ where $\mathscr{M}_{i+1}$ is universal over $\mathscr{M}_i$ and a galois type $p \in S(\bigcup_{i<\omega} \mathscr{M}_i)$ that splits over $\mathscr{M}_i$ for all $i < \omega$.
3. The union of a $< \lambda^+$-increasing chain of galois-saturated models in $\mathbf{K}$ of size λ is saturated for $\lambda > \mathrm{LS}(\mathbf{K})$.

One of the first difficulties in comparing the situations in the AEC case to the first order case is that in the first order case the various conditions are true for all large enough cardinals, where as in the AEC case, one may have a "superstability condition" only in a particular cardinal, or in a range of cardinals (such as the cardinals between $\mathrm{LS}(\mathbf{K})$ and some cardinal λ where the class is λ-categorical). In particular, while there is a stability spectrum theorem for elementary classes, there are only a few specialized, partial stability spectrum results for abstract elementary classes.[1]

It seems likely that condition 1. is too strong a condition to capture "AECs with a very nice stability theory". That is, we want to allow for "superstable" classes to be well behaved only in either one distinguished cardinal or some range of cardinals.

[1] Published in (Grossberg and VanDieren, 2006b) and (Baldwin et al., 2006)

At least under the assumption of JEP, AP and the existence of arbitrarily large models, Condition 1. and 2. are known to hold in a cardinal λ if $\mathbf{K}$ is κ-categorical and $\mathrm{LS}(K) \leq \lambda < \kappa$, while Condition 3. is known to hold in λ where $\mathrm{LS}(K) < \lambda < \kappa$.[1] However, for a non-categorical AEC, it is not known whether these three conditions are or are not equivalent. Indeed, at least two more possible definitions of "superstability for an abstract elementary class" have been suggested. This expands our list of "superstability conditions" to:

i. $\mathbf{K}$ is λ-stable for all $\lambda \geq \mu$.

ii. There are is no chain $(\mathscr{M}_i)_{i<\omega}$ where $\mathscr{M}_{i+1}$ is universal over $\mathscr{M}_i$ and a galois type $p \in S(\bigcup_{i<\omega} \mathscr{M}_i)$ that splits over $\mathscr{M}_i$ for all $i < \omega$.

iii. The union of a $< \lambda^+$-increasing chain of galois-saturated models in $\mathbf{K}$ of size λ is saturated for $\lambda > \mathrm{LS}(\mathbf{K})$.

iv. Uniqueness of limit models in $\lambda \geq \mathrm{LS}(\mathbf{K})$.[2]

v. Existence of a globally superlimit model in $\lambda \geq \mathrm{LS}(\mathbf{K})$.[3]

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[1]See (Shelah, 1999) and (Grossberg and VanDieren, 2006b), proofs are available in (Baldwin, 2009) as well.

[2]See Definition 2.2.2 for the definition of "limit model". For the precise meaning of the phrase "uniqueness of limit models" see Definition 2.2.9. The idea that "uniqueness of limit models" is an appropriate criterion for superstability is implicit in (VanDieren, 2006).

[3]See Definition 2.3.1. Shelah argues on pages 41 and again on page 61 of the introduction to (Shelah, 2009) that the existence of a superlimit model is the proper generalization of superstability from first order model theory to abstract elementary classes.

In general, at least under the assumption of stability, AP, and JEP we know that condition v. and condition iii. are equivalent if $\lambda > \mathrm{LS}(\mathbf{K})$ and λ is regular (see Corollary 2.3.12).[1] Condition iii. implies condition v. for any $\lambda > \mathrm{LS}(\mathbf{K})$ (See Corollary 2.3.10). Condition v. or Condition iii. both imply condition iv., but we do not know if condition iv. implies any other condition above. In particular, even if limit models are unique in $\mathrm{LS}(\mathbf{K})$ we do not know that the union of less than $\mathrm{LS}(\mathbf{K})^+$ limit models in $\mathrm{LS}(\mathbf{K})$ is itself a limit model. We do know that λ^+-categoricity implies iv.[2]

The question of whether a union of limit models is itself a limit model, or more generally, whether a "superlimit-like" model exists is intimately tied to upward categoricity transfer. Building on work of Shelah in (Shelah, 1999) and Grossberg and VanDieren in (Grossberg and VanDieren, 2006c) (Grossberg and VanDieren, 2006a), Lessmann proved the following Theorem in (Lessmann, 2005):

Theorem 1.0.3 (Lessmann) *Let $\mathbf{K}$ be a tame AEC with arbitrarily large models satisfying AP, JEP, and $\mathrm{LS}(\mathbf{K}) = \omega$. If $\mathbf{K}$ is $\aleph_1$-categorical $\mathbf{K}$ is categorical in every uncountable cardinal.*

The proof is an induction that depends, in the base case, on having the ability to build a pair of models with "superlimit-like" properties in the sense of Shelah. Categoricity in $\aleph_1 = \aleph_0{}^+$ (at least, under the assumption of JEP and AP in $\aleph_0$) implies that the unique model in $\aleph_1$ is

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[1] Indeed, the definition of "galois-saturation" is seldom a useful tool for an analysis of models of size $\mathrm{LS}(\mathbf{K})$.

[2] See Theorem 4.0.2, reproduced from (VanDieren, 2012), more details are available in Section 9 of (VanDieren, 2006).

saturated; the existence of a Vaughtian-pair of superlimit-like models in LS($\mathbf{K}$) is used to build a non-saturated model in $\aleph_1$. This theorem is essentially an analogue of this classical theorem of Vaught for elementary classes (see section 4.3 of (Marker, 2002)):

Theorem 1.0.4 (Vaught) *If T has a Vaughtian pair of models there is an $(\aleph_1, \aleph_0)$-model.*

In particular Lessmann proves:

Theorem 1.0.5 (Lessmann) *If $\mathbf{K}$ satisfies AP and JEP in $\aleph_0$ and has Vaughtian pair of galois-saturated models of size at least $\aleph_1$ then $\mathbf{K}$ has an $(\aleph_1, \aleph_0)$-model.*

In Lessmann's proof, countable limit models play the role of countable homogeneous models realizing the same types in the proof of Vaught's theorem. A sufficient condition for building an $(\aleph_1, \aleph_0)$-model is being able to extend some countable Vaughtian pair without ever adding realizations of the type which has no new realizations in the larger of the two models in the countable Vaughtian pair. We make this condition explicit in Definition 2.3.15. In a way, when distilled to its core, Vaught's theorem is an application of the fact that a countable homogeneous model is a locally superlimit model.

More specifically, to prove Vaught's Theorem, one first passes from an arbitrary Vaughtian pair to a Vaughtian pair of $(\mathscr{M}_0, \mathscr{M}_1)$ of countable homogeneous models which realize the same types over the empty set, witnessed by some formula ϕ. It is then quite easy to construct a two cardinal model. Since countable homogeneous models are isomorphic if they realize the same types over the empty set there is an isomorphism $f : \mathscr{M}_0 \to \mathscr{M}_1$. One finds an extension f^* of f to $\mathscr{M}_1$ and lets $\mathscr{M}_2 := f(\mathscr{M}_1)$, which will then be countable, homogeneous, realize the same types over the empty set as $\mathscr{M}_0$, and have $\phi(\mathscr{M}_2) = \phi(\mathscr{M}_1) = \phi(\mathscr{M}_0)$. One continues inductively in this manner. Since the countable union of homogeneous models is homogeneous, and no new type can be realized in the union, the countable union of models defined in this manner is still isomorphic to the model we started with.

This property, that the countable union of "$\prec$"-sequence of countable homogeneous models realizing the same types over the empty set is isomorphic to the first model of the sequence that defines "local superlimit". Studying notions 'around superlimit models' and their relationship to the existence of two-cardinal models of various types is the core focus of this dissertation. We explore one application of "superlimit-like" models in Chapter 3 and another in Chapter 5.

We would like to be able to extend Lessmann's categoricity transfer result to other cardinals. That is, we would like to show λ^+-categoricity in a tame AEC $\mathbf{K}$ with $\mathrm{LS}(\mathbf{K}) = \lambda$ implies categoricity above λ^+. It is understood that the missing step for uncountable λ is constructing the $(\mathrm{LS}(\mathbf{K})^+, \mathrm{LS}(\mathbf{K}))$-model from a Vaughtian pair of saturated models above $\mathrm{LS}(\mathbf{K})$. If one could complete such a construction, the methods of (Grossberg and VanDieren, 2006c) could be applied to achieve upwards categoricity transfer in this context.

In Chapter 3 we show that Definition 2.3.15 is a sufficient condition to extend Theorem 1.0.5 to uncountable cardinals, however we have not yet been able to deduce the existence of models satisfying Definition 2.3.15 from λ^+ categoricity.

Because of its potential usefulness toward proving upwards categoricity transfer, we feel that the existence of "superlimit-like" models, such as models satisfying Definition 2.3.15, might also provide a reasonable definition of superstability for AECs. In Chapter 2 we define relevant notions, e.g "μ-splitting", "limit model", etc., precisely and present some basic results concerning their behavior. In particular, we discuss how both Shelah and Lessmann define "superlimit" and offer our own definition as well.

Shelah of course, has his own program for developing stability theory for AECs, via "good frames in λ"; much of his work on this topic is collected in (Shelah, 2009). A λ-good frame is an AEC that satisfies various axioms including admitting some sort of independence relation. On particular conditions included in the λ-good frame axioms[1] is the existence of a globally superlimit model in λ and the non-existence of long forking chains. From these assumptions one is able to deduce additional nice properties, in particularly, working under strong enough assumptions, one is able to construct a λ^+-good frame by modifying the original AEC. However, since one works with an abstract independence relation, it's unclear when concrete independence relations derived from natural notions like splitting or splintering of types actually satisfy the good frames axioms.

In Chapter 4, we discuss somewhat briefly uniqueness of limit models, which (as we have noted) is another condition that could potentially be taken as a definition of "superstability". We prove a small useful result, namely that a continuous relatively full tower is a limit model, without needing to assume disjoint amalgamation. We also offer some discussion of how this fits into a larger effort to prove that from $\kappa(\mathbf{K}, \mu) = \omega$ uniqueness of limit models holds.

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[1] At least, this is true of the version of the axioms given on page 138 of (Shelah, 2009).

Initially we had hoped that uniqueness of limit models (Conjecture 4.2.5), should it hold for a class $\mathbf{K}$, could be applied to generalize Lessmann's categoricity transfer theorem to AECs with uncountable Löwenheim number, however we were ultimately unable to do so. We discuss the difficulties we encountered in Section 4.3 of Chapter 4.

In Chapter 5 we again apply a "superlimit like" notion, this time to build an $(\aleph_2, \aleph_0)$-gap in an AEC by using a simplified morass. We discuss some progress we have made in trying to prove an analogue of Jensen's Gap-2 Transfer Theorem for elementary classes in the context of AECs. We discover, that while the argument employs set theoretic machinery, that the classical proof of the theorem is in fact dependent on model theoretic properties of first order logic that do not hold in arbitrary AECs. The proof that, under the assumption that $\mathbf{V} = \mathbf{L}$, the existence of a (κ^{++}, κ)-gap in a model of a first order theory T implies the existence of an $(\aleph_2, \aleph_0)$-gap proceeds via four main steps:

1. Build an isomorphic pair of structures in κ.
2. Obtain a Vaughtian pair (in κ) and an elementary embedding that codes some combinatorial information; expand the language to code the Vaughtian pair and combinatorial information into an expanded first order theory.
3. Find a "nice" pair of countable homogeneous models of the expanded theory.
4. Construct an $(\aleph_2, \aleph_0)$-gap using properties of a simplified morass.

The main theorem of Chapter 5, Theorem 5.4.2, is an analogue of step 4. in the context of an AEC whose members are ordered structures. Another way to look at the main body of our work in Chapter 5 is that we prove the existence of an $(\aleph_2, \aleph_0)$-model given nice enough combinatorial conditions in a pair of countable models and the existence of a simplified Morass.

In the context of an ordered AEC (or arbitrary elementary class) we are able complete step 1 (this proof, like the proof in the first order case, utilizes **GCH**, see Proposition 5.7.11). From 3., we can prove 4. And from a pair of "extra nice" structures in κ (a sort of "2+") we can derive 3. Unfortunately, in the non-elementary case, we have not been able to go from the conclusion of step 1. to an "extra nice" pair of structures in κ. Dealing with unordered AEC is problematic in itself, though we prove some small partial results using Shelah's presentation theorem.

The classical gap-two transfer theorem of Jensen utilized countable homogeneous models as a tool in the construction. The definition of "countable homogeneous model" is, classically, given in terms of partial elementary maps, which are implicitly tied to the syntax of first-order logic. Since AECs in general will not interact in a nice way with first order syntactic notions we define a purely semantic notion of homogeneity instead, which we term "galois homogeneity". This notion generalizes the classical model theoretic notion of "homogeneity" after one fixes a distinguished monster model. Perhaps undesirably, however, this notion is intimately tied to the choice of monster model, $\mathfrak{C}$. As such, to develop a useful theory, one must first fix a $\mathfrak{C}$ model and give up the flexibility to change monster models at a later time.

CHAPTER 2

SUPERSTABILITY FOR ABSTRACT ELEMENTARY CLASSES

For the definition of Abstract Elementary Class (AEC) we suggest that one read (Baldwin, 2009), though many basic results are also available in a particularly readable form in (Lessmann, 2005) as well. Most of the original definitions and many basic results are due to Shelah, who has conveniently published a number of the key papers together in (Shelah, 2009). Other notions one should consult existing literature for include Löwenheim Number, galois type, galois stability, the amalgamation property (AP), tameness, and joint embedding property (JEP).[1]

I will define certain results where the terminology is either uncommon or inconsistent in existing literature or otherwise potentially confusing.

Definition 2.0.1 *A model $\mathscr{M}$ is μ-universal over $\mathscr{M}_0$ if the following hold:*

1. $\mathscr{M}_0 \prec_{\mathbf{K}} \mathscr{M}$
2. *Given any $\mathscr{M}_1$ where $|M_1| \leq \mu$ and $\mathscr{M}_0 \prec_{\mathbf{K}} \mathscr{M}_1$ there is some strong embedding $f : \mathscr{M}_1 \to \mathscr{M}$ fixing $\mathscr{M}_0$.*

If $\mathscr{M}$ is $|M|$ universal over $\mathscr{M}_0$ we say $\mathscr{M}$ is universal over $\mathscr{M}_0$.

We will apply the following basic result quite often.

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[1]JEP and AP are defined to be natural generalizations the corresponding notions in first order model theory.

Proposition 2.0.2 *Let $\mathscr{M}_1 \prec_{\mathbf{K}} \mathscr{M}_2 \prec_{\mathbf{K}} \mathscr{M}_3$. If $\mathscr{M}_3$ is universal over $\mathscr{M}_2$ and $\mathscr{M}_2$ is universal over $\mathscr{M}_1$, then $\mathscr{M}_3$ is universal over $\mathscr{M}_1$.*

proof. Suppose that $\mathscr{M}_1 \prec_{\mathbf{K}} \mathscr{N}$. By AP we can find an amalgam $\mathscr{N}'$ of $\mathscr{M}_2$ and $\mathscr{N}$ over $\mathscr{M}_1$ such that $\mathscr{M}_2$ is embedded into $\mathscr{N}'$ by the inclusion map. Thus $\mathscr{M}_2 \prec_{\mathbf{K}} \mathscr{N}'$, but then by universality of $\mathscr{M}_3$ over $\mathscr{M}_2$ there exists some strong embedding $g : \mathscr{N}' \to \mathscr{M}_3$ fixing $\mathscr{M}_2$. In particular g fixes $\mathscr{M}_1$, so $\mathscr{M}_3$ is universal over $\mathscr{M}_1$. $\square$

2.1 Splitting

In AECs there is a notion of "splitting" which generalizes splitting over models in elementary classes, namely:

Definition 2.1.1 *A galois type $p \in S(\mathscr{M})$* μ-splits over $\mathscr{N}$ *if the following conditions hold:*

1. $\mathscr{N} \prec_{\mathbf{K}} \mathscr{M}$
2. *There exists some $\mathscr{M}_0 \in \mathbf{K}_\mu$ and strong embedding $f : \mathscr{M}_0 \to \mathscr{M}$ such that:*
   (a) $\mathscr{N} \prec_{\mathbf{K}} \mathscr{M}_0 \prec_{\mathbf{K}} \mathscr{M}$
   (b) $\mathscr{N} \prec_{\mathbf{K}} f(\mathscr{M}_0) \prec_{\mathbf{K}} \mathscr{M}$
   (c) $p \restriction f(\mathscr{M}_0) \neq f(p \restriction \mathscr{M}_0)$

For general AECs, splitting seems to provide the most "forking-like" notion available in absence of some syntactic notion of forking. Chapter 12 of (Baldwin, 2009) is one source amongst others which provides a good exposition of splitting and basic results about the existence and uniqueness of non-splitting extensions, such as:

Lemma 2.1.2 *Let $\mathscr{M} \prec_{\mathbf{K}} \mathscr{M}' \prec_{\mathbf{K}} \mathscr{M}''$ all be of size μ and suppose $\mathscr{M}'$ is μ-universal over $\mathscr{M}$. If $p \in S(\mathscr{M}')$ does not μ-split over $\mathscr{M}$ then p has at least one extension to $S(\mathscr{M}'')$ that does not μ-split over $\mathscr{M}$.*

Though syntactic types over models of a stable first order theory are *always* stationary, the same may not hold for galois types in a stable, non-elementary AEC. The following result provides a sufficient condition for a type in $S(\mathscr{M})$ to be "stationary" in the sense of having a unique non-splitting extension to any model of size $|M|$.

Lemma 2.1.3 *Let $\mathscr{M} \prec_{\mathbf{K}} \mathscr{M}' \prec_{\mathbf{K}} \mathscr{M}''$ all be of size μ and suppose $\mathscr{M}'$ is μ-universal over $\mathscr{M}$. If $p \in S(\mathscr{M}')$ does not μ-split over $\mathscr{M}$, p has at most one extension to $S(\mathscr{M}'')$ that does not μ-split over $\mathscr{M}$.*

Putting Lemmas 2.1.3 and 2.1.2 together we get:

Corollary 2.1.4 *Let $\mathscr{M} \prec_{\mathbf{K}} \mathscr{M}' \prec_{\mathbf{K}} \mathscr{M}''$ all be of size μ and suppose $\mathscr{M}'$ is μ-universal over $\mathscr{M}$. If $p \in S(\mathscr{M}')$ does not μ-split over $\mathscr{M}$, p has a unique extension to $S(\mathscr{M}'')$ that does not μ-split over $\mathscr{M}$.*

There is a natural transitivity property one would want to have for splitting, namely:

Conjecture 2.1.5 *Let $\mathscr{M} \prec_{\mathbf{K}} \mathscr{M}' \prec_{\mathbf{K}} \mathscr{M}''$ all be of size μ and suppose $\mathscr{M}'$ is μ-universal over $\mathscr{M}$. If $p \in S(\mathscr{M}'')$ does not μ-split over $\mathscr{M}'$ and $p \restriction \mathscr{M}'$ does not split over $\mathscr{M}$ then p does not split over $\mathscr{M}$.*

This is left as an exercise (Exercise 12.9) in (Baldwin, 2009); it seems to be quite a difficult exercise. Neither a valid proof nor counterexample is known. One is tempted to offer the following as proof, having just read a similar proof of (Baldwin, 2009)'s Theorem 12.7:

"proof". Note that $p \restriction \mathscr{M}'$ has a unique extension q to $\mathscr{M}''$ that does not split over $\mathscr{M}$. In particular, q does not split over $\mathscr{M}'$ since $\mathscr{M} \prec_{\mathbf{K}} \mathscr{M}'$, but then both p and q are non-splitting extensions of $p \restriction \mathscr{M}''$ that do not split over $\mathscr{M}'$, so $p = q$. "$\square$"

The gap in the proof above is that while p and q both are non-splitting extensions of $p \restriction \mathscr{M}''$ over $\mathscr{M}'$, in order to apply Lemma 2.1.3 one must have the non-splitting occur over some $\mathscr{N}$ which $\mathscr{M}'$ is universal over, e.g. $\mathscr{M}$. A number of other simple, but false proofs have been formulated of Conjecture 2.1.5.

2.2 Saturated Models and Limit Models

We follow Shelah's convention in defining "saturated models" that is:

Definition 2.2.1 *A model $\mathscr{M}$ is* μ-saturated *if for every $\mathscr{N} \prec_{\mathbf{K}} \mathscr{M}$ with $|N| < \mu$ any galois type p over $\mathscr{N}$ is realized in $\mathscr{M}$. We call $\mathscr{M}$* saturated *if $\mathscr{M}$ is $|M|$-saturated. When the cardinal parameter μ is clear from context, we will omit it.*

Choosing this definition, however, has the downside of allowing certain models in the Löwenheim number of an AEC to be saturated if they simply have no strong submodels over which they are required to realize types.[1] Such models may be interesting, but do not "realize many types", as one normally expects when discussing saturation. Furthermore, saturated

[1] Indeed, in an AEC with no models of cardinality strictly less than the Löwenheim-number, all models in the Löwenheim number are saturated.

models can only be shown to be unique up to isomorphism above the Löwenheim-number.[1] Below we define an alternate notion of "a model rich in sub-structures", a "limit model". This notion is, in general, more useful in the Löwenheim-number than saturation is.

Definition 2.2.2 *For μ a cardinal and $\theta < \mu^+$ limit ordinal, a (μ, θ)-*limit model *$\mathscr{M}$ is a continuous union $\mathscr{M} = \bigcup_{i \in \theta} \mathscr{M}_i$ where $\mathscr{M}_{i+1}$ is universal over $\mathscr{M}_i$ and each $\mathscr{M}_i$ is of size μ. We say $\mathscr{M}$ is a limit model over $\mathscr{N}$ if there is such a sequence of $\mathscr{M}_i$ where $\mathscr{M}_0 = \mathscr{N}$.*

Also for brevity's sake we define "limit sequences". Sometime after defining this notion we realized that in the existing literature, a limit sequence is a (μ, α)-sequences for some limit ordinal α and μ is the cardinality of the models in the sequence. While it is undesirable to unnecessarily proliferate technical terminology, it is also convenient to have a brief term that describes exactly as much information about a sequence of models as we care to exhibit.

Definition 2.2.3 *A (μ, β)-*chain *or (μ, β)-*sequence *is a a $\prec_{\mathbf{K}}$-increasing sequence of models $(\mathscr{M}_i)_{i<\beta}$ where for all $i < \beta$, $|M_i| = \mu$ and $\mathscr{M}_{i+1}$ is universal over $\mathscr{M}_i$.*

Definition 2.2.4 *f $(\mathscr{M}_i)_{i<\alpha}$ is a (μ, α) sequence and $\alpha < \mu^+$ is a limit ordinal then $\mathscr{M}_\alpha := \bigcup_{i+1} \mathscr{M}_i$ is clearly a limit model. We say that $(\mathscr{M}_i)_{i<\alpha}$* is a limit sequence for *$\mathscr{M}_\alpha$. If we wish to distinguish the base model of the sequence, we will say $(\mathscr{M}_i)_{i<\alpha}$* is a limit sequence for *$\mathscr{M}_\alpha$* over *$\mathscr{M}_0$.*

[1]For this reason some existing literature, such as (Lessmann, 2005) take the convention that "$\mathscr{M}$ is saturated" implicitly implies that $|M| > \mathrm{LS}(K)$.

The following proposition is proved by a standard back and forth argument.

Proposition 2.2.5 *Suppose $\mathscr{N}$ is a $(\mu, \mathrm{cf}(\alpha))$-limit model over $\mathscr{N}_0$ and $\mathscr{M}$ is a (μ, α)-limit model over $\mathscr{M}_0$. If $\mathscr{M}_0 \cong \mathscr{N}_0$, then $\mathscr{M} \cong \mathscr{N}$. Furthermore, given an isomorphism $f_0 : \mathscr{N}_0 \to \mathscr{M}_0$ one can find an isomorphism $f : \mathscr{N} \to \mathscr{M}$ which extends f_0.*

The following appears as Definition 15.1 of (Baldwin, 2009). It more or less corresponds to "$\kappa^1_\mu(\mathbf{K})$" in terms Definition 4.3 of (Grossberg and VanDieren, 2006b). The definition given in (Baldwin, 2009) differs slightly from (Grossberg and VanDieren, 2006b) in only demanding universality of $\mathscr{M}_{i+1}$ over $\mathscr{M}_i$, not that $\mathscr{M}_{i+1}$ is a limit model over $\mathscr{M}_i$.

Definition 2.2.6 *We define $\kappa(\mathbf{K}, \mu)$ as the least ordinal α, should it exist, such that there does not exist a limit sequence $(\mathscr{M}_i)_{i<\alpha}$ of models $\mathscr{M}_i \in \mathbf{K}_\mu$ such that there is some $p \in S(\bigcup_{i<\alpha} \mathscr{M}_i)$ where p splits over $\mathscr{M}_i$ for all $i < \alpha$.*

This is one possible analogue of the first order notion of $\kappa(T)$ for AECs.[1] Another possible analogue[2] of $\kappa(T)$ is:

Definition 2.2.7 *We define $\overline{\kappa}(\mathbf{K}, \mu)$ as the least cardinal $\overline{\kappa}$, should it exist, such that for any $\mathscr{M} \in \mathbf{K}_\mu$ and $p \in S(\mathscr{M})$ there is some $\mathscr{N}$ where $|N| < \overline{\kappa}$ such that p does not $|N|$-split over $\mathscr{N}$.*

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[1]A definition of $\kappa(T)$ can be found in (Baldwin, 1988), or any other reasonably complete book on first order model theoretic stability theory.

[2]Similarly to Definition 2.2.6, $\overline{\kappa}(\mathbf{K}, \mu)$ is more or less $\overline{\kappa}^1_\mu(\mathbf{K})$ as defined in Definition 4.4 of (Grossberg and VanDieren, 2006b). Again the definition given in (Baldwin, 2009) differs slightly from (Grossberg and VanDieren, 2006b) in only demanding universality of $\mathscr{M}_{i+1}$ over $\mathscr{M}_i$, not that $\mathscr{M}_{i+1}$ is a limit model over $\mathscr{M}_i$.

It is pointed out in (Grossberg and VanDieren, 2006b) that while $\kappa(\mathbf{K},\mu)$ and $\overline{\kappa}(\mathbf{K},\mu)$ agree if $\mathbf{K}$ is an elementary class (this is true as long as $|T| \leq \aleph_0$), that in the non-elementary case it is only known that $\kappa(\mathbf{K},\mu) < \overline{\kappa}(\mathbf{K},\mu)$ (see Proposition 4.5 of (Grossberg and VanDieren, 2006b)). In (Shelah, 1999), Shelah derives a bound on $\kappa(\mathbf{K},\mu)$ from categoricity in a larger cardinal and the existence of arbitrarily large models in an Abstract Elementary class which satisfies AP and JEP. A proof is also available as Theorem 15.3 of (Baldwin, 2009). In general, for non-elementary AECs, no bound on $\overline{\kappa}(\mathbf{K},\mu)$ has yet been derived from categoricity.

In general, a (μ,α) and (μ,β)-limit model may fail to be isomorphic even if they are based over the same model. One can construct an example of non-isomorphic limit models in the archetypal example of the "stable, not superstable" first order theory, namely:

Example 2.2.8 *Let $\mathscr{L} = \{E_i : i < \omega\}$ and let T be the $\mathscr{L}$-theory of infinitely many refining equivalence relations. For a cardinal λ where $\lambda = \lambda^\omega$ there exists a (λ,ω)-limit model and (λ,ω_1)-limit model which are not isomorphic.*

One must work in a cardinal λ where $\lambda^\omega = \lambda$, so that T will be stable in λ. This is necessary so that given $\mathscr{N} \models T$ of cardinality λ one may construct a universal extension of $\mathscr{N}$ of size λ.

Definition 2.2.9 *By the phrase* uniqueness of limit models *we mean that given any limit ordinals $\alpha,\beta < \mu^+$ and two (μ,α) and (μ,β)-limit model* are *isomorphic, even if* $\mathrm{cf}(\alpha) \neq \mathrm{cf}(\beta)$.

Grossberg, VanDieren, and Villaveces make a case that satisfying uniqueness of limit models is a suitable definition for "superstability" in AECs. Indeed in (Grossberg et al., 2012) the following theorem is proved about elementary classes:

Theorem 2.2.10 (Grossberg,VanDieren, Villaveces) *Suppose that T is a* stable *complete first order theory. Let $\mu > 2^{|T|}$. If T is not superstable, then no (μ, ω)-limit model is isomorphic to any (μ, κ)-limit model where $cf(\kappa) \geq \kappa(T)$.*

This means that the classic theorem of first order model theory from (Shelah, 1990) which describes conditions equivalent to superstability can be extended by one new equivalent condition:

Theorem 2.2.11 *The following are equivalent for any* stable *first order theory T:*

1. *T is superstable, that is, T is λ-stable for all $\lambda \geq 2^{\aleph_0}$*
2. *There are no infinite forking chains within models of T, that is $\kappa(\mathbf{K}, \mu) = \omega$ for all μ.*
3. *The union of an increasing chain of saturated models of T is saturated.*
4. *If $\mathscr{M}_1$ and $\mathscr{M}_2$ are respectively (μ, α) and (μ, β)-limit models of T for $\mu \geq 2^{|T|}$ then $\mathscr{M}_1 \cong \mathscr{M}_2$.*

We note that this equivalence only holds true for *stable* first order theories, indeed, limit models will not exist if the theory T is complete and unstable. We hope it now seems clear that uniqueness of limits (proved from categoricity in a successor cardinal in (VanDieren, 2012) and examined in depth in Chapter 4) should be a candidate for a generalization of first order superstability, in addition to demanding $\kappa(\mathbf{K}, \mu) = \omega$, the union of saturated models is saturated, or that a globally superlimit model exists. As of yet there is no known equivalence such as Theorem 2.2.11 for AECs.

2.3 Superlimits

An important related question to "what should define superstability in the context of AECs?" is whether the union of an $\prec_{\mathbf{K}}$-increasing sequence of limit models is a limit model.[1] More generally, an important questions is whether a given AEC admits the existence of a "superlimit model". We reproduce the following definition from Definition 3.3 in (Shelah, 2009), found on page 138.

Definition 2.3.1 (Shelah's superlimit) *A model $\mathscr{M} \in \mathbf{K}_\lambda$ is* λ-locally superlimit *if:*

1. *For any $\mathscr{N} \prec_{\mathbf{K}} \mathscr{M}$ there is $\mathscr{M}' \cong \mathscr{M}$ such that $\mathscr{N} \prec_{\mathbf{K}} \mathscr{M}'$ and $\mathscr{M}' \neq \mathscr{M}$.*
2. *Given any ordinal α and $\prec_{\mathbf{K}}$-increasing sequence of models $(\mathscr{M}_i)_{i<\alpha}$ where $|\alpha| = \lambda$ such that for all $i < \alpha$ $\mathscr{M}_i \cong \mathscr{M}$ then $\bigcup_{i \in I} \mathscr{M}_i \cong \mathscr{M}$*

$\mathscr{M}$ *is* λ-globally superlimit*, or simply λ-superlimit if the above hold and in addition:*

3. *$\mathscr{M}$ is universal in $\mathbf{K}_\lambda$, that is, any $\mathscr{N} \in \mathbf{K}_\lambda$ can be embedded into $\mathscr{M}$.*

We will occassionally use the noun local superlimit *to mean a locally superlimit model and* global superlimit *or simply* superlimit *to mean a globally superlimit model.*[2]

The existence of a superlimit model in $\mathbf{K}$ of cardinality λ is a useful technical condition for proving categoricity transfer results and for building two-cardinal models. An example of an

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[1] This condition appears in (VanDieren, 2006) as "hypothesis 3".

[2] *In interest of strict technical accuracy we point out that this final sentence is not reproduced from (Shelah, 2009).*

ω-locally superlimit model is a countable homogeneous model (which may omit some types) of a first-order theory in a countable language. Indeed in, say Lessmann's analogue of Vaught's Theorem for AECs[1] or our own construction of an $(\aleph_2, \aleph_0)$-model in Theorem 5.4.2, a superlimit model serves as something of a substitute for a countable homogeneous model in the first order proof.

Lessmann, however, defines a slightly different version of "superlimit" than Shelah. The following first appears in (Lessmann, 2005):

Definition 2.3.2 (Lessmann's superlimit) *A model $\mathscr{M} \in \mathbf{K}_\omega$ is a superlimit if $\mathscr{M}$ is an (ω, α)-limit model for some limit ordinal α.*

Since for any countable α an (ω, α)-limit model over $\mathscr{M}$ is isomorphic to an (ω, ω)-limit model $\mathscr{M}$, Definition 2.3.2 can be restated as:

Proposition 2.3.3 *A model $\mathscr{M} \in \mathbf{K}_\omega$ is a superlimit in the sense of Lessmann if and only if it is an $(\omega, \omega \times \omega)$-limit model.*

For this reason, we will refer to such models as $(\omega, \omega \times \omega)$-limit models to avoid confusion with either Shelah's notion of superlimit or our own definition of superlimit, given in Chapter 3.

A countable globally superlimit model is clearly a superlimit model in the sense of Lessmann, however Lessmann and Shelah's definitions of, respectively superlimit and locally superlimit

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[1]See Theorem 2.3.14.

are, to the best of our knowledge, orthogonal in strength. Lessmann's definition is weaker than Shelah's in the sense that the isomorphism type of an $(\omega, \omega \times \omega)$-limit model may not be closed under arbitrary countable unions, merely those countable unions where the models "fit nicely enough together". Shelah, by contrast, demands the isomorphism type of a countable locally superlimit model be closed under arbitrary countable unions. However, Shelah does not demand that a locally superlimit model is a limit model (indeed,a countable atomic model of a first order theory with non-atomic models is an example of a locally superlimit model which is not a limit model).

We can define a notion of "Vaughtian pair" and "two cardinal model" for AECs in a fairly natural way. We use the same definition available (amongst many sources) in (Baldwin, 2009) as Definition 13.2 for Vaughtian pair. For the classical definitions of Vaughtian Pair and two cardinal model see (Chang and Keisler, 1977) or (Marker, 2002).

Definition 2.3.4 *Let $p \in S(\mathscr{M})$ be a type over a model of size λ. A (p, μ) Vaughtian Pair is a pair of models, $\mathscr{N}_1, \mathscr{N}_2$ of size μ such that $\mathscr{M} \prec_{\mathbf{K}} \mathscr{N}_1 \prec_{\mathbf{K}} \mathscr{N}_2$ and $\mathscr{N}_1 \subsetneq \mathscr{N}_2$ where p has a non-algebraic extension to $\mathscr{N}_1$ not realized in $\mathscr{N}_2$.*

Definition 2.3.5 *A two cardinal model in λ is a model $\mathscr{M} \in \mathbf{K}_\lambda$ where there exists a type p over $\mathscr{N} \prec_{\mathbf{K}} \mathscr{M}$ where $|N| < |M|$ and p is not realized in $\mathscr{M}$.*

In certain cases we are interested in being more definite about how much the model $\mathscr{M}$ differs in size from the model $\mathscr{N}$ over which $\mathscr{M}$ omits a type. First we define "invariant set";

this only makes sense when considered with respect to some fixed monster model $\mathfrak{C}$. Invariant sets will be discussed in much greater detail in Chapter 5.

Definition 2.3.6 *A subset $X \subseteq \mathfrak{C}$ is* invariant *if for any $\gamma \in \mathrm{aut}(\mathfrak{C})$ $\gamma(X) \subseteq X$.*

Definition 2.3.7 *A* (κ, λ)-model over μ *is a model $\mathscr{M}$ such that there is a set X invariant over some model $\mathscr{N} \prec_{\mathbf{K}} \mathscr{M}$ with $|N| = \mu$ such that $X \cap \mathscr{M} = \lambda$. A (κ, λ)-model (with no μ mentioned) is a (κ, λ)-model over $\emptyset$. If κ is the n-fold cardinal successor to λ we refer to this as an n-gap.*

In (Lessmann, 2005), Lessmann shows that, at least in the context of a theory with countable Löwenheim number, AP, and JEP, $(\omega, \omega \times \omega)$-limit models satisfy a slightly weakened version of the first item of Definition 2.3.1. This allows him to prove a nice categoricity transfer theorem, thus extending work of VanDieren in (VanDieren, 2006) and Grossberg and VanDieren in (Grossberg and VanDieren, 2006a). As a step towards this result Lessmann proves a version of Vaught's Two Cardinal Theorem for AECs which is interesting to us in its own right. More specifically, Lessmann constructs an $(\aleph_1, \aleph_0)$-model in $\aleph_1$ given a Vaughtian pair of saturated models in $\aleph_1$.

Lessmann's innovation in his definition of superlimit is not to demand that all unions of superlimits yield another superlimit, only those unions in which the models fit together in a nice enough way. It is fairly easy to show, if one has full uniqueness of limit models in $\mu > \mathrm{LS}(\mathbf{K})$

that limit models of size μ are saturated[1]. In the elementary class case, superstability then implies that the union of limit models is a limit model since a union of saturated models is saturated. We know of no proof that, in the general AEC case,[2] $\prec_{\mathbf{K}}$-increasing sequence $(\mathscr{M}_i)_{i<\omega}$ of $(\omega, \omega \times \omega)$-limit models, or more generally (μ, θ)-limit models, is itself a limit model. As such, there is no existing proof that Lessmann's notion of superlimit is either a local or global superlimit in the sense of Shelah.[3]

On the other hand, Shelah's definition does not imply that if $\mathscr{M}$ is locally superlimit that $\mathscr{M}$ will be a limit model. In particular, the Example 2.3.16 (presented later in this chapter) is an example of a locally superlimit model that is *not* a superlimit. However, if $\mathscr{M}$ is a globally superlimit model then we can demonstrate that $\mathscr{M}$ *is* a limit model.

Lemma 2.3.8 *Assume* $\mathbf{K}$ *is a* λ *galois stable AEC which satisfies JEP and AP in* λ*. If* $\mathscr{M}_0$ *is a* λ*-globally superlimit model there exists* $\mathscr{M}_1 \cong \mathscr{M}_0$ *and there exists* $\mathscr{N}_1$ *universal over* $\mathscr{M}_0$ *where* $\mathscr{M}_0 \prec_{\mathbf{K}} \mathscr{N}_1 \prec_{\mathbf{K}} \mathscr{M}_1$*.*

proof. By λ-stability, AP, and JEP we may assume the existence of a λ^+ saturated structure $\mathfrak{C}$. We suppose $\mathscr{M}_0$ is a λ-globally superlimit model. By λ-stability we can find a $\mathscr{N}_1$ which is λ-

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[1]This is also true in an elementary class when "saturation" means a model is saturated with respect to syntactic types.

[2]However, for a discussion of a case where the union of saturated models is saturated, (and thus, in certain cases, the union of limit models is a limit model) see Chapter 15 of (Baldwin, 2009), and Theorem 15.8 in particular. Here Baldwin clarifies the presentation of ideas originating in (Shelah, 1999).

[3]In an elementary class, limit models are saturated, and the countable union of countably many saturated models is saturated. So in an elementary class, Lessmann and Shelah's notion of "superlimit" do agree. However, outside of the elementary case, the question remains open.

universal over $\mathscr{M}_0$ of size λ. By the universality of $\mathscr{M}_0$ there exists an embedding $f : \mathscr{N}_1 \to \mathscr{M}_0$, we extend f to $F \in \mathrm{aut}(\mathfrak{C})$. Let $\mathscr{M}_1 := F^{-1}(\mathscr{M}_1)$. $\square$

Proposition 2.3.9 *Assume* $\mathbf{K}$ *is a* λ *galois stable AEC which satisfies JEP and AP in* λ. *Let* $\alpha < \lambda^+$ *be a limit ordinal. If* $\mathscr{M}_0$ *is a* λ*-globally superlimit model then* $\mathscr{M}_0$ *is a* (λ, α)*-limit model.*

proof. Let $\mathscr{M}_0$ be a λ-globally superlimit model. We construct sequences $(\mathscr{M}_i)_{i<\alpha}$ and $(\mathscr{N}_i)_{i<\alpha}$ of models inductively that satisfy the following properties:

1. $\mathscr{M}_i \cong \mathscr{M}_0$
2. $\mathscr{N}_{i+1}$ is universal over $\mathscr{M}_i$ of size λ.
3. $\mathscr{M}_i \prec_{\mathbf{K}} \mathscr{N}_{i+1} \prec_{\mathbf{K}} \mathscr{M}_{i+1}$

For the base case $\mathscr{M}_0$ is given and we can find $\mathscr{M}_1, \mathscr{N}_1$ by applying Lemma 2.3.8. Given $\mathscr{M}_i, \mathscr{N}_i$ again apply Lemma 2.3.8 to find some $\mathscr{N}_{i+1}$ universal over $\mathscr{M}_i$ and $\mathscr{M}_{i+1} \cong \mathscr{M}_i \cong \mathscr{M}_0$ such that $\mathscr{N}_{i+1} \prec_{\mathbf{K}} \mathscr{M}_{i+1}$.

At limit stages $\gamma \leq \alpha$ let $\mathscr{M}_\gamma := \bigcup_{i<\gamma} \mathscr{M}_i = \bigcup_{i<\gamma} \mathscr{N}_i$. Because $\mathscr{M}_0$ is a λ-globally superlimit model $\mathscr{M}_0 \cong \bigcup_{i<\gamma} \mathscr{M}_i$.

$\mathscr{M}_\alpha := \bigcup_{i<\alpha} \mathscr{M}_i = \bigcup_{i<\alpha} \mathscr{N}_i$ is obviously a (λ, α)-limit model. Since $\mathscr{M}_0$ is a λ-globally superlimit model, $\mathscr{M}_0 \cong \mathscr{M}_\alpha$. $\square$

It is worth noting also that if a λ-globally superlimit model exists then uniqueness of limit models in λ is actually quite easy to show. We do so below in Corollary 2.3.10. In general, proving that a (λ, α)-limit model and (λ, β)-limit model where $\mathrm{cf}(\alpha) \neq \mathrm{cf}(\beta)$ are isomorphic is a highly non-trivial result. The harder, general case is discussed in Chapter 4 building upon the work done in (Grossberg et al., 2012).

Corollary 2.3.10 *Assume* $\mathbf{K}$ *is a* λ *galois stable AEC which satisfies JEP and AP in* λ*. Let* $\alpha < \lambda^+$ *be a limit ordinal. If there exists a* λ*-globally superlimit model* $\mathscr{M}_0$ *then for any* $\alpha < \lambda^+$ *any* (λ, α)*-limit model is isomorphic to* $\mathscr{M}_0$*. In particular given* $\alpha, \beta < \lambda^+$ *any two* (λ, α)*,*(λ, β)*-limit models are isomorphic.*

proof. Fix $\alpha < \lambda^+$. By Proposition 2.3.9 $\mathscr{M}_0$ is a (λ, α)-limit model. Let $\mathscr{N}$ be a (λ, α)-limit model over $\mathscr{N}_0$. Since $\mathscr{M}_0$ is λ-universal we may assume without loss of generality that $\mathscr{N} \prec_{\mathbf{K}} \mathscr{M}_0$. Since we can always, if necessary, repeat the construction used to prove Proposition 2.3.9 and get a model $\mathscr{M}_\alpha \cong \mathscr{M}_0$, we may as well assume that $\mathscr{M}_0$ is in fact a (λ, α)-limit model over $\mathscr{N}_0$. By Proposition 2.2.5 $\mathscr{M}_0 \cong \mathscr{N}$. □

In fact this proof technique is applicable to a number of superstability questions, in particular the following:

Proposition 2.3.11 *Suppose that* $\mathbf{K}$ *is a* λ*-galois stable AEC which satisfies JEP and AP in* λ*, where* λ *is regular. If there exists a* λ*-globally superlimit model* $\mathscr{M}_0$ *then* $\mathscr{M}_0$ *is saturated.*

proof. Note that since λ is regular the union $\bigcup_{i<\lambda} \mathscr{N}_i$ of $\prec_{\mathbf{K}}$-increasing sequence of saturated models $\mathscr{N}_i \in \mathbf{K}_\lambda$ is saturated. Since λ is regular and $\mathbf{K}$ is λ-stable, a saturated model exists. Build interleaved sequences of models $(\mathscr{M}_i)_{i<\lambda}$ and $(\mathscr{N}_i)_{i<\lambda}$ such that:

1. $\mathscr{M}_i$ is a λ-superlimit.

2. $\mathscr{N}_i$ is saturated and $\mathscr{M}_i \prec_{\mathbf{K}} \mathscr{N}_i$.

Note that by the definition of λ-superlimit $\mathscr{M}_0 \cong \bigcup_{i<\lambda} \mathscr{M}_i = \bigcup_{i<\lambda} \mathscr{N}_i$. Since $\bigcup_{i<\lambda} \mathscr{N}_i$ is saturated, $\mathscr{M}_0$ is saturated. $\square$

The regularity of λ is important, without this, we cannot be certain that any union of saturated models will be saturated. The following Corollary is easily deducible:

Corollary 2.3.12 *Suppose that* $\mathbf{K}$ *is a* λ*-galois stable AEC which satisfies JEP and AP in* $\lambda > \mathrm{LS}(\mathbf{K})$*, where* λ *is regular. Then the following are equivalent:*[1]

iii. The union of a $< \lambda^+$*-increasing chain of galois-saturated models in* $\mathbf{K}$ *of size* λ *is saturated for* $\lambda > \mathrm{LS}(\mathbf{K})$.

v. Existence of a globally superlimit model in $\lambda \geq \mathrm{LS}(\mathbf{K})$.[2]

proof. The implication iii. $\implies$ v. is always trivial. By Proposition 2.3.11 if there exists a λ-superlimit model $\mathscr{M}_0$ then $\mathscr{M}_0$ is saturated. So by the definition of λ-superlimit, the union of a $< \lambda^+$-increasing chain of galois-saturated models in $\mathbf{K}$ of size λ is saturated for $\lambda > \mathrm{LS}(\mathbf{K})$. $\square$

The following theorems of Lessmann build on work of Grossberg and VanDieren in (Grossberg and VanDieren, 2006a) (see Chapter 3 for more details).

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[1] This numbering was chosen to match the list of "superstability conditions" given in the introduction.

[2] *See Definition 2.3.1.*

Theorem 2.3.13 (Lessmann) *Let* $\mathbf{K}$ *be a tame AEC with arbitrarily large models satisfying AP, JEP, and* $\mathrm{LS}(\mathbf{K}) = \omega$*. If* $\mathbf{K}$ *is* $\aleph_1$*-categorical, then* $\mathbf{K}$ *is categorical in every uncountable cardinal.*

Theorem 2.3.14 (Lessmann) *Let* $\mathbf{K}$ *be an AEC satisfying AP in* λ*, JEP in* λ*,* $\mathrm{LS}(\mathbf{K}) = \omega$*, and that there are only countably many galois types over any countable model. If there exists a Vaughtian pair of saturated models in* $\aleph_1$ *then there exists a two cardinal model in* $\aleph_1$*.*

An obstacle to easily extending this theorem to AECs with uncountable Löwenheim number is one cannot generalize the proof that countable limit models are closed under "nice enough" countable unions. Indeed, this problem seems hard even when even assuming that limit models of a given cardinality are unique up to isomorphism.

We offer a framework below which abstracts the transfer from a Vaughtian pair of limit models in $\aleph_0$ to a two cardinal model in $\aleph_1$ in Lessmann's proof of Theorem 1.0.5 to the transfer from a Vaughtian pair in a sufficiently nice pair of models in λ to a two cardinal model in λ^+ in any "superstable" AEC.[1]

Definition 2.3.15 *We define a* (p_0, λ)-superlimit class *as a class of* $\prec_{\mathbf{K}}$*-increasing sequences of models in* $\mathbf{K}_\lambda$ *(of length at least 2 and less than* λ^+*) with distinguished models* $\mathscr{N}_0, \mathscr{N}_1 \in \mathbf{K}^L_\lambda$ *the following properties:*

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[1]This is most interesting when $\lambda = \mathrm{LS}(\mathbf{K}) > \aleph_0$ so that the neither the ambient assumptions of Grossberg-VanDieren in (Grossberg and VanDieren, 2006a) nor Lessmann in (Lessmann, 2005) are not satisfied. In this case, the additional hypothesis of λ^+ categoricity and tameness would allow us to prove a new upwards categoricity transfer theorem.

1. *$(\mathscr{N}_0, \mathscr{N}_1)$ are a p_0-Vaughtian Pair in λ, such that:*

   (a) *p_0 is a quasiminimal extension of some $p \in S(\mathscr{M})$*[1].

   (b) *$\mathscr{N}_0 \cong \mathscr{N}_1$ over $\mathscr{M}$.*

   (c) *There is a unique quasiminimal extension of p to any model $\mathscr{N}_i$ where for some θ, $i < \theta < \lambda^+$, $(\mathscr{N}_i)_{i<\theta} \in \mathbf{K}^L_\lambda$.*

   (d) *Any non-algebraic extension of p_0 to a model $\mathscr{N}_i$ where for some θ, $i < \theta < \lambda^+$, $(\mathscr{N}_i)_{i<\theta} \in \mathbf{K}^L_\lambda$ is big.*

2. *$(\mathscr{N}_0, \mathscr{N}_1) \in \mathbf{K}^L_\lambda$ and for every sequence $(\mathscr{N}_i')_{i<\theta} \in \mathbf{K}^L_\lambda$, $\mathscr{N}_0' = \mathscr{N}_0, \mathscr{N}_1' = \mathscr{N}_1$.*[2]

3. *If $(\mathscr{N}_i)_{i<\theta} \in K^L_\lambda$ then $\mathscr{N}_i \cong \mathscr{N}_0$ over $\mathscr{M}$.*

4. *If $\theta < \lambda^+$ and $(\mathscr{N}_i)_{i<\theta} \in \mathbf{K}^L_\lambda$ and $\mathscr{N}_\theta = \bigcup_{i<\theta} \mathscr{N}_i$. then $(\mathscr{N}_i)_{i\leq\theta} \in \mathbf{K}^L_\lambda$.*

5. *If $\theta < \lambda^+$ and $(\mathscr{N}_i)_{i<\theta} \in \mathbf{K}^L_\lambda$ where $\theta = \alpha + 1$, then there is an $\mathscr{N}_{\alpha+1}$ such that there exists an isomorphism $f : \mathscr{N}_1 \to \mathscr{N}_{\alpha+1}$ where $f(\mathscr{N}_0) = \mathscr{N}_0$ and f fixes $\mathscr{M}$.*

We call $(\mathscr{N}_i)_{i<\theta} \in K^L_\lambda$ an adequate sequence.

We offer some examples of superlimit classes below:

Example 2.3.16

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[1] *We allow $\mathscr{M} = \emptyset$ to be a possibility.*

[2] *We make this restriction since the sequences useful for proving Theorem 3.3.1 are the sequences which contain $(\mathscr{N}_0, \mathscr{N}_1)$ as an initial segment. It would do no harm to omit our demand that* every *sequence $\mathbf{K}^L_\lambda$ begin with $\mathscr{N}_0$ followed by $\mathscr{N}_1$ as long $(\mathscr{M}_0, \mathscr{N}_1) \in \mathbf{K}^L_\lambda$.*

1. *Let $\mathscr{L} := \{U_i : i < \omega\}$, where each U_i is a unary predicate. Let T be the first order theory containing the following sentences/axiom schema:*

   - $\forall x\, (U_{i+1}(x) \to U_i(x))$
   - $\exists^\infty x\, (\neg U_{i+1}(x) \wedge U_i(x))$ *for all* $i < \omega$.
   - $\forall x\, U_0(x)$.

   Let $\mathbf{K} := \mathrm{mod}(T)$, *let* $\prec_{\mathbf{K}} := \prec$, *let* $p(x) = \{U_i(x) : i < \omega\}$. $p(x)$ *has only one non-algebraic extension to any model* $\mathscr{M}$. *If* $\mathscr{N}_0 \prec_{\mathbf{K}} \mathscr{N}_1$ *are two models of size* λ *which both omit* p *such that for all* $i < \omega$ $|U_i(\mathscr{N}_0)| = |U_i(\mathscr{N}_1)| = \lambda$, *then the class of all* $\prec$*-extensions of* $\mathscr{N}_1$ *of cardinality* λ *which are isomorphic to* $\mathscr{N}_1$ *is a* p*-superlimit class in* λ. *Note that the class of models isomorphic to* $\mathscr{N}_0$ *is a class of locally superlimit models in the sense of Shelah which are* not *universal. Hence* $\mathscr{N}_0$ *is not a globally superlimit model. It also is not a limit model.*

In Chapter 3 we discuss extensively ω-limit models (which we recall are what Lessmann called "superlimit" models) in an AEC $\mathbf{K}$ with $\mathrm{LS}(K) = \omega$ and $\kappa(\mathbf{K}, \mu) = \omega$, which are another example of a superlimit class. In general, ω-limit models are not a superlimit in the sense of Shelah. Despite this, ω-limit models are "close enough" to being a superlimit that Lessmann was able to prove his theorem with them. Our Definition 2.3.15 is to abstracts Lessmann's method in an effort to provide a more general proof.

Another place where superlimit models show up is in proving the gap-2 transfer theorem, in the guise of countable homogeneous models of a first order theory. We discuss this in Chapter 5.

Theorem 2.3.17 (Jensen) *If **V=L**, then a first order theory T has a (κ^{++}, κ) model it has an $(\aleph_2, \aleph_0)$ model.*

In trying to generalize this theorem to AEC it is tempting to use countable limit models as a substitute for countable homogeneous models of a first order theory, however one runs into difficulty in defining orders "compatible enough" with the structure of the existing AEC to prove a real generalization of the first order theorem. Unfortunately, the presence of an order also suggests the class of structures will likely be unstable, so in Chapter 5 we work without the assumption of stability. In particular, this means that limit models may not exist. As a substitute we introduce a notion of "galois homogeneous" model which generalizes the classical definition of "homogeneous" model. "Galois homogeneous" models are local superlimits in the sense of Definition 2.3.1, but in fact it turns out that slightly stronger homogeneity properties are needed than just being a local superlimit. "Galois homogeneity" is sufficient but may not be necessary. Weaker, but more complicated conditions for building an $(\aleph_2, \aleph_0)$-model are given in Assumptions 5.3.3 and 5.3.7.

A related notion to "limit model" is that of a "special model".

Definition 2.3.18 *We say a model $\mathscr{M} \in \mathbf{K}_\lambda$ is* α-special *over $\mathscr{N}$ if there exists a sequence of models $(\mathscr{M}_i)_{i \in \lambda \times \alpha}$ with $\mathscr{M}_0 = \mathscr{N}$ where $\mathscr{M}_{i+1}$ realizes every galois type over $\mathscr{M}_i$.*

Being α-special is a useful sufficient condition for being a limit model. The following appears as Theorem 10.9 of (Baldwin, 2009).

Theorem 2.3.19 *Let* $\mathbf{K}$ *be* μ*-galois stable with the amalgamation property. Suppose* $|M| = \mu$ *and* $\theta < \kappa^+$*, then a* θ*-special model over* $\mathscr{M}$ *is a* (μ, θ) *limit model.*

An easy corollary of this theorem is that if K is stable in μ we can construct (μ, θ)-limit models over any model of size μ.

Corollary 2.3.20 *Let* K *be* μ*-galois stable with the amalgamation property and joint embedding property. Suppose* $\mathscr{M}_0 \in K$ *is of cardinality* μ *and* $|A| \leq \mu$*,* $A \subset \mathfrak{C}$ *then for some* $\theta < \mu^+$ *there is a* (μ, θ)*-limit model* $\mathscr{M}$ *in* K *over* $\mathscr{M}_0$ *with* $A \subset \mathscr{M}$*.*

proof. Let $\mathscr{M}_0'$ be any model such that $\mathscr{M}_0 \prec_{\mathbf{K}} \mathscr{M}_0'$ and $A \subset \mathscr{M}_0'$. By μ-stability there are only μ-types over $\mathscr{M}_0'$, so we can find $\mathscr{M}_1$ of cardinality μ realizing every galois type over $\mathscr{M}_0'$. So by induction we can instruct a θ-special model $\mathscr{M}$ over $\mathscr{M}_1$, since $\mathscr{M}_1$ is universal over $\mathscr{M}_0$, $\mathscr{M}$ is a limit model over $\mathscr{M}_0$ as well. By Theorem 2.3.19 $\mathscr{M}$ is a (θ, μ)-limit. $\square$

CHAPTER 3

ANALYSIS OF LESSMANN FOR $\aleph_1$

Lessmann's result in (Lessmann, 2005) is primarily focused on extending the main result of (Grossberg and VanDieren, 2006a), namely:

Theorem 3.0.1 (Grossberg, VanDieren) *Suppose K is categorical in λ^+ and ξ-tame for* $\mathrm{LS}(K) < \xi < \lambda$*, K has arbitrarily large models, AP, and JEP. Then K is categorical in all $\kappa \geq \lambda^+$. If $\lambda = LS(K)$ then if K is λ and λ^+ categorical then K is categorical in all $\kappa > \lambda^+$.*

The additional strength of the result in Lessmann is that he is able to remove the need for categoricity in the Löwenheim number if $\mathrm{LS}(K) = \omega$. However, along the way, Lessmann's proof of this fact yields the following theorem, which is interesting in its own right (which we recall from the previous chapter).

Theorem 3.0.2 (Lessmann) *Let $\mathbf{K}$ be an AEC satisfying AP in ω, JEP in ω,* $\mathrm{LS}(\mathbf{K}) = \omega$*, and that there are only countably many galois types over any countable model. If there exists a Vaughtian pair of saturated models in $LS(\mathbf{K})^+$ then there exists a two cardinal model in* $\mathrm{LS}(\mathbf{K})^+$.

It was our hope that this theorem would generalize to the context of "superstable AEC". That is, we had hoped that we could prove the following conjecture:

Conjecture 3.0.3 *Let* $\mathbf{K}$ *be an AEC satisfying AP in* $\mathrm{LS}(\mathbf{K})$*, JEP in* $\mathrm{LS}(\mathbf{K})$*, and* $\mathbf{K}$ *is "superstable". If there exists a Vaughtian pair of saturated models in* $\mathrm{LS}(\mathbf{K})^+$ *then there exists a two cardinal model in* $\mathrm{LS}(\mathbf{K})^+$*.*

Our initial hope was to use uniqueness of limit models, discussed in further in Chapter 4, to extend Lessmann's result to AEC with arbitrary Löwenheim number. Unfortunately we were unable to utilize either the theorem or the construction used to prove it to accomplish this goal. We discuss the difficulties we encountered in this endeavor in Chapter 4, Section 4.3. Upon further investigation, we were able to directly generalize many of Lessmann's results to the case where $LS(K) > \omega$. However, the key combinatorial argument that shows (ω, ω)-limits models are super-limit models is dependent on combinatorial properties unique to countable ordinals. While we were unable to extend Lessmann's result as desired, we have been able to isolate, as Definition 2.3.15 key sufficient conditions for proving Conjecture 3.0.3. In Section 3.3 we offer a proof of Conjecture 3.0.3 from the existence of a p_0 superlimit Class. In Section 3.4 we deduce the result Lessmann already proved from our abstract framework.

3.1 Big types and Quasiminimality

We examine further the properties of big and quasiminimal types in this section, with an eye toward using these properties to prove results in the vein of Theorem 2.3.14.

Assumption 3.1.1 *Throughout this chapter we fix an AEC* K *where* $\mathrm{LS}(K) = \lambda$*,* K *satisfies AP and JEP in* λ*,* K *has models in* λ^+*, and* K *is* λ*-stable. We fix also a "*λ*-monster model"* $\mathfrak{C}$*. Note that we are* not *assuming* λ^+*-categoricity.*

We move on now to "quasiminimality". While non-splitting types have some degree of stationarity, we introduce the notion of "quasiminimal type" which has a greater degree of stationarity. First we introduce big types and what it means for a type to be "non-algebraic" in an AEC since these notions are referred to in the definition of "quasiminimal type".

Definition 3.1.2 *A galois type $p \in S(\mathscr{M})$ is* non-algebraic[1]

Definition 3.1.3 *Let $p \in S(\mathscr{M})$, where $|\mathscr{M}| = LS(\mathbf{K})$, we say p* is big *if p has a non-algebraic extension to any $\mathscr{M}'$ where $|\mathscr{M}'| = \mathrm{LS}(K)$.*

Definition 3.1.4 *A type $p \in S(\mathscr{M})$, where $|\mathscr{M}| = \lambda$, is* quasiminimal *if p has a unique big extension in $S(\mathscr{M}')$ for every $\mathscr{M}'$ of cardinality λ with $\mathscr{M} \prec_{\mathbf{K}} \mathscr{M}'$.*

It follows clearly from the definition that if $p \in S(\mathscr{M})$ is big then so is any restriction $p \restriction \mathscr{N}$ where $\mathscr{N} \prec_{\mathbf{K}} \mathscr{M}$. It's also fairly clear that if p has a non-algebraic extension to a model of universal over $\mathscr{M}$, p is big, in fact this condition is necessary as well as sufficient. This next proposition also justifies the terminology "big", as we show that a type being big is equivalent to being realized many times in the monster model. The following appears as Proposition 2.3 of (Lessmann, 2005).

Proposition 3.1.5 *Let $p \in S(\mathscr{M})$ where $|M| = \lambda$, the following are equivalent:*

---

[1] There is some potential for confusion as the notion of "non-algebraic" does not generalize the classical notion of "algebraic" for first-order syntactic types. For example, if T is the (incomplete) first order theory of fields, then $\mathrm{tp}^{\mathrm{syntactic}}(i/\mathbb{Q})$ will be "algebraic" in the classical sense, while $\mathrm{tp}^{\mathrm{ga}}(i/\mathbb{Q})$ is still unrealized in $\mathbb{Q}$. Of course, if a galois type is algebraic then so is the first order type of the necessarily unique realization, since a galois type over a model always has a unique realization.

1. p is big

2. p has a non-algebraic extension to some $\mathscr{M}'$ universal over $\mathscr{M}$.

3. p is realized λ^+ many times in $\mathfrak{C}$.

proof. (1) $\implies$ (2) is clearly implied by the definition and stability. That is, if $p \in S(\mathscr{M})$ we can build a model $\mathscr{N}$ that is universal over p by utilizing λ-stability. By definition of big, p has a non-algebraic extension to $\mathscr{N}$.

(2) $\implies$ (3) is also fairly easy to prove using stability. If we assume that $p \in S(\mathscr{M})$ is realized λ-times only in $\mathfrak{C}$ we can show p cannot have a non-algebraic extension to any $\mathscr{M}'$ universal over $\mathscr{M}$. Suppose for contradiction that this is not the case; that is, that the realizations of p in $\mathfrak{C}$ are a set A where $|A| = \lambda$ but p still has a non-algebraic extension to $\mathscr{M}'$ where $\mathscr{M}'$ is universal over $\mathscr{M}$. Fix $\mathscr{N}$ where $M \cup A \subseteq \mathscr{N}$. By universality of $\mathscr{M}'$ over $\mathscr{M}$, there is an embedding $f : \mathscr{N} \to \mathscr{M}'$ fixing $\mathscr{M}$. Since p has a non-algebraic extension to some $q \in S(\mathscr{M}')$ there is $a \in \mathfrak{C} \setminus M')$ such that a realizes q. In particular, a realizes p. Clearly $a \notin f(\mathscr{N})$, thus $a \notin A$. This contradicts that A contained all realizations of p in $\mathfrak{C}$.

(3) $\implies$ (1) follows since any model $\mathscr{M}'$ of cardinality λ cannot contain all of the λ^+ realizations of p. $\square$

Lemma 3.1.6 *Suppose that $\mathscr{M}'$ is universal over $\mathscr{M}$, if $p \in S(\mathscr{M}')$ does not split over $\mathscr{M}$ then p is big.*

proof. Let $\mathscr{M}''$ be universal over $\mathscr{M}'$, by Lemma 2.1.2 there exists an extension q of p to $\mathscr{M}''$ that does not split over $\mathscr{M}$. In particular, q is non-algebraic, so by Proposition 3.1.5, p is big. □

Recall that $\kappa(\mathbf{K}, \mu)$ was defined in Definition 2.2.6.

Lemma 3.1.7 *If $\kappa(\mathbf{K}, \lambda) \leq \mu$, $p \in S(\mathscr{N})$ is non-algebraic and $\mathscr{N}$ is a (λ, θ)-limit model where $\theta \leq \mu$, then $p \in S(\mathscr{N})$ is big.*

proof. Let $(\mathscr{N}_i)_{i<\theta}$ be a limit-sequence for $\mathscr{N}$. Since $\kappa(\mathbf{K}, \mu) \leq \mu$ we can find $i < \theta$ such that p does split over $\mathscr{N}_i$, note that $\mathscr{N}$ is universal over $\mathscr{N}_i$ so by Lemma 3.1.6, p is big. □

Below we show that λ-stability implies that existence of big types and quasiminimal extensions. This is remarked on in (Grossberg and VanDieren, 2006a) (remark 2.8) and appears in (Lessmann, 2005) with a proof as Proposition 2.4.

Proposition 3.1.8 *There exists a big type $p \in S(\mathscr{M})$ for every $\mathscr{M} \in K_\lambda$, furthermore if $p \in S(\mathscr{M})$ is* already *big and $\mathscr{M} \prec_{\mathbf{K}} \mathscr{M}'$ where $\mathscr{M}, \mathscr{M}' \in K_\lambda$ then there is a big type $p' \in S(\mathscr{M}')$ that extends p.*

proof. By λ-stability if $\mathscr{M} \in K_\lambda$ we can find a model $\mathscr{N} \in K_\lambda$ that is universal over $\mathscr{M}$, then Proposition 3.1.5 implies that for any non-algebraic q type over $\mathscr{N}$, $q \upharpoonright \mathscr{N}$ is big. In particular, if $p \in S(\mathscr{M})$ is big, then by Proposition 3.1.5 we could take q to be a non-algebraic extension of p. If $\mathscr{M}' \in K_\lambda$ is any other extension of $\mathscr{M}$, then there is an embedding $f : \mathscr{M}' \to \mathscr{N}$ fixing $\mathscr{M}$. $f^{-1}(q)$ is a big extension of p to $\mathscr{M}'$. □

In (Shelah, 1999) Shelah proved that a quasiminimal type exists over any saturated model above LS($\mathbf{K}$) (this result is also available in (Baldwin, 2009) as Theorem 12.23). We prove an analog of this theorem for limit models; first we introduce a notion of "sufficient coherence" amongst types so that we can carry the necessary construction through limit stages. It was observed by Laskowski and/or Lessmann and/or Baldwin that this "sufficient coherence" could be extracted from the inductive conditions given in Shelah's argument for Claim 3.3 of (*Shelah*, 1999). Namely:

Definition 3.1.9 *Let $(\mathscr{M}_i)_{i<\gamma}$ be an increasing $\prec_{\mathbf{K}}$-chain of models. A* coherent chain *of Galois types is an increasing chain of types $p_i \in S(\mathscr{M}_i)$ with distinguished realizations $a_i \in \mathfrak{C}$ where $a_i \models p_i$ such that there exists a family of automorphisms $(f_{i,j})_{i<j<\gamma}$ with the following properties:*

1. $f_{i,j} \in \mathrm{aut}(\mathfrak{C}/\mathscr{M}_i)$.
2. $f_{i,j}(a_j) = a_i$
3. *For* $i < j < k < \gamma$ $f_{i,k} = f_{i,j}f_{j,k}$

A proof of the following appears in (Baldwin, 2009) Theorem 11.3.1, amongst other sources. The result is originally claimed without proof in (Shelah, 2001). The same result appears and is proved in (Grossberg and VanDieren, 2006a) as Lemma 2.12, and in (Lessmann, 2005) as Proposition 0.13 as well. A similar result that shows increasing chains of non-splitting types (instead of coherent sequences) admit an upper bound appears as Corollary I.4.14 of Theorem I.4.10 in (VanDieren, 2006).

Theorem 3.1.10 *If $(p_i)_{i<\delta}$, $p_i \in S(\mathscr{M}_i)$, is a coherent chain of Galois types, there exists a $p_\delta \in S(\mathscr{M}_\delta)$ that extends p_i for all $i < \delta$ such that $(p_i)_{i\leq\delta}$ is a coherent-chain.*

It should be noted, however, that there may not be a unique choice for p_δ. In general, given a sequence of galois types $(p_i)_{i<\delta}$ there may be many types $q \in S(\mathscr{M}_\delta)$ such that $q \models p_i$, for $i < \delta$. Because of this fact, coherence is a stronger condition on a sequence of types than merely being an increasing sequence; given an arbitrary increasing sequence of types $(q_i)_{i<\delta}$ there may be no $q \in \mathscr{M}_\delta$ such that $q_\delta \models q_i$ for $i < \delta$. In (Baldwin and Shelah, 2008) Baldwin and Shelah show that it is consistent with **ZFC** that there is an increasing chain of types $(p_i)_{i<\omega_2}$ but no $q \models p_i$ for all $i < \omega_2$. Koerwien has provided us with an example axiomatizable in $\mathscr{L}_{\omega_1,\omega}$ where AP does not hold and under **CH** there is an increasing sequence of types $(p_i)_{i<\omega_1}$ but no $q \models p_i$ for all for all $i < \omega_1$. We briefly sketch this example now.

Example 3.1.11 (Koerwien) *Let $\mathscr{L} = \{P, Q, R\} \cup \{E_n\}_{n<\omega}$ where P and Q are unary predicates interpreted as disjoint sets which partition the universe of an element of* **K**. *Each E_n is a binary relation interpreted as equivalence relations on P such that for each E_n is divided into two infinite E_{n+1} classes. We demand that if for all $n < \omega$, $E_n(x, y)$ holds, then $x = y$. There is a natural identification then between an element a in some structure in* **K** *and an element of 2^ω. Let R be an extensional binary relation which is a subset of $P \times Q$. Let $A_q := \{p \in P : R(q, p)\}$. We demand that the sets "A_q" are dense in the sense that A_q intersects every E_n-class for each $n < \omega$. That is, for some $q \in Q$, $p \in P$, and $n < \omega$ there must be some $p' \in P$ such that $E_n(p, p')$ and $R(q, p')$. such that $R(q, p)$ holds.*

The class described above can be axiomatized in $\mathscr{L}_{\omega_1.\omega}$. The maximal size of any model in $\mathbf{K}$ is $2^{\aleph_0}$. Consider the type $p_{\mathscr{M}}(x)$, over some model $\mathscr{M} \in \mathbf{K}$, of an element x such that "x is in Q, $x \notin M$, and for any $p \in P(\mathscr{M})$, $\neg R(x,p)$". Since the A_q are dense, for any q, q' A_q and $A_{q'}$ in a countable structure of Koerwien's class are isomorphic as trees. Even more strongly given any two realizations $q, q' \models p_{\mathscr{M}}$ where $|A_q| = |A_{q'}| = \omega$, A_q is isomorphic to $A_{q'}$ over $\mathscr{M}$.

Fix some maximal model $\mathscr{M}_{\omega_1}$ (of size continuum) in Koerwien's class. Under **CH**, $\mathscr{M}_{\omega_1}$ admits a filtration of length ω_1 by countable models. Fix some filtration $(\mathscr{M}_i)_{i<\omega_1}$ of $\mathscr{M}_{\omega_1}$ such that $p_i := p_{\mathscr{M}_i}$ is realized in $p_{\mathscr{M}_{i+1}}$. While each p_i is consistent (and p_{i+1} extends p_i), since $\mathscr{M}_{\omega_1}$ is maximal there cannot be an element a which realizes every p_i simultaneously.

We prove now that quasiminimal types exist. The following theorem appears as Proposition 3.2 in (Grossberg and VanDieren, 2006a) and as Propositions 1.5 (the countable case) and 2.6 (the uncountable case) in (Lessmann, 2005), the basic method is present in Theorem 9.5 part 5 of (Shelah, 1999). The proof offered below deals with details omitted in the uncountable case of (Lessmann, 2005). The proof uses the same underlying ideas as the proof in (Grossberg and VanDieren, 2006a).

Proposition 3.1.12 *Suppose* $\mathbf{K}$ *is* λ*-galois stable. There exists* $\mathscr{M} \in K_\lambda$ *and* $p \in S(\mathscr{M})$ *such that* p *is quasiminimal. Furthermore if* $p \in S(\mathscr{M})$ *is big then there is* $\mathscr{M}' \in K_\lambda$ *such that there exists* $p' \in S(\mathscr{M}')$ *extending* p *which is quasiminimal.*

proof. Once again the proposition follows from λ-stability. This result is proved in the same way that one proves λ-stability implies the non-existence of long splitting chains (see for example Lemma 12.2 of (Baldwin, 2009)). Since we have shown big types exist, its sufficient to prove the existence of quasiminimal extension. For the purpose of contradiction suppose $p \in S(M)$ is a big type (so $|M| = \lambda$) and there is no quasiminimal extension of p to any $M' \in K_\lambda$ which is a strong extension of M.

Fix κ to be minimal such that $\lambda^\kappa > \lambda^+$. We will construct a family of models $(\mathscr{M}_i)_{i<\kappa}$ in $\mathbf{K}_\lambda$ and a collection of types $(\mathrm{tp}^{\mathrm{ga}}(a_\nu/\mathscr{M}_{|\nu|}))_{\sigma \in 2^{<\kappa}}$, with automorphisms $(f_{\sigma,\tau})_{\sigma,\tau \in 2^{<\kappa}}$ with the following properties:

1. $\mathscr{M}_\emptyset = \mathscr{M}$, $p_\emptyset = p$.
2. $\mathscr{M}_{i+1} \in \mathbf{K}_\lambda$ is universal over $\mathscr{M}_i$.
3. $p_\sigma := \mathrm{tp}^{\mathrm{ga}}(a_\sigma/\mathscr{M}_{|\sigma|})$ is a big type.
4. $p_{\sigma^\frown 0} \neq p_{\sigma^\frown 1}$.
5. For each $\sigma < 2^{<\kappa}$, $(f_{\eta,\nu})_{\eta \subseteq \tau \subseteq \sigma}$ and $(\mathrm{tp}^{\mathrm{ga}}(a_\eta/\mathscr{M}_{|\eta|}))_{\eta \subseteq \sigma}$ form a coherent system. That is:
   (a) $f_{\eta,\nu} \in \mathrm{aut}(\mathfrak{C}/\mathscr{M}_\eta)$.
   (b) $f_{\eta,\nu}(a_\nu) = a_\eta$
   (c) For $\eta \subseteq \nu \subseteq \sigma$, $f_{\eta,\sigma} = f_{\eta,\nu} f_{\nu,\sigma}$

The base case is determined by condition 1.

Suppose we have defined $(\mathscr{M}_j)_{j \leq i}$, $(\mathrm{tp}^{\mathrm{ga}}(a_\sigma/\mathscr{M}_{|\sigma|}))_{|\sigma| \leq i}$ and $(f_{\sigma,\tau})_{\sigma \subseteq \tau, |\tau| \leq i}$. Note that $|2^i| < \lambda$; we will define approximations $\mathscr{M}^\sigma_{i+1}$ of $\mathscr{M}_{i+1}$ for $\sigma \in 2^i$. We also define approximations $p'_{\sigma^\frown 0}, p'_{\sigma^\frown 1} \in S(\mathscr{M}^\sigma_{i+1})$ of the types $p_{\sigma^\frown 0}, p_{\sigma^\frown 1}$ we want to define in our inductive construction. In order to do so, we fix some well-ordering α_i of 2^i and proceed to define the approximations $\mathscr{M}^{\sigma_k}_{i+1}$ and types $p'_{\sigma_k^\frown 0}, p'_{\sigma_k^\frown 1}$ by induction on $k < \alpha_i$.

First we note that p_{σ_0} is a big extension of p. By hypothesis, p has no quasiminimal extensions, so there is some $\mathscr{M}^{\sigma_0}_{i+1}$ where $\mathscr{M}_{i+1} \prec_{\mathbf{K}} \mathscr{M}^{\sigma_0}_{i+1}$ such that p_{σ_0} has two big extensions $p'_{\sigma_0\frown 0} \neq p'_{\sigma_1\frown 1}$ over $\mathscr{M}^{\sigma_0}_{i+1}$. Suppose we have defined $\mathscr{M}^{\sigma_j}_{i+1}$, $p_{\sigma_j\frown 0}$, and $p_{\sigma_j\frown 1}$ for $j \leq k$. By Proposition 3.1.8 $p_{\sigma_{k+1}}$ has a big extension q_{k+1} to $\mathscr{M}^{\sigma_k}_{i+1}$. Since p has no quasiminimal extension there must be a $\mathscr{M}^{\sigma_{k+1}}_{i+1}$ extending $\mathscr{M}^{\sigma_k}_{i+1}$ and big types $p'_{\sigma_{k+1}\frown 0} \neq p'_{\sigma_{k+1}\frown 1}$ over $\mathscr{M}^{\sigma_{k+1}}_{i+1}$ which both extend q_{k+1}. In particular, both $p'_{\sigma_{k+1}\frown 0}$ and $p'_{\sigma_{k+1}\frown 0}$ extend p_σ. At limit stages γ let $\mathscr{M}_\gamma = \bigcup_{j<\gamma} \mathscr{M}^{\sigma_j}_{i+1}$. Again there is a big extension q_γ of p_{σ_γ} to $\mathscr{M}_\gamma$. q_γ is not quasiminimal so there is some model $\mathscr{M}^{\sigma_\gamma}_{i+1}$ extending $\mathscr{M}_\gamma$ and distinct big types $p'_{\sigma_\gamma\frown 0}, p'_{\sigma_\gamma\frown 1} \in S(\mathscr{M}_\gamma)$ extending p_{σ_γ}.

Let $\mathscr{M}_{i+1}$ be a universal extension of $\mathscr{M}_i$ with $\bigcup_{j<\alpha_i} \mathscr{M}^{\sigma_j}_{i+1} \prec_{\mathbf{K}} \mathscr{M}_{i+1}$. For $n = 0, 1$ let $p_{\sigma\frown n}$ be a big extension of $p'_{\sigma\frown n}$ to $\mathscr{M}_{i+1}$. Fix realizations $a_{\sigma\frown 0}, a_{\sigma\frown 1} \in \mathfrak{C}$ of, respectively, $p_{\sigma\frown 0}, p_{\sigma\frown 1}$. Since $p_{\sigma\frown n}$ extends p_σ (for $n = 0, 1$) there are automorphisms $f_{\sigma,\sigma\frown n} \in \operatorname{aut}(\mathfrak{C}/\mathscr{M}_i)$ such that $f_{\sigma\frown n}(a_{\sigma\frown n}) = a_\sigma$ for $n = 0, 1$. For $\tau \subseteq \sigma$ and $n = 0, 1$ define $f_{\tau,\sigma\frown n} := f_{\tau,\sigma} f_{\sigma,\sigma\frown n}$.

At limit stages γ, let $\mathscr{M}_\gamma = \bigcup_{i<\gamma} \mathscr{M}_i$, note that $\mathscr{M}_\gamma$ is a limit model. Let $\sigma \in 2^\gamma$, by induction that $(p_\eta, f_{\eta,\nu})_{\eta\subseteq\nu\subsetneq\sigma}$ form a coherent-chain. So by Theorem 3.1.10 there is a a_σ and maps $f_{\eta,\sigma}$ so that $(p_\eta, f_{\eta,\nu})_{\eta\subseteq\nu\subseteq\sigma}$ form a coherent-chain. Since $\mathscr{M}_\gamma$ is a limit model, Lemma 3.1.7 implies that for each $\sigma \in 2^\gamma$, p_σ is big.

This completes the desired construction. By applying Theorem 3.1.10 to $(p_\sigma, f_{\sigma,\tau})_{\sigma\subseteq\tau\subseteq\nu}$ for each $\nu \in 2^\kappa$, we construct $2^\kappa \geq \lambda^+$ galois types over a model $\bigcup_{i<\kappa} \mathscr{M}_i \in \mathbf{K}_\lambda$, contradicting λ-stability. $\square$

Proposition 3.1.13 *Suppose that $p \in S(\mathscr{M})$ is quasiminimal and that $f : \mathscr{M} \to \mathscr{N}$ is an isomorphism, then $f(p)$ is also quasiminimal.*

proof. Suppose not, then $q := f(p)$ has two big extensions $q_1 \neq q_2$ over some $\mathscr{N}'$ where $\mathscr{N} \prec_{\mathbf{K}} \mathscr{N}'$. Extend f^{-1} to g to an automorphism of $\mathfrak{C}$ and let $p_1 = g(q_1), p_2 = g(q_2)$. If $p_1 = p_2$ then there is some $h \in \mathrm{aut}(\mathfrak{C}/g(\mathscr{N}'))$ such that for some $a_1 \models p_1, a_2 \models p_2$, $h(a_1) = a_2$. But notice that $g^{-1}(a_i) \models q_i$ and $g^{-1}hg : a_1 \mapsto a_2$ while fixing $\mathscr{N}'$, so $q_1 = q_2$, resulting in a contradiction. □

3.2 Vaughtian Pairs

We note that the analogue of Vaught's theorem that Lessmann proved in (Lessmann, 2005), has as it's hypothesis not just the existence of a Vaughtian pair in a larger cardinal, but a Vaughtian pair of saturated models. This assumption is made because in Proposition 3.2.2 we show that if a p-Vaughtian pair of saturated models exists then p is a big type. If p is *not* big, then it is quite easy to produce p-Vaughtian pairs.

In the first order case, merely demanding a syntactic type p is non-algebraic sufficient to show that p has arbitrarily many realizations. In the AEC case, however, this is not the case. In the following example due to David Kueker, we describe a p-Vaughtian pair of non-saturated models where p is non-algebraic (in the AEC sense) but p has exactly one realization in the monster model. In certain sources, such as (Baldwin, 2009), a Vaughtian pair of saturated models is referred to as a "true Vaughtian pair".

Example 3.2.1 *Let K be an AEC in vocabulary $\mathscr{L} = \{U\}$ where U is a unary predicate. The models of K are those $\mathscr{L}$ structures $\mathscr{M}$ such that $\neg U(\mathscr{M})$ is an infinite set and $|U| \leq 1$, where $\prec_{\mathbf{K}}$ is $\mathscr{L}$-substructure*

In the above example any two models $\mathscr{M} \prec_{\mathbf{K}} \mathscr{M}'$ that have $|U| = 0$ form a p-Vaughtian pair, where p is the type of the unique element $a \in \mathfrak{C}$ such that $\mathfrak{C} \models U(a)$. However, if p clearly is *not* big. In particular, though p is not algebraic in the sense that p is not realized in either model of the Vaughtian pair, p has only one realization in *any* $\mathscr{L}$-structure in $\mathbf{K}$. If, however, we assume that both models $\mathscr{M}, \mathscr{M}'$ are saturated, then we are able to rule out such pathological examples.

We now show that if there is a p-Vaughtian pair $(p, \mathscr{M}, \mathscr{M}')$ where $\mathscr{M}, \mathscr{M}'$ are $|\mathrm{dom}(p)|^+$-saturated then p is in fact a big type.[1]

Proposition 3.2.2 *Let $(p, \mathscr{M}, \mathscr{M}')$ be a p-Vaughtian pair where $|M| = |M'| \geq \lambda$ are $\max(|\mathrm{dom}(p)|, \lambda)^+$-saturated. Then p is a big type.*

proof. Let $(p, \mathscr{M}, \mathscr{M}')$ be as in the hypothesis to the proposition. Let $\mathscr{N} := \mathrm{dom}(p)$ and let $\kappa := \max(|N|, \lambda)^+$. By the definition of Vaughtian pair, p has a non-algebraic extension to $\mathscr{M}$, say $q = \mathrm{tp}^{\mathrm{ga}}(a/\mathscr{M})$ for some $a \in \mathfrak{C}$.

Suppose for contradiction that p has less than λ realizations in $\mathfrak{C}$, then let $A \subseteq \mathscr{M}$ be the set of realizations of p in $\mathscr{M}$, note $|A| \leq \kappa$. By downward Löwenheim-Skolem we can find $\mathscr{N}'$

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[1] Our result is quite close to Proposition 1.16 of (Lessmann, 2005) but we are careful to keep track of exactly how much saturation is required.

such that $N \cup A \subseteq \mathscr{N}' \prec_{\mathbf{K}} \mathscr{M}$ and $|N'| \leq \kappa$. Now consider $p' = \mathrm{tp}^{\mathrm{ga}}(a/\mathscr{N}')$; by κ^+-saturation of $\mathscr{M}$, p' is realized by some $a' \in M$. Since p' extends p, $a' \models p$ as well, but $a \notin A$, contradicting that A contained all realizations of p in $\mathscr{M}$.

Thus p must be a big type. □

We will show in the next corollary that the previous proposition allows us to assume without loss of generality that, given a p-Vaughtian pair of sufficiently saturated models, p is quasiminimal.

Corollary 3.2.3 *Let $(p, \mathscr{M}, \mathscr{M}')$ be a p-Vaughtian pair where $|M| = |M'| \geq \lambda$ are $\max(|\mathrm{dom}(p)|, \lambda)^+$-saturated. Then there is a quasiminimal type q extending p such that $(q, \mathscr{M}, \mathscr{M}')$ is a Vaughtian Pair.*

proof. By Proposition 3.2.2, p is big. Thus, by Proposition 3.1.12 p extends to a quasiminimal type q. By $\max(|\mathrm{dom}(p)|, \lambda)^+$-saturation of $\mathscr{M}$ we may assume that $\mathrm{dom}(q) \prec_{\mathbf{K}} \mathscr{M}$. Now, since q extends p any realization of q also realizes p, hence $q(\mathscr{M}') \subseteq p(\mathscr{M}') \subseteq \mathscr{M}$, as required for $(q, \mathscr{M}, \mathscr{M}')$ to be a q-Vaughtian pair. □

3.3 An Abstract Version of Lessmann's Two Cardinal Theorem

In this section we show that the existence of p_0-superlimit class in λ implies the existence of a two cardinal model in λ^+.

Theorem 3.3.1 *Let $\mathbf{K}$ be an AEC with $\mathrm{LS}(\mathbf{K}) \leq \lambda$, let K^L_λ be a (p, λ)-superlimit class[1], then there is a two-cardinal model $\mathscr{N}$ in K_{λ^+}. In particular, $\mathscr{N}$ will omit p.*

---

[1] *(see Definition 2.3.15)*

proof.

Suppose that $\mathscr{N}_0, \mathscr{N}_1 \in \mathbf{K}^L_\lambda$ are the distinguished models in the superlimit class, let $p \in S(\mathscr{M})$ be as in Definition 2.3.15. We construct adequate sequences[1], along with a sequence of types $(p_i)_{i<\lambda^+}$ and isomorphisms $(f_i)_{0<i<\lambda^+}$ inductively as follows:

- Begin with $(\mathscr{N}_0, \mathscr{N}_1)$. Note that $(\mathscr{N}_0, \mathscr{N}_1) \in \mathbf{K}^L_\lambda$ by axiom 2 of the superlimit class definition. Let $f_1 : \mathscr{N}_0 \to \mathscr{N}_1$ be an isomorphism that fixes $\mathscr{M}$.
- Suppose that we have defined $(\mathscr{N}_i)_{i<\theta} \in K^L_\lambda$ for a limit θ. Then by axiom 4 of the superlimit class definition if $\mathscr{N}_\theta = \bigcup_{i<\theta} \mathscr{N}_i$ then $(\mathscr{N}_i)_{i\le\theta} \in \mathbf{K}^L_\lambda$. Let $f_\theta : \mathscr{N}_0 \to \mathscr{N}_\theta$ be an isomorphism that fixes $\mathscr{M}$.
- Suppose that we have defined $(\mathscr{N}_i)_{i\le\theta} \in \mathbf{K}^L_\lambda$, for $\theta \ge 1$. Note that by superlimit class axiom 5 we can find $\mathscr{N}_{\theta+1}$ and an isomorphism $f_{\theta+1} : \mathscr{N}_1 \to \mathscr{N}_{\theta+1}$ such that $f_{\theta+1}(\mathscr{N}_0) = \mathscr{N}_\theta$ and $f_{\theta+1}$ fixes $\mathscr{M}$.

Define p_i to be the quasiminimal extension of p to $\mathscr{N}_i$.

Claim 3.3.2 *p_i cannot be realized in $\mathscr{N}_{i+1}$.*

Note that by construction we have an isomorphism $f_{i+1} : \mathscr{N}_1 \to \mathscr{N}_{i+1}$ such that $f(\mathscr{N}_0) = \mathscr{N}_i$ and f fixes $\mathscr{M}$. Since p_0 is not realized in $\mathscr{N}_1$, $f(p_0)$ cannot be realized in $\mathscr{N}_{i+1}$. Both p_i and $f(p_0)$ are quasiminimal extensions of p to $\mathscr{N}_i$, since f fixes $\mathscr{M}$. But, since p has a unique quasiminimal extension to $\mathscr{N}_i$ by axiom 1, $p_i = f(p_0)$. Hence p_i cannot be realized in $\mathscr{N}_{i+1}$.

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[1]Also defined in Definition 2.3.15

Let $\mathscr{N} = \bigcup_{i<\omega_1} \mathscr{N}_j$; We now argue that $\mathscr{N}$ omits p_0. Suppose for contradiction $a \in \mathscr{N}$ realizes p_0. Clearly $a \notin \mathscr{N}_0$, since p_0 is a non-algebraic type over $\mathscr{N}_0$. So for some $j < \omega_1$, $a \in \mathscr{N}_{j+1} \setminus \mathscr{N}_j$. By axiom 1d of the superlimit class, $q := \mathrm{tp}^{\mathrm{ga}}(a/\mathscr{N}_{j+1})$ is big and q extends p_0, so by quasiminimality of p_0, $q = p_j$, but p_j is not realized in N_{j+1} by Claim 3.3.2. So we arrive at a contradiction. Hence $\mathscr{N}$ omits p_0. $\square$

3.4 A Superlimit in $\aleph_0$

Having developed an abstract framework and proved an analogue of Vaught's Two Cardinal Theorem in this framework, we now offer an alternate proof of Lessmann's Two Cardinal Theorem by showing that if there is a Vaughtian pair of $\aleph_1$ saturated models then there is a p_0 such that the subclass of (ω, ω)-limit models is a p_0-Superlimit class for $\aleph_0$. For sake of brevity, will often refer to (ω, ω)-limit models as ω-limit models for the remainder of this section.

It's easy to see, by Proposition 2.2.5, that any two ω-limit models over isomorphic base models are isomorphic, indeed all countable limit models over isomorphic base models are isomorphic. The difficulty is finding a condition which will guarantee the union of a countable $\prec_{\mathbf{K}}$-increasing sequence of ω-limit models is a limit model; providing a method for doing so is a key insight of (Lessmann, 2005).

It's perhaps worth noting that in (Lessmann, 2005) the analysis is done using $\omega \times \omega$-limit models. The advantage of using $\omega \times \omega$-limit models, instead of ω-limit models or any randomly chosen countable limit ordinal is that given two limit sequences $(\mathscr{M}_i)_{i<\omega\times\omega}$ and $(\mathscr{N}_i)_{i<\omega\times\omega}$ over isomorphic bases, one can not only construct an isomorphism $f : \bigcup_{i<\omega\times\omega} \mathscr{M}_i \to \bigcup_{i<\omega\times\omega} \mathscr{N}_i$, but also have $f \restriction \bigcup_{i<n\times\omega} \mathscr{M}_i$ be an isomorphism onto $\bigcup_{i<n\times\omega} \mathscr{N}_i$ for any $n < \omega$. However, we proceed with the weaker uniqueness properties held by ω-limit models, though it has the unfortunate consequence of making some arguments more cumbersome.

We now define a partial order on countable limit sequences. While not exactly the same as Lessmann's partial order in (Lessmann, 2005), which he writes as"$<^*$", we draw inspiration from his definition. Shortly after his Proposition 1.12 in (Lessmann, 2005), Lessmann defined:

Definition 3.4.1 *Let $(\mathscr{M}_i)_{i<\alpha}$ and $(\mathscr{N}_i)_{i<\beta}$ be ω-limit sequences. $(\mathscr{M}_i)_{i<\alpha} <^* (\mathscr{N}_i)_{i<\beta}$ when for any $i < \alpha$ there exists some $j < \beta$ such that $\mathscr{N}_j$ is universal over $\mathscr{M}_i$.*

We demand more uniformity in our definition:

Definition 3.4.2
- *Given two limit sequences, $(\mathscr{M}_i)_{i<\alpha}$, $(\mathscr{N}_i)_{i<\beta}$ we say that $(\mathscr{N}_i)_{i<\beta}$ k-*dominates $(\mathscr{M}_i)_{i<\alpha}$ *if for all $i < \alpha$ there exists $k < \omega$ such that $\mathscr{N}_{i+k}$ is a proper universal extension of $\mathscr{M}_i$ and $\bigcup_{i<\omega} \mathscr{N}_i$ is a proper extension of $\bigcup_{i<\omega} \mathscr{M}_i$. We write $(\mathscr{M}_i)_{i<\alpha} <^k_L (\mathscr{N}_i)_{i<\beta}$.*
- *Given two limit sequences, $(\mathscr{M}_i)_{i<\alpha}$, $(\mathscr{N}_i)_{i<\beta}$ we say that $(\mathscr{N}_i)_{i<\beta}$* eventually -dominates $(\mathscr{M}_i)_{i<\alpha}$ *if for some k $(\mathscr{M}_i)_{i<\alpha} <^k_L (\mathscr{N}_i)_{i<\beta}$. We write $(\mathscr{M}_i)_{i<\alpha} <_L (\mathscr{N}_i)_{i<\beta}$.*

It is perhaps worth noting that the ordering above is obviously transitive, hence is a partial order on limit-sequences. The only natural way apparent to define k-domination on limit models, $\mathscr{M}$ and $\mathscr{M}'$ seems to be "$\mathscr{M}'$ k-dominates $\mathscr{M}$ if there exist limit sequences $(\mathscr{M}_i)_{i<\alpha} <^k_L (\mathscr{M}'_i)_{i<\beta}$ such that $\mathscr{M} = \bigcup_{i<\alpha} \mathscr{M}_i, \mathscr{M}' = \bigcup_{i<\beta} \mathscr{M}'_i$". While this provides a well-defined relation of limit models, we were unable to demonstrate this notion is transitive. Given limit models $\mathscr{M}, \mathscr{M}', \mathscr{M}''$ with $\mathscr{M}''$ k-dominating $\mathscr{M}'$ and $\mathscr{M}'$ k-dominating $\mathscr{M}$, we are not aware of a proof that $\mathscr{M}''$ k-dominates $\mathscr{M}$. The key obstacle is being able to answer the following question positively:

Question 3.4.3 *If you are given two limit models $\mathscr{M}$ and $\mathscr{N}$ and respective limit sequences $(\mathscr{M}_i)_{i<\alpha}, (\mathscr{N}_i)_{i<\beta}$ such that $(\mathscr{M}_i)_{i<\alpha} <^k_L (\mathscr{N}_i)_{i<\beta}$ as well as an additional limit sequence $(\mathscr{M}_i')_{i<\alpha'}$ for $\mathscr{M}$, can you find a limit sequence $(\mathscr{N}_i')_{i<\beta'}$ for $\mathscr{N}$ such that for some k', $(\mathscr{M}_i')_{i<\alpha'} <^{k'}_L (\mathscr{N}_i')_{i<\beta'}$?*

The fact $<_L$ may not be a partial order on limit models is what motivates us to define a superlimit class in terms of adequate sequences, instead of axiomatizing the properties of "$<''_L$ as a partial order. We define a notion of "adequate sequence" below that we will eventually prove is an adequate sequence in the sense of Definition 2.3.15.

Definition 3.4.4 *Let α be an ordinal such that $1 < \alpha < \omega_1$. We say a sequence of limit models, indexed by their superscripts*[1] *$(\mathscr{N}^i)^{i<\alpha}$ is an $\mathscr{N}^0$-*adequate sequence *if there exist limit sequences $(\mathscr{N}^i_n)_{n<\omega}$ for every $i < \alpha$ such that for all $i < j < \alpha$ $(\mathscr{N}^i_n)_{n<\omega} <_L (\mathscr{N}^j_n)_{n<\omega}$ where $\mathscr{N}^i_0 = \mathscr{N}^0_0$ for all $i < \alpha$.*

Below we reprove a special case of Proposition 2.2.5 because we want to notice a slightly stronger condition holds of the isomorphism constructed than is typically relevant. The proof is a standard back-and-forth argument.

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[1] *We choose to index an adequate sequence $(\mathscr{N}^i)^{i<\alpha}$ of limit models by superscripts so that when we fix limit sequences witnessing each $\mathscr{N}^i$ is a limit model we may write them as an array $(\mathscr{N}^i_n)^{i<\alpha}_{n<\omega}$.*

Lemma 3.4.5 *Given limit-sequences $(\mathscr{M}_i)_{i<\omega}$ and $(\mathscr{N}_i)_{i<\omega}$ there exists an isomorphism $f : \bigcup_{i<\omega}\mathscr{M}_i \to \bigcup_{i<\omega}\mathscr{N}_i$ and $f(\mathscr{M}_n) \prec_{\mathbf{K}} \mathscr{N}_{n+1}$. Furthermore, given an isomorphism $h : \mathscr{M}_0 \to \mathscr{N}_0$ then f may be taken to extend h. Finally, if $\mathscr{M}_0 = \mathscr{N}_0$ then f can be chosen to fix $\mathscr{M}_0 = \mathscr{N}_0$ pointwise.*

proof. Let $(\mathscr{M}_i)_{i<\omega}, (\mathscr{N}_i)_{i<\omega}$ be as in the hypothesis to the lemma. We will construct f as the limit of a sequence of maps $(f_i)_{i<\omega}$ constructed inductively, where each $f_i : \mathscr{M}_i \to \mathscr{N}_{i+1}$. We also construct a sequence of maps $(g_i)_{i<\omega}$ where $g_i : \mathscr{N}_{i+1} \to \mathscr{M}_{i+1}$ and $g_{i+1}f_i = \mathrm{Id}_{\mathscr{M}_i}$.

In the case $\mathscr{M}_0 = \mathscr{N}_0$ we simply choose $f_0 := \mathrm{Id}_{\mathscr{M}_0} = \mathrm{Id}_{\mathscr{N}_0}$, in the case $h : \mathscr{M}_0 \to \mathscr{N}_0$ is an isomorphism we take $f_0 := h$. Otherwise, we notice that by joint embedding we can find an $\mathscr{M}$ where $|M| = \omega$ such that $\mathscr{M}_0$ and $\mathscr{N}_0$ can both be embedded in $\mathscr{M}$. Without loss of generality $\mathscr{N}_0 \prec_{\mathbf{K}} \mathscr{M}$.

Since $\mathscr{N}_1$ is universal over $\mathscr{N}_0$ we can embed $\mathscr{M}$ into $\mathscr{N}_1$ over $\mathscr{N}_0$, so without loss of generality $\mathscr{M} \prec_{\mathbf{K}} \mathscr{N}_1$. Since $\mathscr{N}_1$ is universal over $\mathscr{N}_0$ it follows that $\mathscr{N}_1$ is universal over $\mathscr{M}$. We let $f_0 : \mathscr{M}_0 \to \mathscr{M}$ be a strong embedding of $\mathscr{M}_0$ into $\mathscr{M}$. Note that $f(\mathscr{M}_0) \prec_{\mathbf{K}} \mathscr{M} \prec_{\mathbf{K}} \mathscr{N}_0$. Extend f_0 to $f_0^* \in \mathrm{aut}(\mathfrak{C})$. $\mathscr{M}_0 \prec_{\mathbf{K}} {f_0^*}^{-1}(\mathscr{N}_1)$, so since $\mathscr{M}_1$ is universal over $\mathscr{M}_0$ there is an embedding h of ${f_0^*}^{-1}(\mathscr{N}_1)$ into $\mathscr{M}_1$ that fixes $\mathscr{M}_0$. Let $g_0 = h{f_0^*}^{-1} \restriction \mathscr{N}_1$. Since f_0 was defined only on $\mathscr{M}_0$, which is fixed by the embedding h, g_0 extends f_0^{-1}.

Suppose that we have defined $(f_i)_{i\leq n}, (g_i)_{i\leq n}$. We now must define f_{n+1} and g_{n+1}:

Inductively, $g_n : \mathscr{N}_{n+1} \to \mathscr{M}_{n+1}$. Extend g_n to an automorphism $g_n^* \in \mathrm{aut}(\mathfrak{C})$, since $\mathscr{N}_{n+2}$ is universal over $\mathscr{N}_{n+1}$ we can find some embedding h of ${g^*}_n^{-1}(\mathscr{M}_{n+1})$ into $\mathscr{N}_{n+2}$ over $\mathscr{N}_{n+1}$, let

$f_{n+1} = hg^{*-1}_{n}$. Note that since h fixed $\mathscr{N}_{n+1}$ which contains the image of f_n, f_{n+1} extends f_n. Note that f_{n+1} also extends g_n^{-1}, since $\mathrm{dom}(g_n) = \mathscr{N}_{n+1}$ which, we have noted, was fixed by h.

Similarly, let f^*_{n+1} be an extension of f to an automorphism of $\mathfrak{C}$, let h be an embedding of $f^{*-1}_{n+1}(\mathscr{N}_{n+1})$ into $\mathscr{M}_{n+2}$ over $\mathscr{M}_{n+1}$, and let $g_{n+1} := hf^{*-1}_{n+1} \restriction \mathscr{M}_{n+1}$. Again, since h fixes $\mathscr{M}_{n+1}$, which contains the image of g_n, g_{n+1} extends g_n. Also, since $\mathrm{dom}(f_{n+1}) = \mathscr{M}_{n+1}$, which is fixed by h, g_{n+1} extends f_n^{-1}. $\square$

We now prove closure under $<_L$-chains. Unfortunately, we have not been able to generalize this result to (λ, θ)-limit models for $\theta < \lambda^+$ where $\lambda = LS(K) > \omega$. One can demonstrate an analogue of this result holds for $\lambda > \omega$ when $\theta = \mathrm{cf}(\lambda)$; however, the argument below is not valid if $\theta \neq \mathrm{cf}(\lambda)$. We only prove the proposition for ω-limit models, since this is the class where we are desire to apply the result.

Proposition 3.4.6 *Given a $<_L$-sequence of ω-limit sequences*[1] $(\mathscr{M}_i^j)_{i<\omega}^{j<\alpha}$ *for α a countable ordinal, $\bigcup_{j<\alpha} \mathscr{M}_\omega^j$ is an ω-limit model. Furthermore, there is a limit sequence $(\mathscr{M}_i^\alpha)_{i<\omega}$ for $\mathscr{M}^\alpha$ such that $(\mathscr{M}_i^j)_{i<\omega}^{j\leq\alpha}$ is a $<_L$-sequence and $\mathscr{M}_0^\alpha = \mathscr{M}_0^0$.*

proof. If α is not a limit ordinal then the result is trivial. If $\alpha = \beta + 1$ then $(\mathscr{M}_i^\beta)_{i<\omega}$ satisfies the desired conditions.

So suppose α is a countable limit ordinal. We must find a limit sequence $(\mathscr{M}_i^\alpha)_{i\in\omega}$ to witness that $\mathscr{M}_\omega^\alpha := \bigcup_{i<\alpha} \mathscr{M}_\omega^i$ is an ω-limit model.

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[1] *For a fixed j, $(\mathscr{M}_i^j)_{i<\omega}$ is a limit sequence.*

Since α is a countable ordinal we may fix a cofinal sequence $(\alpha_i)_{i<\omega}$. Without loss of generality we assume $\alpha_0 = 0$. We define $(\mathscr{M}_n^\alpha)_{n<\omega}$ and a sequence of integers $(\ell_n)_{n<\omega}$ inductively with the following properties:

1. $\mathscr{M}_0^\alpha = \mathscr{M}_0^0$, $\ell_0 = 0$.

2. $\mathscr{M}_n^\alpha = \mathscr{M}_{\ell_n}^{\alpha_n}$

3. $\mathscr{M}_{n+1}^\alpha$ is universal over $\mathscr{M}_n^\alpha$

4. For all j where $0 \leq j < n$, there exists $k \leq \ell_n$, $(\mathscr{M}_i^{\alpha_j})_{i<\omega} <_L^k (\mathscr{M}_i^{\alpha_n})_{i<\omega}$

The base case is determined by the first condition above. Suppose that we have already chosen ℓ_m and $\mathscr{M}_{\ell_m}^{\alpha_m}$ for $0 \leq m \leq n$; we must define ℓ_{n+1} and $\mathscr{M}_{\ell_{n+1}}^{\alpha_{n+1}}$. Since the $(\mathscr{M}_i^j)_{i\in\omega}^{j<\alpha}$ is a $<_L$ sequence, for each $i, j < n+1$ there is some $k_{i,j} < \omega$ such that $(\mathscr{M}^{\alpha_i})_{i<\omega} <_L^{k_{i,j}} (\mathscr{M}^{\alpha_j})_{i<\omega}$. Choose $\ell_{n+1} = \max(\{k_{i,j} : 0 < i < j < n+1\} \cup \{n\})$.

Since ℓ_{n+1} is at least $k_{n,n+1}$, $\mathscr{M}_{\ell_{n+1}}^{\alpha_{n+1}}$ will be universal over $\mathscr{M}_{\ell_n}^{\alpha_n}$, thus $(\mathscr{M}_n^\alpha)_{n<\omega}$ is a limit sequence as desired. All that remains is to show that if $j < \alpha$ then $(\mathscr{M}_n^j)_{n<\omega} <_L (\mathscr{M}_n^\alpha)_{n<\omega}$. That is, if we fix $j < \alpha$ and we must find $k < \omega$ such that for any $n < \omega$ $\mathscr{M}_{n+k}^\alpha$ is universal over $\mathscr{M}_n^j$.

Fix m so that $\alpha_m < j < \alpha_{m+1}$, let $k := k_{j,\alpha_{m+1}}$. Note that $(\ell_n)_{n<\omega}$ is increasing, so we can find some N such $N > m+1$ and $\ell_N > k$. We know then that for any $n < \omega$ that $\mathscr{M}_{n+k}^{\alpha_{m+1}}$ is universal over $\mathscr{M}_n^j$ and we know for n where $N \leq n < \omega$ that $\mathscr{M}_{n+1}^\alpha = \mathscr{M}_{\ell_{n+1}}^\alpha$ is universal over $\mathscr{M}_{n+k}^{\alpha_{m+1}}$. Since $\ell_N > k$, it follows that for all $n < \omega$, $\mathscr{M}_{n+\ell_N}^{\alpha_m+1}$ is universal over $\mathscr{M}_n^j$, thus $(\mathscr{M}^j)_n <_L^{\ell_N} (\mathscr{M}_n^\alpha)_{n<\omega}$. $\square$

Corollary 3.4.7 *Suppose $(\mathscr{N}^j)^{i<\alpha}$ is an $\mathscr{N}_0$-adequate sequence for some countable limit ordinal α. If $\mathscr{N}^\alpha := \bigcup_{j<\alpha} \mathscr{N}^j$ then $(\mathscr{N}^\alpha)^{i\leq\alpha}$ is an $\mathscr{N}_0$-adequate sequence and $\mathscr{N}^0 \cong \mathscr{N}^\alpha$ over $\mathscr{N}_0^0$.*

proof. Since $(\mathscr{N}^j)^{i<\alpha}$ is $\mathscr{N}^0$-adequate there exist limit sequences $(\mathscr{N}_n^j)_{n<\omega}^{j<\alpha}$ where $\mathscr{N}_n^j = \mathscr{N}_0^0$ and if $i < j < \alpha$ then $(\mathscr{N}_n^i)_{n<\omega} <_L (\mathscr{N}_n^j)_{n<\omega}$. So by Proposition 3.4.6 we can find a limit sequence $(\mathscr{N}_n^\alpha)_{n<\omega}$ such that $\mathscr{N}_0^\alpha = \mathscr{N}_0^0 = \mathscr{N}_0$, for all $i < \alpha$, $(\mathscr{N}_n^i)_{n<\omega} <_L (\mathscr{N}_n^\alpha)_{n<\omega}$, and $\mathscr{N}^\alpha := \bigcup_{n<\omega} \mathscr{N}_n^\alpha = \bigcup_{n<\omega, j<\alpha} \mathscr{N}_n^j$. By Proposition 3.4.5 $\mathscr{N}^\alpha \cong \mathscr{N}^0$ over $\mathscr{N}_0^0$. $\square$

Proposition 3.4.8 *Suppose that $(\mathscr{N}^j)^{j\leq\alpha}$ is a $(\mathscr{N}^0)$-adequate sequence for $\alpha \geq 1$, then there exists $\mathscr{N}^{\alpha+1}$ such that $(\mathscr{N}^j)^{j\leq\alpha+1}$ is a $\mathscr{N}^0$-adequate sequence and there is an isomorphism $f : \mathscr{N}^1 \to \mathscr{N}^{\alpha+1}$ such that $f(\mathscr{N}^0) = \mathscr{N}^\alpha$ and f fixes $\mathscr{N}_0^0$-pointwise.*

proof. Suppose $(\mathscr{N}^j)^{j\leq\alpha}$ is an $\mathscr{N}^0$-adequate sequence and fix limit sequences $(N_n^j)_{n<\omega}^{j\leq\alpha}$ which witness this. By Proposition 3.4.6 we know that there is an isomorphism $f : \mathscr{N}^\alpha \to \mathscr{N}^0$ which fixes $\mathscr{N}_0^0$ pointwise and satisfies $f(\mathscr{N}_n^\alpha) \prec_{\mathbf{K}} \mathscr{N}_{n+1}^0$. Extend f to $F \in \mathrm{aut}(\mathfrak{C}/\mathscr{N}_0^0)$ and let $\mathscr{N}^{\alpha+1} := F^{-1}(\mathscr{N}^1), \mathscr{N}_n^{\alpha+1} := F^{-1}(\mathscr{N}_n^1)$. Since F fixes $\mathscr{N}_0^0$, $\mathscr{N}_0^{\alpha+1} = \mathscr{N}_0^0$. Clearly the $\mathscr{N}_n^{\alpha+1}$ witness that $\mathscr{N}^{\alpha+1}$ is a limit model over $\mathscr{N}_0^0$, since $\mathscr{N}^1$ is a limit model over $\mathscr{N}_0^0$.

We must argue that for any $i < \alpha+1$, $(\mathscr{N}_n^i)_{n<\omega} <_L (\mathscr{N}_n^{\alpha+1})_{n<\omega}$. Fix $k < \omega$ such that $(\mathscr{N}_n^0)_{n<\omega} <_L^k (\mathscr{N}_n^1)_{n<\omega}$. So for $n < \omega$, $\mathscr{N}_{n+k}^1$ is universal over $\mathscr{N}_n^0$. We know that $F(\mathscr{N}_n^\alpha) \prec_{\mathbf{K}} \mathscr{N}_{n+1}^0$, we know $\mathscr{N}_{n+1+k}^1$ is universal over $\mathscr{N}_{n+1}^0$, hence also $F(\mathscr{N}_n^\alpha)$. Thus, it follows that $F^{-1}(\mathscr{N}_{n+1+k}^1) = \mathscr{N}_{n+1+k}^{\alpha+1}$ is universal over $F^{-1}(F(\mathscr{N}_n^\alpha)) = \mathscr{N}_n^\alpha$. So $(\mathscr{N}_n^\alpha)_{n<\omega} <_L^{k+1} (\mathscr{N}_n^\alpha)_{n<\omega}$.

Fix $i < \alpha + 1$, we know that for some $\ell < \omega$, $(\mathscr{N}_n^i)_{n<\omega} <_L^\ell (\mathscr{N}_n^i)_{n<\omega}$. It should be clear that $(\mathscr{N}_n^i)_{n<\omega} <_L^{\ell+1} (\mathscr{N}_n^i)_{n<\omega}$. $\square$

Below we provide a reformulation of Lessmann's theorem in our terminology. In it's original form it appears in the body of the proof of Proposition 1.18. We have isolated it and stated it separately below.

Lemma 3.4.9 (Lessmann) *If there is a p-Vaughtian pair of $\aleph_1$-saturated models $\mathscr{M}_0, \mathscr{M}_1 \in K_\mu$ then there is a p_0-Vaughtian pair of ω-models $(\mathscr{N}^0, \mathscr{N}^1)$ in $\aleph_0$ such that:*

1. *p_0 is a big extension of some quasiminimal $p_{\mathscr{M}}$ where $\mathscr{N}^0, \mathscr{N}^1$ are limit models over $\mathscr{M}$.*
2. *There are limit sequence $(\mathscr{N}_n^0)_{n<\omega} <_L^0 (\mathscr{N}_n^1)_{n<\omega}$ where $\mathscr{N}_0^0 = \mathscr{N}_0^1 = \mathscr{M}$.*

proof. Suppose that $(p, \mathscr{M}_0, \mathscr{M}_1)$ is a Vaughtian pair of $\aleph_1$-saturated models in μ. By Corollary 3.2.3, we may replace p by $p_{\mathscr{M}} \in S(\mathscr{M})$ where $p_{\mathscr{M}}$ is quasiminimal and $\mathscr{M} \prec_{\mathbf{K}} \mathscr{M}_0$, $|M| = \omega$. Let $A = p_{\mathscr{M}}(\mathscr{M}_1)$. Since $(p_{\mathscr{M}}, \mathscr{M}_0, \mathscr{M}_1)$ is a Vaughtian Pair, $A = p_{\mathscr{M}}(\mathscr{M}_0)$.

We will inductively construct limit sequences $\mathscr{N}_n^0, \mathscr{N}_n^1$ to satisfy the following properties:

1. $\mathscr{N}_0^0 = \mathscr{N}_1^1 = \mathscr{M}$
2. $\mathscr{N}_n^0 \prec_{\mathbf{K}} \mathscr{M}_0, \mathscr{N}_n^1 \prec_{\mathbf{K}} \mathscr{M}_1$.
3. $A \cap \mathscr{N}_n^1 \subseteq N_{n+1}^0$.
4. $\mathscr{N}_n^1$ is universal over $\mathscr{N}_n^0$.

$\square$

Let $\mathscr{N}_0^0 = \mathscr{N}_1^1 = \mathscr{M}$. Given $(\mathscr{N}_j^i)_{j\leq n}$ we define $\mathscr{N}_{n+1}^0$ and $\mathscr{N}_{n+1}^1$ as follows. By downward Löwenheim-Skolem we can find $\mathscr{N} \prec_{\mathbf{K}} \mathscr{M}_0$ such that $A \cap N_n^1 \subseteq N_{n+1}^0$ and $|N| = \omega$. By stability and $\aleph_1$-saturation of $\mathscr{M}_0$ we can find $\mathscr{N}_{n+1}^0$ that is universal over $\mathscr{N}_n^0$ and contains $\mathscr{N}$. By stability and $\aleph_1$-saturation of $\mathscr{M}^1$ we can choose $\mathscr{N}_{n+1}^1 \prec_{\mathbf{K}} \mathscr{M}_1$ to be universal over both $\mathscr{N}_n^0$ and $\mathscr{N}_n^1$.

Set $\mathscr{N}^i := \bigcup_{n<\omega} \mathscr{N}_n^i$ for $i = 0, 1$. By Proposition 3.1.8 $p_{\mathscr{M}}$ has a big extension $p_0 \in S(\mathscr{N}^0)$. By quasiminimality of $p_{\mathscr{M}}$, p_0 is quasiminimal. We argue that p_0 is not realized in $N^1 \setminus N^0$. If $a \in N^1 \setminus N^0$, for some $n < \omega$, $a \in N_n^1$, but then $a \in N_{n+1}^0$, contradicting that $a \notin N^0$.

We note that by condition 4, $(\mathscr{N}_n^0)_{n<\omega} <_L^0 (\mathscr{N}_n^1)_{n<\omega}$.

Corollary 3.4.10 *If there is a p-Vaughtian pair of $\aleph_1$-saturated models $(\mathscr{M}_0, \mathscr{M}_1)$ in K then there is a p_0-Vaughtian pair of ω-limit models $(\mathscr{N}^0, \mathscr{N}^1)$ in $\aleph_0$ such that:*

1. *p_0 is a quasiminimal extension of some $p_{\mathscr{M}} \in S(\mathscr{M})$*
2. *$\mathscr{N}^0 \cong \mathscr{N}^1$ over $\mathscr{M}$.*
3. *The is a unique quasiminimal extension of $p_{\mathscr{M}}$ to any model $\mathscr{N}^i$ where for some θ, $i < \theta < \lambda^+$ $(\mathscr{N}^i)$ is an $\mathscr{N}^0$-adequate sequence.*
4. *Any non-algebraic extension of p_0 to a model $\mathscr{N}^i$, where for some θ, $i < \theta < \lambda^+$ $(\mathscr{N}^i)$ is an $\mathscr{N}^0$-adequate sequence, is big.*

proof. By Lemma 3.4.9 we can find a p_0-Vaughtian pair $(\mathscr{N}^0, \mathscr{N}^1)$ where p_0 extends a quasiminimal type $p_{\mathscr{M}}$. It follows that p_0 is itself quasiminimal. Furthermore, since $p_{\mathscr{M}}$ is already quasiminimal, there is a unique quasiminimal extension of $p_{\mathscr{M}}$ to any model $\mathscr{M} \in \mathbf{K}_\omega$

such that $\mathscr{N}^0 \prec_{\mathbf{K}} \mathscr{M}$. In particular this holds if $\mathscr{M} = \mathscr{N}^i$ where for some θ, $i < \theta < \lambda^+$ $(\mathscr{N}^i)^{i<\theta}$ is an $\mathscr{N}^0$-adequate sequence.

By 3.1.7 any non-algebraic type over a limit model is big, so in particular this holds for any non-algebraic extension of p_0 to a model $\mathscr{N}^i$, where for some θ, $i < \theta < \lambda^+$ $(\mathscr{N}^i)^{i<\theta}$ is an $\mathscr{N}^0$-adequate sequence.

Since $(\mathscr{N}_n^0)_{n<\omega} <_L^0 (\mathscr{N}_n^1)_{n<\omega}$ are $\mathscr{N}^0$-adequate, by Proposition 3.4.5 they are isomorphic over $\mathscr{N}_0^0 = \mathscr{N}_0^1 = \mathscr{M}$. □

Theorem 3.4.11 *If there exists a p-Vaughtian pair of $\aleph_1$-saturated models then*

$$K_\omega^L := \{(\mathscr{M}^i)^{i<\alpha} \ : 0 < \alpha < \omega_1,\ (\mathscr{M}^i)^{i<\alpha} \text{ is an } \mathscr{N}^0 - \text{ adequate sequence and } \mathscr{M}^1 = \mathscr{N}^1\}$$

is an ω-superlimit class, where p_0, $\mathscr{N}^0$, and $\mathscr{N}^1$ are as in Lemma 3.4.9 and Corollary 3.4.10 .

proof. By Corollary 3.4.10, all the conditions of superlimit axiom 1 are satisfied. By definition of K_ω^L, axiom 2 holds. By Lemma 3.4.5 axiom 3 holds. By Corollary 3.4.7 axiom 4 holds. By Proposition 3.4.8 axiom 5 holds as well. □

We finish by deducing the result Lessmann originally proved in (Lessmann, 2005) as Proposition 1.18:

Corollary 3.4.12 (Lessmann) *If there exists a p-Vaughtian pair of saturated models in $\aleph_1$ then there exists a two cardinal model in $\aleph_1$.*

proof. By the Theorem 3.4.11, if there exists a p-Vaughtian pair of saturated models there is a p_0 such that $\mathbf{K}_\omega^L$ is a superlimit class in $\aleph_0$. By Theorem 3.3.1 there is a two cardinal model in $\aleph_1$. $\square$

CHAPTER 4

UNIQUENESS OF LIMIT MODELS

As noted in Chapter 2 there is a nice equivalence between superstability and uniqueness of limits in elementary classes. There is, then, a natural desire to have an analogous result for abstract elementary classes. Various published (and a few unpublished) works have addressed this question, namely (Villaveces and Shelah, 1999), (VanDieren, 2006), (Grossberg et al., 2012). and (Zambrano and Villaveces, 2010). (Shelah, 2008) investigates uniqueness of limit models in the more abstract framework of good frames; while good results are obtained by this method, it's less clear when a concrete AEC with μ-splitting satisfies Shelah's "good frame" axioms. Outside of good frames, as of yet, there has not yet been a satisfactory proof that for any $\sigma_1, \sigma_2 < \mu^+$ any (μ, σ_1)-limit is isomorphic to a (μ, σ_2) limit model except under the relatively strong hypothesis of μ^+ categoricity (plus other conditions). At the same time, there is no known counterexample of two non-isomorphic limit models in an Abstract Elementary Class that should qualify as "superstable". (Example 2.2.8 takes place in a strictly stable elementary class). The following question, which is settled by Theorem 2.2.11 in the elementary case, remains open in the abstract case:

Question 4.0.1 *Let K be an Abstract Elementary Class and $\mu > LS(K) > \aleph_0$, suppose that K satisfies the μ-amalgamation property and the μ-joint embedding property. If K is μ-Galois* *stable and* $\kappa(\mathbf{K},\mu)=\omega$*, are any two* $(\mu,\sigma_1),(\mu,\sigma_2)$*-limit models over* $\mathscr{M}$ *are isomorphic over* $\mathscr{M}$*?*

The strongest published result is VanDieren's work in the erratum to (VanDieren, 2006). The following theorem appears in (VanDieren, 2012):

Theorem 4.0.2 (GCH,$\phi_{\mu^+}(S^{\mu^+}_{\mathrm{cf}(\mu)})$**)(VanDieren)** [1] *Assume that* $\mathbf{K}$ *is a* μ^+*-categorical abstract elementary class with no maximal models, for some* $\mu \geq \mathrm{LS}(\mathbf{K})$*. Further assume that* ***GCH*** *and* $\phi_{\mu^+}(S^{\mu^+}_{\mathrm{cf}(\mu)})$ *hold. Let* θ_1 *and* θ_2 *be limit ordinals* $< \mu^+$ *. Under Hypothesis 1,2 if* $\mathscr{M}_1$ *and* $\mathscr{M}_2$ *are* (μ,θ_1)*- and* (μ,θ_2)*-limit models over* $\mathscr{M}$*, respectively, then there exists an isomorphism* $f:\mathscr{M}_1\cong\mathscr{M}_2$ *such that* $f\restriction\mathscr{M}$ *is the identity mapping.*

We offer a small result that is potentially useful in the program of deriving uniqueness of limit models from some superstability-like assumption in μ, namely Proposition 4.1.5, which eliminates the need to assume disjoint amalgamation to prove one of the necessary steps of the proof. Since disjoint amalgamation is provable from μ^+ categoricity and the existence of arbitrarily large models, there is no immediate strengthening of the result in (VanDieren, 2006), however there is potential application toward answering Question 4.0.1 positively.

In this chapter we sketch the new result in Section 4.1 , then explain how the result fits into the general program of research toward answering Question 4.0.1 in Section 4.2. Finally, we end with a brief discussion regarding the applicability of uniqueness of limits towards upward

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[1] *Details on the set-theoretic principle "*$\phi_{\mu^+}(S^{\mu^+}_{\mathrm{cf}(\mu)})$*" can be found in (VanDieren, 2006) or Appendix C of (Baldwin, 2009).*

categoricity transfer[1]. There is no new research present in Section 4.2 merely a discussion of existing work and how Proposition 4.0.1 fits into the broader program of work on uniqueness of limit models. Existing work done towards the goal of proving uniqueness of limit models has proceeded through a careful analysis of the situation in a fixed cardinal μ.[2] For this reason it is convenient to fix some default values for certain parameters and omit them from discussion for the rest of this chapter, unless they deviate from these default values.

Notation 4.0.3 *Unless otherwise stated, for the remainder of this section all models are assumed to be of size μ, limit models are (μ, θ)-limit models for some $\theta < \mu^+$, and all splitting discussed is μ-splitting.*

4.1 Towers

We define a "tower", which is an increasing sequence of limit models, along with a fixed sequence of non-splitting extensions of fixed sequence of types. Towers were introduced by Shelah and Villaveces in (Villaveces and Shelah, 1999), though similar objects were previously defined by Shelah in (Shelah, 2001). In (Zambrano and Villaveces, 2010) Zambrano and Villaveces generalize towers to the context of metric AECs.

Definition 4.1.1 *1. Let I be an infinite well order. A* tower *is a triple $(\bar{\mathscr{M}}, \bar{a}, \bar{\mathscr{N}})$ where $\bar{\mathscr{M}} = \langle \mathscr{M}_i : i \in I \rangle$ is a $\prec_{\mathbf{K}}$ increasing sequence of limit models of cardinality μ, $\bar{a} =$*

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[1]in particular, the kind of results we explore in Chapter 3) in Section 4.3

[2]Ideally, such an analysis would be then be applicable to any $\mu \geq \mathrm{LS}(\mathbf{K})$, currently the existing theorems are only valid for a $\mu \geq \mathrm{LS}(\mathbf{K})$ such that $\mathbf{K}$ is μ^+ categorical.

$\langle a_i : i \in I\rangle$ *and* $\bar{\mathscr{N}} = \langle \mathscr{N}_i : i \in I\rangle$ *where* $a_i \in \mathscr{M}_{i+1} \setminus \mathscr{M}_i$, $\mathscr{M}_i$ *is a limit over* $\mathscr{N}_i$ *and* $\mathrm{tp}^{\mathrm{ga}}(a_i/\mathscr{M}_i)$ *does not* μ*-split over* $\mathscr{N}_i$*. Let* $\mathbf{K}^*_{\mu,I}$ *denote the class of such towers.*

2. *We say a tower* $(\bar{\mathscr{M}}, \overline{a}, \bar{\mathscr{N}}) \in \mathbf{K}^*_{\mu,I}$ *is* continuous *if for a limit* $i \in I$ $\mathscr{M}_i = \bigcup_{j<i} \mathscr{M}_j$*.*

Note that while each model $\mathscr{M}_i$ in a tower $(\bar{\mathscr{M}}, \overline{a}, \bar{\mathscr{N}})$ is a limit model, the union $\bigcup_{i\in I} \mathscr{M}_i$ of models in the tower need not be a limit model, that is, there is no requirement that for $i < j$, $\mathscr{M}_j$ is universal over $\mathscr{M}_i$. We also do not demand that $(\bar{\mathscr{M}}, \overline{a}, \bar{\mathscr{N}})$ be continuous, that is, we do not demand that for a limit ordinal $i \in I$ $\mathscr{M}_i = \bigcup_{j<i} \mathscr{M}_j$. However, by adding the additional assumptions that a tower is "relatively full" (which we define later) and continuous we can show that the union of the models comprising such a tower *is* a limit model. We index towers by generic well orders instead of ordinals because at various points we will want to take extensions of a well-order I that are *not* end extensions, which is often assumed to be the natural notion of extension for ordinals.

Continuity is a significant property of a tower. Unfortunately it does not seem to be possible to work *only* with continuous towers, since certain extension results proved in (Grossberg et al., 2012) or (VanDieren, 2006) are proved in a manner that destroys continuity of a tower. On the other hand certain results, in particular our Proposition 4.1.5 or the analogous Proposition II.6.7 in (VanDieren, 2006) require continuity of the tower.

We note that stability implies the existence of a tower in $\mathbf{K}^*_I$ for any well ordered I.

Proposition 4.1.2 *If I is well-ordered and $\kappa(\mathbf{K},\mu) \le \mu$*[1] *then there exists some $(\bar{\mathscr{M}}, \bar{a}, \bar{\mathscr{N}}) \in \mathbf{K}^*_I$.*

proof. We will construct $(\bar{\mathscr{M}}, \bar{a}, \bar{\mathscr{N}})$ by induction on $i \in I$. Let $\theta = \kappa(\mathbf{K},\mu)$; by hypothesis $\theta \le \mu$.

By Corollary 2.3.20 there exists a limit model $\mathscr{M}^0 = \bigcup_{i<\theta} M^0_i$. If $p \in S(\mathscr{M}^0)$ is any non-algebraic type, then $\kappa(\mathbf{K},\mu) = \theta$ implies that p does not split over $\mathscr{M}^0_i$ for some $n < \theta$. Let $\mathscr{N}_0 = \mathscr{M}^0_i$, let a_0 be any realization of p.

Suppose that for $j \le k$ we have defined models $\mathscr{M}^j, \mathscr{N}_j, a_j$ with a corresponding limit sequences $(\mathscr{M}^j_i)_{i<\theta}$ for each $\mathscr{M}^j$. By Corollary 2.3.20 there is an $\mathscr{M}^{j+1}$ containing a_j which is a (μ,θ)-limit model over $\mathscr{M}^j$, as in the base case. Fix a limit sequence $(\mathscr{M}^{j+1}_i)_{i<\theta}$ for $\mathscr{M}^{j+1}$ where $\mathscr{M}^{j+1}_0 = \mathscr{M}^j$. Since $\kappa(\mathbf{K},\mu) = \theta$ there is some $\mathscr{M}^j_i$ over which $\mathrm{tp}^{\mathrm{ga}}(a_i/\mathscr{M}^i)$ does not split such that $\mathscr{M}^i$ is universal over $\mathscr{N}^i$. Fix a_{i+1} as a realization of any non-algebraic type in $\mathscr{M}^{i+1}$.

Suppose that we have defined $\mathscr{M}^j, \mathscr{N}_j, a_j$ for $j < i$, where k is a limit ordinal, along with limit sequences $(\mathscr{M}^j_i)_{i<\theta}$ for each $j < k$ such that $\mathscr{M}^{j+1}_0 = \mathscr{M}^j$. Let $\mathscr{M}_i := \bigcup_{j<i} \mathscr{M}_j = \bigcup_{j<k, i<\theta} \mathscr{M}^j_i$, if we define $\mathscr{M}_\alpha = \mathscr{M}^j_i$ where $\alpha < k \times \theta$ satisfies $\alpha = i \times j$, then we see that $\mathscr{M}_\alpha$ is a $(\mu, k \times \theta)$-limit model. Choose a_k to realize any non-algebraic type over $\mathscr{M}_k$, by $\kappa(\mathbf{K},\mu) = \theta$, for some $\alpha < \theta$, $\mathrm{tp}^{\mathrm{ga}}(a_k/\mathscr{M}_k)$ does not split over $\mathscr{M}_\alpha$, let $\mathscr{N}_k = \mathscr{M}_\alpha$. □

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[1] *We note this explicitly here because we commonly assume $\kappa(\mathbf{K},\mu) = \omega$ through this chapter.*

In the following definition the "$\mathfrak{St}$" stands for "stationary", since such a type has a unique non-splitting extension (see Lemma 2.1.3).

Definition 4.1.3

$$\mathfrak{St}(\mathscr{M}) := \left\{ (p, \mathscr{N}) : \begin{array}{l} \mathscr{M} \text{ is universal over} \mathscr{N} \\ p \in S(\mathscr{M}) \text{ is non} - \text{algebraic} \\ p \text{ does not split over } \mathscr{N} \end{array} \right\}$$

In (Villaveces and Shelah, 1999) and (Grossberg et al., 2012), the following notion was defined in terms of "parallel types", we have reformulated the notion in terms of non-splitting extensions.

Definition 4.1.4 *Suppose that I is a well-ordered set with a cofinal sequence $\langle a_i : i \in \theta \rangle \subset I$ and for each $\alpha \in \theta$ there are $\mu \cdot \omega$ elements between i_α and $i_{\alpha+1}$, then we say a tower $(\bar{\mathscr{M}}, \overline{a}, \bar{\mathscr{N}}) \in \mathbf{K}^*_{\mu,I}$ is σ-*relatively full with respect to* $\langle \mathscr{M}_i^\gamma : \gamma \in \sigma \rangle$, if each $(\mathscr{M}_i^\gamma)_{i \in I}$ witnesses that $\mathscr{M}_i$ is a (μ, σ)-limit model and for every $\gamma \in \sigma$ and every $(p, \mathscr{M}_i^\gamma) \in \mathfrak{St}(\mathscr{M}_i)$ for $i_\alpha < i < i_{\alpha+1}$ there is some $j \in I$ such that $i \leq j < i_{\alpha+1}$ such that $(\mathrm{tp}^{\mathrm{ga}}(a_j/\mathscr{M}_j), \mathscr{N}_j)$ is the non-splitting extension of $(p, \mathscr{M}_i^\gamma)$.*

Our proof of the following Proposition differs from the similar result Theorem 4 in (Grossberg et al., 2012). Our proof requires no use of disjoint amalgamation.

Proposition 4.1.5 *If $\kappa(\mathbf{K},\mu) = \omega$*[1]*, a relatively full, continuous tower $(\bar{\mathscr{M}}, \bar{a}, \bar{\mathscr{N}}) \in \mathbf{K}^*_{\mu,I}$, where I is as in definition 4.1.4 and $\theta < \mu^+$ is a limit ordinal such that $\theta = \mu \cdot \theta$, then $\bigcup_{i \in I} \mathscr{M}_i$ is a θ-limit model.*

proof.

We will show that $\mathscr{M}^* = \bigcup_{i \in I} \mathscr{M}_i$ is θ-special, hence, by 2.3.19 a (μ, θ)-limit.

Note that between each i_α and $i_{\alpha+1}$ there are $\mu \cdot \omega$ elements, as in definition 4.1.4. Note that $\theta = \mu \cdot \theta$, thus $\mathscr{M}^* = \bigcup_{i \in I} \mathscr{M}_i = \bigcup_{i_\alpha : \alpha \in \theta} \mathscr{M}_{i_\alpha} = \bigcup_{i_\alpha : \alpha \in \mu \cdot \theta} \mathscr{M}_{i_\alpha}$. So it enough to show:

Claim 4.1.6 *$\mathscr{M}_{i_{\alpha+1}}$ realizes every Galois type over $\mathscr{M}_{i_\alpha}$.*

Fix $p \in S(\mathscr{M}_{i_\alpha})$ since $\mathscr{M}_{i_\alpha} \prec_{\mathbf{K}} \mathscr{M}_{i_{\alpha+1}}$, we may as well assume p is non-algebraic, by the assumption that $\kappa(\mathbf{K}, \mu) = \omega$ we can find some $\mathscr{M}^\gamma_{i_\alpha}$ for $\gamma < \sigma$ such that p does not split over $\mathscr{M}^\gamma_{i_\alpha}$. Now we note:

- $\mathscr{M}^\gamma_{i_\alpha} \prec_{\mathbf{K}} \mathscr{M}_{i_\alpha}$
- $\mathscr{M}_{i_\alpha}$ is universal over $\mathscr{M}^\gamma_{i_\alpha}$
- $p \in S(\mathscr{M}_{i_\alpha})$ is non-algebraic.
- p does not split over $\mathscr{M}^\gamma_{i_\alpha}$.

It follows that $(p, \mathscr{M}^\gamma_{i_\alpha}) \in \mathfrak{St}(\mathscr{M}_{i_\alpha})$, so by relative fullness we can find j such that $i_\alpha \leq j < i_{\alpha+1}$ and $\mathrm{tp}^{\mathrm{ga}}(a_j/\mathscr{M}_j)$ is a non-splitting extension of p, so $a_j \in \mathscr{M}_{j+1} \prec_{\mathbf{K}} \mathscr{M}_{i_{\alpha+1}}$ realizes p.

So $\mathscr{M}^*$ is θ-special, hence a (μ, θ)-limit, by Theorem 2.3.19. $\square$

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[1] *We implicitly also assume AP and JEP.*

4.2 A discussion of existing work on uniqueness of limit models

We turn now to a discussion of existing work and how Proposition 4.1.5 fits into this existing work. The first work on this topic was done by Shelah and Villaveces in (Villaveces and Shelah, 1999). Unfortunately there were gaps in the proof of some results in this paper, so the main theorem cannot be considered "proved". In (Villaveces and Shelah, 1999) additional set theoretic assumptions beyond **ZFC** were made in order to deal with lack of amalgamation. In (VanDieren, 2006), working under the same set theoretic assumptions, VanDieren endeavored to resolve the issues encountered by Shelah and Villaveces in (Villaveces and Shelah, 1999). A proof of the following conjecture was published in (VanDieren, 2006), though another gap was subsequently found in a key proposition in this article was eventually found as well.

Conjecture 4.2.1 (GCH, $\phi_{\mu^+}(S^{\mu^+}_{\mathrm{cf}(\mu)})$) *Assume that* $\mathbf{K}$ *is a* λ*-categorical abstract elementary class for some* $\mathrm{LS}(\mathbf{K}) \leq \mu < \lambda$ *with no maximal models, for some* $\mu \geq \mathrm{LS}(\mathbf{K})$*. Let* θ_1 *and* θ_2 *be limit ordinals* $< \mu^+$ *. Under Hypothesis 1 or 2 if* $\mathscr{M}_1$ *and* $\mathscr{M}_2$ *are* (μ, θ_1)*- and* (μ, θ_2)*-limit models over* $\mathscr{M}$*, respectively, then there exists an isomorphism* $f : \mathscr{M}_1 \cong \mathscr{M}_2$ *such that* $f \restriction \mathscr{M}$ *is the identity mapping.*

Where Hypothesis 1 and 2 are: (these are truncated literal quotes from (VanDieren, 2012)):

- **Hypothesis 1:** Every continuous tower inside $\mathfrak{C}$[1] has an amalgamable extension inside $\mathfrak{C}$.

[1]It should be noted that the model $\mathfrak{C}$ in (VanDieren, 2006) is not a monster in the same sense as we assume since failure of amalgamation may prevent model homogeneous structures from existing.

- **Hypothesis 2:** For $\mu < \lambda$, the class of amalgamation bases in $\mathbf{K}$ is closed under unions of $\prec_{\mathbf{K}}$-increasing chains of length $< \mu^+$.

Unfortunately there is a gap in a key step of the proof of this conjecture, namely the proof that so called "reduced towers" are continuous.

Definition 4.2.2 *A tower $(\bar{\mathscr{M}}, \bar{a}, \bar{\mathscr{N}}) \in \mathbf{K}^*_{\mu,I}$ is* reduced *if for every $(\bar{\mathscr{M}}', \bar{a}', \bar{\mathscr{N}}') \in \mathbf{K}^*_{\mu,I}$ where $(\bar{\mathscr{M}}, \bar{a}, \bar{\mathscr{N}}) < (\bar{\mathscr{M}}', \bar{a}', \bar{\mathscr{N}}')$ and for all $i \in I$:*

$$\mathscr{M}'_i \cap \bigcup_{j \in I} \mathscr{M}_j = \mathscr{M}_i$$

Definition 4.2.3 *We say that a sequence of towers, $((\bar{\mathscr{M}}^\gamma, \bar{a}^\gamma, \bar{\mathscr{N}}^\gamma))_{\gamma<\delta}$ is* continuous *if for all $\alpha < \delta$ $(\bar{\mathscr{M}}^\alpha, \bar{a}^\alpha, \bar{\mathscr{N}}^\alpha)$ is the limit of the sequence $((\bar{\mathscr{M}}^\gamma, \bar{a}^\gamma, \bar{\mathscr{N}}^\gamma))_{\gamma<\alpha}$.*

The original idea of the proof offered in (VanDieren, 2006) is to take a number of discontinuous extensions of towers in order to build a tower that is relatively full. The extension process produces discontinuous towers, but then continuity is recovered by taking reduced extensions of each discontinuous tower. Thus, if reduced towers are continuous, the final tower is both relatively full and continuous, hence Proposition 4.1.5 applies. Unfortunately, the only hypothesis known, as of yet, to imply reduced towers are continuous is μ^+-categoricity.

An erratum was published to (VanDieren, 2006) in which the strongest, verified result known at this date is stated, namely this theorem reproduced from (VanDieren, 2012):

Theorem 4.2.4 (GCH,$\phi_{\mu^+}(S^{\mu^+}_{\mathrm{cf}(\mu)})$)(VanDieren) *Assume that $\mathbf{K}$ is a μ^+-categorical abstract elementary class with no maximal models, for some $\mu \geq \mathrm{LS}(\mathbf{K})$. Further assume that* ***GCH***

and $\phi_{\mu^+}(S^{\mu^+}_{\mathrm{cf}(\mu)})$ *hold. Let* θ_1 *and* θ_2 *be limit ordinals* $< \mu^+$ *. Under Hypothesis 1,2 if* $\mathscr{M}_1$ *and* $\mathscr{M}_2$ *are* (μ, θ_1)*- and* (μ, θ_2)*-limit models over* $\mathscr{M}$*, respectively, then there exists an isomorphism* $f : \mathscr{M}_1 \cong \mathscr{M}_2$ *such that* $f \restriction \mathscr{M}$ *is the identity mapping.*

Since the function of the set theory in Theorem 4.2.4 is to enable one to work in a context where the amalgamation property fails to hold in μ, one would hope that the assumption of either μ-AP or full AP can eliminate the need for set theoretic assumptions and indeed it does. However, one also hopes to prove a uniqueness theorem from weaker "stability-type" hypotheses than μ^+-categoricity. The following result is claimed in the preprint (Grossberg et al., 2012):

Conjecture 4.2.5 *If* $\mathbf{K}$ *satisfies AP and JEP as well as the conditions specified in 4.2.6 then for* $\mu > \mathrm{LS}(\mathbf{K})$ *and limit ordinals* $\theta_1, \theta_2 < \mu^+$ *any* (μ, θ_1) *and* (μ, θ_2)*-limit models over* $\mathscr{M}$ *are isomorphic over* $\mathscr{M}$*.*

Condition 4.2.6 *For any sequence* $(\mathscr{M}_i)_{i<\alpha}$ *of limit models of cardinality* μ *and* $p \in S(\mathscr{M}_\alpha)$ *where* $\mathscr{M}_\alpha = \bigcup_{i<\alpha} \mathscr{M}_i$*:*

1. *If for all* $i < \alpha$*,* $p \restriction \mathscr{M}_i$ *does not split over* $\mathscr{M}_0$ *then* p *does not split over* $\mathscr{M}_0$*.*
2. *There exists* $i < \alpha$ *such that* p *does not split over* $\mathscr{M}_i$*.*[1]

These conditions are stronger than necessary. VanDieren has observed that $\kappa(\mathbf{K}, \mu) = \omega$ suffices for the arguments presented in (Grossberg et al., 2012), however the Conditions in 4.2.6

[1] *This is similar to* $\kappa(\mathbf{K}, \mu) = \omega$*, but omits the requirement that* $\mathscr{M}_{i+1}$ *be universal over* $\mathscr{M}_i$ *and adds the condition that every* $\mathscr{M}_i$ *must be a limit model.*

are the ones stated in the preprint as of January 12, 2012 and prior versions. Note that by the following theorem of Shelah (15.3 of (Baldwin, 2009)) we know that categoricity above μ (and the existence of arbitrarily large models) implies $\kappa(\mathbf{K}, \mu) = \omega$.

Theorem 4.2.7 *Assume K is λ-categorical for regular cardinal λ where $LS(K) < \mu < \lambda$ and that arbitrarily large models in $\mathbf{K}$ exist, then $\kappa(\mathbf{K}, \mu) = \omega$.*

From $\kappa(\mathbf{K}, \mu)$ one may easily deduce a weaker form of 4.2.6.1. That is:

Proposition 4.2.8 *If $\kappa(\mathbf{K}, \mu) = \omega$ and $(\mathscr{M}_i)_{i\in\alpha}$ witness $\mathscr{M}_\alpha = \bigcup_{i<\alpha} \mathscr{M}_i$ is a (μ, α)-limit model then if $p \in S(\mathscr{M}_\alpha)$ satisfies that $p \restriction \mathscr{M}_i$ does not split over $\mathscr{M}_0$ for all $i < \alpha$, p does not split over $\mathscr{M}_0$.*

proof. Suppose that $\mathscr{M}_\alpha$, $(\mathscr{M}_i)_{i<\alpha}$, $p \in S(\mathscr{M}_\alpha)$ are as above. Since $\kappa(\mathbf{K}, \mu) = \omega$, p does not split over $\mathscr{M}_i$ for some $i < \alpha$. We know by hypothesis that $p \restriction \mathscr{M}_i$ and $p \restriction \mathscr{M}_{i+1}$ do not split over $\mathscr{M}_0$. There exists some extension q of $p \restriction \mathscr{M}_i$ to $S(\bigcup_{i<\alpha} \mathscr{M}_i)$ which does not split over $\mathscr{M}_0$. Note by the monotonicity of splitting $q \restriction \mathscr{M}_{i+1}$ does not split over $\mathscr{M}_i$. But then both p and q are non-splitting extensions of $p \restriction \mathscr{M}_{i+1}$ to $S(\bigcup_{i<\alpha} \mathscr{M}_i)$ which do not split over $\mathscr{M}_i$. So by 2.1.3, $p = q$, in particular p does not split over $\mathscr{M}_0$. $\square$

The assumption of AP, JEP and $\kappa(\mathbf{K}, \mu) = \omega$ seem in many ways to be the proper assumptions to prove Conjecture 4.2.5. Work was done in (Grossberg et al., 2012) which is valid under these assumptions, though there is a gap in one of the key propositions which prevents the stated main theorem of (Grossberg et al., 2012) from being proved. That is to say, Conjecture 4.2.5 remains unproved.

The next theorem follows from the valid parts of (Grossberg et al., 2012) by applying some of the work in (VanDieren, 2012). As of yet, this remains unpublished in the currently available version of (Grossberg et al., 2012).

Theorem 4.2.9 (Grossberg, VanDieren, Villaveces) *Let* $\mathbf{K}$ *be an Abstract Elementary Class and* $\mu > LS(K)$*, suppose that* $\mathbf{K}$ *satisfies amalgamation property, joint-embedding property, and has arbitrarily large models. If* $\mathbf{K}$ *is* μ^+*-categorical then any two* $(\mu, \sigma_1), (\mu, \sigma_2)$*-limit models over* $\mathscr{M}$ *are isomorphic over* $\mathscr{M}$*.*

It should be noted, that disjoint amalgamation over limit models is assumed in (Grossberg et al., 2012), but this follows from categoricity in μ^+. Weak disjoint amalgamation is proved from categoricity in Fact I.3.15 of (VanDieren, 2006). This result originally appeared in (Villaveces and Shelah, 1999), but the proof in (VanDieren, 2006) is simplified from the version in (Villaveces and Shelah, 1999).

The gap in the current version of (Grossberg et al., 2012) is in the proof that so called "reduced towers" are continuous. One may substitute the assumption that reduced towers are continuous for μ^+-categoricity. In this case one cannot apply the existing proof used in (VanDieren, 2006) or (Grossberg et al., 2012) that a relatively full tower is a limit model, without also assuming (weak) disjoint amalgamation over limit models. However, in Proposition 4.1.5 we proved that a relatively full tower is a limit model assuming only JEP, AP, and $\kappa(\mathbf{K}, \mu) = \omega$. Thus we can also deduce the following result easily from (Grossberg et al., 2012):

Corollary 4.2.10 *If the following hold:*

1. $\mathbf{K}$ *satisfies AP and JEP in* $\mu \geq \mathrm{LS}(\mathbf{K})$

2. $\mathbf{K}$ *is* μ *stable*

3. $\kappa(\mathbf{K}, \mu) = \omega$

4. *For any limit sequence* $(\mathscr{M}_i)_{i<\alpha}$ *of models of cardinality* μ *where* $|\alpha| \leq \mu$ *and* $p \in S(\bigcup_{i<\alpha} \mathscr{M}_i)$, *if* $p \restriction \mathscr{M}_i$ *does not split over* $\mathscr{M}_0$ *for all* $i < \alpha$ *then* p *does not split over* $\mathscr{M}_0$.

5. *Reduced towers in* $\mathbf{K}^*_\mu$ *are continuous.*

then for any ordinals $\sigma_1, \sigma_2 < \mu^+$, *any two* $(\mu, \sigma_1), (\mu, \sigma_2)$*-limit models over* $\mathscr{M}$ *are isomorphic over* $\mathscr{M}$.

Two final questions we raise briefly is:

Question 4.2.11 *Suppose that for any* $\theta_1, \theta_2 < \mu^+$ *any two* (μ, θ_1) *and* (μ, θ_2)*-limit models are isomorphic. Does this imply* $\kappa(K, \mu) = \omega$*?*

And the same question, relativised to stable classes:

Question 4.2.12 *Suppose* $\mathbf{K}$ *is stable in* μ *and for any* $\theta_1, \theta_2 < \mu^+$ *any two* (μ, θ_1) *and* (μ, θ_2)*-limit models are isomorphic. Does this imply* $\kappa(K, \mu) = \omega$*?*

By Theorem 2.2.11 we know that in the case of elementary classes the answer to Question 4.2.12 is “yes”, however in arbitrary AEC, this question is, as far as we know, still open.

4.3 Applying Uniqueness of Limit Models to Gap Transfer Theorems

Our original motivation in looking at uniqueness of limit models was that we wanted to be able to extend Lessmann's argument for building a two cardinal in $\aleph_1$ (Theorem 3.4.11) to larger uncountable cardinals. Unfortunately, this is something we've been unable to do. It is perhaps worthwhile discussing the difficulties we encountered in attempting to do so.

The proof of Proposition 3.4.6 is essentially a diagonalization argument; we are picking a diagonal of sorts out of an $\omega \times \alpha$ array where α is countable. However, when encountering a $\beta \times \alpha$ array (as when one is considering a sequence of α (μ, β)-limit models) if $\mathrm{cf}(\beta) \neq \mathrm{cf}(\alpha)$ there's obviously no way to simply pick out a diagonal of the resulting array. For simplicity's sake, assume β and α are regular. If $\beta > \alpha$ then one may preserve eventual domination (see Definition 3.4.2), however, one builds only a (μ, α)-limit model. On the other hand, if $\alpha > \beta$, one builds a (μ, β)-limit model before getting through the array, thus, one loses eventual domination.

Of course, if Conjecture 4.2.5 held, this would allow us to rewrite (μ, β)-limit models as (μ, α)-limit models, but this still seems insufficient to carry the argument through. For the sake of being more concrete, assume we are trying to transfer a Vaughtian pair of limit models in $\aleph_1$ to a two cardinal model in $\aleph_2$. Given a pair of (ω_1, ω)-limit models $(\mathscr{N}_n^0)_{n<\omega} <_L^k (\mathscr{N}_n^1)_{n<\omega}$ one can proceed inductively through any countable stage of the construction detailed in Proposition 3.4.6. One can then apply Conjecture 4.2.5 to rewrite each $(\mathscr{N}_n^\alpha)_{n<\omega}$ as an (ω_1, ω_1)-limit model, however these isomorphisms may be non-uniform, hence, they need not preserve "eventual domination".

Thus, the essential tool required to prove a theorem like Theorem 3.4.11 in a larger uncountable cardinal seems to be not *uniqueness of limits* but rather that *the union of some sequence of (μ, β)-limit models is itself a (μ, β)-limit model.* In other words, uniqueness of limit models in μ does not imply (as far as we know) that some (subclass) of limit models in μ will be a superlimit class in μ, and it is the latter condition which we know implies the existence of a two cardinal model in μ^+, not the former.

CHAPTER 5

TOWARDS GAP-2 TRANSFER VIA A SIMPLIFIED MORASS

In this chapter we explore generalizations of the first order gap-2 transfer theorem to non-elementary classes. For first order logic, Jensen was able to prove a gap-2 transfer theorem. That is, consistently with **ZFC**, if you have a model of size κ^{++} where the realizations of some formula ϕ are of size κ then there's an elementarily equivalent structure of size $\aleph_2$ which has countably many realizations of ϕ (assuming the language is countable). We believe it is a natural question whether any sort of similar transfer theorem can be proved for abstract elementary classes. We are able to extend part of the construction to all sets invariant over the empty set (which motivates our definition of (κ, λ)-model) and we are able to show that our work can be used to derive Jensen's gap-2 Theorem in the case of an elementary class where the invariant set in question is defined by a first-order formula.

We recall now the definitions of invariant set (Definition 2.3.6) and (κ, λ)-model (Definition 2.3.7) from Chapter 2 for the reader's convenience.

Definition 5.0.1 (2.3.6) *A subset $X \subseteq \mathfrak{C}$ is* invariant over $\mathscr{N}$ *if for any $\gamma \in \mathrm{aut}(\mathfrak{C}/\mathscr{N})$ $\gamma(X) \subseteq X$. If for any $\gamma \in \mathrm{aut}(\mathfrak{C})$ $\gamma(X) \subseteq X$ we say X* is invariant.

The following notation may be redundant, but it is motivated by the analogy to the realizations of a formula, galois type, or syntactic type.

Notation 5.0.2 *For $\mathscr{M} \prec_{\mathbf{K}} \mathfrak{C}$ we write "$X(\mathscr{M})$" for the set $X \cap M$.*

Definition 5.0.3 (2.3.7) *A* (κ, λ)-model over μ *is a model* $\mathscr{M}$ *of size* κ *such that there is a set* X *invariant over some model* $\mathscr{N} \prec_{\mathbf{K}} \mathscr{M}$ *with* $|N| = \mu$ *such that* $X(\mathscr{M}) = \lambda$*. A* (κ, λ)*-model (with no* μ *mentioned) is a* (κ, λ)*-model over* $\emptyset$*. If* κ *is the* n*-fold cardinal successor to* λ *we refer to this as an* n*-gap.*

Of course, as mentioned in Chapter 2, this only makes sense when considered with respect to some fixed monster model $\mathfrak{C}$. So:

Assumption 5.0.4 *We work in a fixed monster model* $\mathfrak{C}$*.*

In (Devlin, 1984) the first order gap-2 transfer theorem (using a simplified morass) is accomplished through the following steps:[1]

1. Build an isomorphic pair of structures in κ.
2. Obtain a Vaughtian pair (in κ) and an elementary embedding that codes some combinatorial information; expand the language to code the Vaughtian pair and combinatorial information into an expanded first order theory.
3. Find a "nice" pair of countable homogeneous models of the expanded theory.
4. Construct an $(\aleph_2, \aleph_0)$-gap using properties of a simplified morass.

We attempt to follow the general outline of the first order proof, but encounter difficulty with step 2. First we carry out step, 4., in the context of an ordered AEC in Section 5.3.

[1] The original proof of the gap-2 transfer theorem is due to Jensen and the argument via a simplified Morass is likely due to Vellemann, though it is also possible the proof is due to either Devlin or Jensen.

We then turn toward the earlier steps of the construction. We can complete step 1. in the AEC context and do so in section 5.7. We can obtain a Vaughtian pair of structures in κ in Proposition 5.7.11, however the trick of coding combinatorial information using an expanded theory in an expanded language seems quite difficult in the non-elementary case. We discuss step 2 in depth in subsection 5.7.2.

Using a stronger condition than conclusion of 2 we are able to derive condition 3., however we have not been able to derive this condition from the existence of a (κ^{++}, κ)-model. From this stronger version of condition 2., however, we are able to derive condition 3. in subsection 5.7.3.

Terminology is a minor obstacle to providing an easy to follow and at the same time technically accurate description of our result in the abstract elementary case. Our problem is that, following the conventions of existing literature, we have defined "Vaughtian pair" for AECs in terms of galois types. Our construction, however, is strong enough not just to keep the set of realizations of a galois type small, but to keep the realizations of *any* invariant set small.[1] At same time, we work with pairs of models $(\mathscr{M}_1, \mathscr{M}_2)$ where for some invariant set $X \subseteq \mathscr{M}$, $M_1 \cap X = M_2 \cap X$ and $\mathscr{M}_1$ is a proper strong substructure of $\mathscr{M}_2$. There is a natural desire to call such a model a "Vaughtian pair" but we have already defined this notion to mean p-Vaughtian pair for some galois type p. However, we believe that by introducing the notion of X-Vaughtian

[1]Indeed, it is because the realizations of a first-order formula (without parameters) and and the realizations of some galois type (again with no parameters) are both invariant that we are able to subsume the construction of an $(\aleph_2, \aleph_0)$-model in the elementary class case in our main theorem, Theorem 5.4.2.

pair we can avoid confusion with p-Vaughtian pairs and still maintain a reasonable analogy to the first order object.

Definition 5.0.5 *We say a pair of models* $(\mathscr{M}_1, \mathscr{M}_2)$ *in* $\mathbf{K}$ *are an* X-Vaughtian pair *if the following hold:*

1. $\mathscr{M}_1$ *is a proper strong substructure of* $\mathscr{M}_2$.
2. $X \subseteq \mathfrak{C}$ *is invariant*[1], *that is for any* $\gamma \in \mathrm{aut}(\mathfrak{C})$, $\gamma(X) = X$.
3. $X \cap \mathscr{M}_1 = X \cap \mathscr{M}_2$.

Our work in this chapter is a generalization of the first order gap two theorem only in the sense that from the original arguments given by Jensen and written up in (Devlin, 1984) one can produce the necessary combinatorial conditions to do our argument in Section 5.3. As such, this work may be better understood as providing a sufficient condition for the existence of $(\aleph_2, \aleph_0)$-model as opposed to a true "gap-2 transfer" theorem for AECs.

More specifically, we are able to show that the construction of an $(\aleph_2, \aleph_0)$-model can be carried out from the assumption of the existence of a sufficiently nice countable X-Vaughtian pair. Furthermore, we are able to derive the existence of a "sufficiently nice" countable X-Vaughtian pair from the existence of a "sufficiently nice" X-Vaughtian pair in κ, as well as derive the existence of a Vaughtian pair in κ from the existence of (κ^{++}, κ)-model. We are not, however, able to derive the existence of "sufficiently nice" X-Vaughtian pair in κ from an arbitrary (κ^{++}, κ) model.

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[1]See Definition 2.3.6

Our notion of "sufficiently nice" X-Vaughtian pair includes a weak homogeneity condition on the models in the X-Vaughtian pair. Deriving the existence of nice enough structures is possible from the assumption of stronger homogeneity assumptions, namely the notion of "galois homogeneity" which we introduce in this chapter. Before we can proceed through any of these arguments, we must first develop some basic properties of invariant sets and introduce the simplified morass. We outline the sections of this chapter below:

Chapter Outline:

§5.1 - We introduce the set theoretic object used in the main construction, the simplified morass.

§5.2 - We introduce a notion of "invariant set" for AEC and observe some basic properties.

§5.3 - We introduce a sufficient condition for the construction of an $(\aleph_2, \aleph_0)$-model and prove some basic results useful in the construction of the $(\aleph_2, \aleph_0)$-model.

§5.4 - We construct the $(\aleph_2, \aleph_0)$-model.

§5.5 - We introduce notions of "1-transitive" and "galois homogeneous" and describe and show that a sufficiently nice X-Vaughtian pair of galois homogeneous structures is a sufficient condition for the the construction of an $(\aleph_2, \aleph_0)$-model.

§5.6 - We give an example of a PC-Γ class where there does not exist a galois homogeneous extension of a particular model.

§5.7 - We discuss steps 2., 3., and 4. in the outline of the proof of the first order gap-2 transfer theorem. In §5.7.1 we discuss using the presentation theorem to add an ordering to the language if we start with an AEC that does not already contain a suitable ordering in it's signature. In §5.7.2 we discuss the difficulties inherent in translating step 2. to the AEC case. In §5.7.3 we give a sufficient condition on models of size κ from which we can construct an $(\aleph_2, \aleph_0)$-model.

§5.8 - We make some closing remarks.

5.1 Simplified morass

In order to prove the gap-2 transfer Theorem for first order logic, a combinatorial object called a "morass" is used. The original definition and application of a morass to cardinality transfer questions is due to Jensen, who showed that such an object exists if **V=L**. Later work by Vellemann showed that the existence of a morass is equivalent to the existence of a "simplified morass", which has a simpler axiomatic definition. Thus, if we wish to show that **V=L** implies the gap-2 transfer Theorem, there's no loss of generality in working with the simplified morass instead of Jensen's original notion. It is perhaps worth noting, that the main theorem of this chapter, 5.4.2, as well as the classical result it generalizes, requires *only* the existence of a simplified morass. **GCH** is, however, required for certain other steps of the construction, in particular Proposition 5.7.11 and its first order analogue.

For our purposes, the simplified morass is a "black box" we will use to prove a two-cardinal theorem for AEC. For a more in-depth discussion of the simplified morass, Jensen's morass, and the construction of the simplified morass from a morass see (Devlin, 1984). We present a definition below. Since we never deal directly with Jensen's original morass, any subsequent reference to a morass refers to a simplified morass.

The simplified morass is an ω_1-long sequence of countable ordinals θ_α which terminates with $\theta_{\omega_1} = \omega_2$. For any of the $\theta_\alpha, \theta_\beta$ there is a countable collection of order-embeddings $\mathscr{F}_{\alpha,\beta}$ that take θ_α into θ_β, which satisfy coherence properties similar to the properties of directed system; the $\mathscr{F}_{\alpha,\beta}$ differ from the maps in a directed system in that there are always at least 2 distinct maps in $\mathscr{F}_{\alpha,\beta}$. We reproduce the definition of the simplified morass from (Devlin, 1984) below:

Definition 5.1.1 *A* simplified morass *consists of an increasing sequence of ordinals $(\theta_\alpha)_{\alpha\leq\omega_1}$, a collection of maps $\mathscr{F}_{\alpha,\beta}$ for each $\alpha < \beta \leq \omega_1$ from θ_α to θ_β, and for each $\alpha < \omega_1$ a distinguished element $\delta_\alpha \in \theta_\alpha$ satisfying the following properties:*

(M1) $\theta_0 = 1, \theta_{\omega_1} = \omega_2$, *if* $\alpha < \omega_1$ *then* $\theta_\alpha < \omega_1$.

(M2) If $\alpha < \beta < \omega_1$ *then* $|\mathscr{F}_{\alpha,\beta}| \leq \omega$.

(M3) If $\alpha < \beta < \gamma$ *then* $\mathscr{F}_{\alpha,\gamma} = \{fg : g \in \mathscr{F}_{\alpha,\beta}, f \in \mathscr{F}_{\beta,\gamma}\}$

(M4) If $\alpha < \omega_1$ *then* $\mathscr{F}_{\alpha,\alpha+1} = \{\mathrm{Id}_{\theta_\alpha}, f\}$ *where f is an order-preserving map and* $f \restriction \delta_\alpha = \mathrm{Id}_{\delta_\alpha}$ *and* $f(\delta_\alpha) \geq \theta_\alpha$

(M5) For all limit ordinals $\alpha \leq \omega_1$ *and* $\beta_i < \alpha$ *for* $i = 1, 2$*: if* $f_i \in \mathscr{F}_{\beta_i,\alpha}$ *there is a* γ *and* β_i *where* $\beta_i < \gamma < \alpha$ *and functions* $h_i \in \mathscr{F}_{\beta_i,\gamma}$ *and* $g \in \mathscr{F}_{\gamma,\alpha}$ *such that* $f_i = gh_i$.

(M6) For all $\beta > 0, \theta_\beta = \bigcup\{f(\theta_\alpha) : \alpha < \beta, f \in \mathscr{F}_{\alpha,\beta}\}$

5.2 Invariant sets

Definition 5.2.1 *A subset* $X \subseteq \mathfrak{C}$ *is* invariant *if for any* $\gamma \in \mathrm{aut}(\mathfrak{C})$ $\gamma(X) = X$.

Note that by definition above, if $\mathscr{M}_0 \subset \mathscr{M}_1 \prec_{\mathbf{K}} \mathfrak{C}$ then $X(\mathscr{M}_0) \subseteq X(\mathscr{M}_1)$. Observe the following trivial observation:

Fact 5.2.2 *Let $\mathscr{M}_0 \prec_{\mathbf{K}} \mathscr{M}_1$. If $X(\mathscr{M}_1) \subseteq X(\mathscr{M}_0)$ then $X(\mathscr{M}_1) = X(\mathscr{M}_0)$.*

The following is easy to deduce but informative:

Proposition 5.2.3 *If $X \subset \mathfrak{C}$ and for all $\phi \in \mathrm{aut}(\mathfrak{C})$, $\phi(X) \subseteq X$, then X is invariant.*

proof. Suppose $X \subset \mathfrak{C}$ and for all $\phi \in \mathrm{aut}(\mathfrak{C})$, $\phi(X) \subseteq X$. Fix $\phi \in \mathrm{aut}(\mathfrak{C})$. Suppose $x \in X \setminus \phi(X)$. Note $\phi^{-1}(x) \notin (X)$. But this contradicts that $\phi^{-1}(X) \subseteq X$. □

Since it is obvious that invariance of X implies for all $\phi \in \mathrm{aut}(\mathfrak{C})$, $\phi(X) \subseteq X$ it follows that:

Corollary 5.2.4 *If $X \subset \mathfrak{C}$ then X is invariant if and only if for all $\phi \in \mathrm{aut}(\mathfrak{C})$, $\phi(X) \subseteq X$,*

In this section we will work extensively with a notion not commonly explored in the study of AECs, namely galois types over $\emptyset$. In common practice, one demands that a galois type be defined over a model as parameter set. But if one considers a galois type p to be an orbit of $\mathrm{aut}(\mathfrak{C}/\mathrm{dom}(p))$ then there is no reason one cannot consider the case that $\mathrm{dom}(p) = \emptyset$.

Notation 5.2.5 $\mathrm{tp}^{\mathrm{ga}}(a)$ *is simply the class of elements $\{b \in \mathfrak{C} : \exists \gamma \in \mathrm{aut}(\mathfrak{C}),\ \gamma(b) = a\}$.*

The reason why we are interested in galois types over the empty set is that they are invariant. See the (trivial) proposition below.

Proposition 5.2.6 *Suppose that $f : \mathscr{M} \to \mathscr{N}$ is a strong embedding and that p is a galois type over the empty set, then $p = f(p)$.*

proof. Suppose that $a \models p$ and that $b \models f(p)$. Since we assume a monster model $\mathfrak{C}$ exists, we may extend f to $F \in \mathrm{aut}(\mathfrak{C})$, consider $F(a)$. Clearly $F(a) \models f(p)$, so there exists $\gamma \in \mathrm{aut}(\mathfrak{C})$

such that $\gamma(F(a)) = b$. But then note that $F^{-1}\gamma^{-1}(b) = a$, so $b \models p$. Thus we see that $p = f(p)$.

$\square$

Corollary 5.2.7 *If* $\mathrm{dom}(p) = \emptyset$ *then p is invariant.*

We give some examples of invariant sets below:

Example 5.2.8

1. *The set of realizations of a formula of first order logic (in $\mathfrak{C}$) is invariant over its parameters.*
2. *The set of realizations of a syntactic type (again in $\mathfrak{C}$) is invariant over its parameter set.*
3. *The set of realizations of a galois type is invariant over its domain. In particular this holds for a galois type p with* $\mathrm{dom}(p) = \emptyset$ *(See 5.2.7),*
4. *If* $\mathrm{dom}(p)$ *is countable, there is an AEC* $\mathbf{K}_{\mathrm{dom}(p)}$ *which consists of the elements of* $\mathbf{K}$ *for which* $\mathrm{dom}(p)$ *is a strong substructure; it should be clear that this is also an AEC in which* $\mathrm{dom}(p)$ *is invariant.*

It is a classical result of model theory that any boolean combination of invariant sets remains invariant.

Proposition 5.2.9 *Let (X_i) be a collection of invariant sets, then $\bigcup_{i \in I} X_i$, $\bigcap_{i \in I} X_i$, and (for any $i \in I$) $\mathfrak{C} \setminus X_i$ is invariant.*

The proof is a simple exercise in elementary set theory. Another fairly obvious observation is that any invariant set is a union of galois types.

Fact 5.2.10 *Let X be a set invariant over $\mathscr{M}$, where either $\mathscr{M} \prec_{\mathbf{K}} \mathfrak{C}$ or $\mathscr{M} = \emptyset$. $X = \bigcup_{a \in X} \mathrm{tp}^{\mathrm{ga}}(a/\mathscr{M})$.*

The following lemma is key to performing the successor step in the inductive construction of an $(\aleph_2, \aleph_0)$-model.

Lemma 5.2.11 *Let $\mathscr{M}_0 \prec_{\mathbf{K}} \mathscr{M}_1 \prec_{\mathbf{K}} \mathfrak{C}$ and let X be an invariant set such that $X(\mathscr{M}_0) \supseteq X(\mathscr{M}_1)$. Let $\phi : \mathscr{M}_1 \to \mathscr{M}_0$ be an isomorphism. Let $\phi^* \in \mathrm{aut}(\mathfrak{C})$ extend ϕ. If $\mathscr{M}_2 := \phi^{*-1}(\mathscr{M}_1)$ then $X(\mathscr{M}_2) = X(\mathscr{M}_1) = X(\mathscr{M}_0)$.*

proof. Let ϕ, ϕ^*, $\mathscr{M}_0$, $\mathscr{M}_1$, $\mathscr{M}_2$, and X be as above. Since $\phi : \mathscr{M}_0 \to \mathscr{M}_1$, it follows that:

$$\phi^*(X(\mathscr{M}_1)) = \phi^*(X \cap M_1) = \phi^*(X) \cap \phi^*(M_1) = X \cap M_0 = X(\mathscr{M}_0)$$

We consider two cases:

- Case $x \in M_1$. If $x \in X(\mathscr{M}_1) \subseteq X(\mathscr{M}_0)$ then, clearly $x \in X(\mathscr{M}_0)$.
- Case $x \in M_2 \setminus M_1$. We argue that $x \notin X(\mathscr{M}_2)$. If $x \in X(\mathscr{M}_2)$, since $\phi(\mathscr{M}_1) = \mathscr{M}_0$ it follows that $\phi^*(x) \in M_1 \setminus M_0$. In particular $\phi^*(x) \notin M_0$. By invariance of X, $\phi^*(x) \in X$. So $\phi^*(x) \in X \cap M_1 = X(\mathscr{M}_1) \subseteq X(\mathscr{M}_0) \subseteq M_0$. But this contradicts that $\phi^*(x) \notin M_0$.

Thus it follows that $X(\mathscr{M}_2) \subseteq X(\mathscr{M}_1) \subseteq X(\mathscr{M}_0)$.

Note that since $\mathscr{M}_0 \prec_{\mathbf{K}} \mathscr{M}_1$, it follows $\phi^{-1}(\mathscr{M}_0) = \mathscr{M}_1 \prec_{\mathbf{K}} \phi^{*-1}(\mathscr{M}_1) = \mathscr{M}_2$. So $X(\mathscr{M}_0) \subseteq X(\mathscr{M}_1) \subseteq X(\mathscr{M}_2)$. Putting this together with Fact 5.2.2 it follows that $X(\mathscr{M}_0) = X(\mathscr{M}_1) = X(\mathscr{M}_2)$. $\square$

Proposition 5.2.12 *Let X be invariant and let $(\mathscr{M}_i)_{i\in I}$ be a $\prec_{\mathbf{K}}$-increasing sequence of models in $\mathfrak{C}$ then $X(\bigcup_{i\in I}\mathscr{M}_i) = \bigcup_{i\in I}(\mathscr{M}_i)$.*

proof.

$$X\left(\bigcup_{i\in I}\mathscr{M}_i\right) = X\cap\bigcup_{i\in I}\mathscr{M}_i = \bigcup_{i\in I} X\cap\mathscr{M}_i = \bigcup_{i\in I} X(\mathscr{M}_i)$$

□

Corollary 5.2.13 *Let X be invariant and let $(\mathscr{M}_i)_{i\in I}$ be a $\prec_{\mathbf{K}}$-increasing sequence of models in $\mathfrak{C}$ such that for $i,j\in I$, $X(\mathscr{M}_i) = X(\mathscr{M}_j)$. Then for any $j\in I$, $X(\bigcup_{i\in I}\mathscr{M}_i) = X(\mathscr{M}_j)$.*

The following lemma allows us to assume certain direct limits can be chosen within an already fixed monster model. In general one cannot always assume that a directed system of models each of which is a strong substructure of $\mathfrak{C}$, that the direct limit of this directed system is also a strong substructure of the $\mathfrak{C}$. This observation was made earlier by VanDieren in (VanDieren, 2006) in the inductive proof of Theorem III.10.1, in particular in Subclaim III.10.4. The claim below differs slightly from the claim in (VanDieren, 2006) and omits many extraneous details (relevant to (VanDieren, 2006) but not to our construction) of the argument in (VanDieren, 2006).

Notation 5.2.14 *Let $\coprod_{i\in I} S_i$ denote the disjoint union of the set of sets $\{S_i\}_{i\in I}$.*

Lemma 5.2.15 *Let I be a non-empty countable directed set. Let $(\mathscr{M}_i, f_{i,j})_{i\le j, i,j\in I}$ be a directed system of models in $\mathbf{K}$ and strong embeddings where $|I|, |M_i| \le \kappa < |\mathfrak{C}|$ for some cardinal κ. Suppose there is a set $X\subseteq M_i$ for all $i\in I$, such that for any $i<j$ $f_{i,j}\restriction X = \mathrm{Id}_X$. Then*

there exists a direct limit $(\mathscr{M}_\infty, f_{i,\infty})_{i\in I}$ *of the directed system where* $\mathscr{M}_{i_0} \prec_{\mathbf{K}} \mathscr{M}_\infty \prec_{\mathbf{K}} \mathfrak{C}$ *and* $f_{i,\infty} \restriction X = \mathrm{Id}_X$.

proof. Recall that abstractly, the direct limit of the directed system $(\mathscr{M}_i, f_{i,j})_{i,j})_{i\leq j,i,j\in I}$ is:

$$\coprod_{i\in I} M_i/\sim$$

where $\sim$ identifies elements $a \sim b$ where $a \in M_i, b \in M_j$ if there exists $k \in I$ such that $f_{i,k}(a) = f_{j,k}(b)$. Fix some $i_0 \in I$. Since every map in our directed system, by virtue of being a strong embedding, is injective, there is no loss of generality in identifying $\mathscr{M}_{i_0}$ with its image under $f_{i_0,\infty}$, so that $\mathscr{M}_{i_0} \prec_{\mathbf{K}} \mathscr{M}_\infty$. Since $\mathfrak{C}$ is κ^+-model homogeneous, we can find a strong embedding g of $\mathscr{M}_\infty$ into $\mathfrak{C}$ fixing $\mathscr{M}_{i_0}$.

Let $\mathscr{M}'_\infty := g(\mathscr{M}_\infty)$. Let $f'_{i,\infty} = gf_{i,\infty}$ for $i \in I$. It should be clear that $(\mathscr{M}'_\infty, f'_{i,\infty})_{i\in I}$ is a direct limit of the directed system $(\mathscr{M}_i, f_{i,j})_{i\leq j,i,j\in I}$. Furthermore, since g fixes $M_{i_0} \supseteq X$, each $f_{i,\infty} \restriction X = \mathrm{Id}_X$. □

We now extend our previous observations to apply to direct limits.

Lemma 5.2.16 *Let* X *be invariant and let* $(\mathscr{M}_i, \sigma_{i,j})_{i<j,i,j\in I}$ *be a directed system such that for any* $i < j$, *where* $i, j \in I$, $\sigma_{i,j} \restriction X(\mathscr{M}_i)) = \mathrm{Id}_{X(\mathscr{M}_i)}$. *Then there exists a direct limit* $\mathscr{M}_\infty$ *of the directed system* $(\mathscr{M}_i, \sigma_{i,j})_{i<j,i,j\in I}$ *such that for any* $i \in I$, $X(\mathscr{M}_\infty) = X(\mathscr{M}_i)$.

proof. By Lemma 5.2.15, without loss of generality, we can choose a direct limit $\mathscr{M}_\infty$ such that the canonical embeddings $\sigma_{i,\infty} : \mathscr{M}_i \to \mathscr{M}_\infty$ each fix $X(\mathscr{M}_i)$. Since $\mathscr{M}_\infty = \bigcup_{i\in I} \sigma_{i,\infty}(\mathscr{M}_i)$. So by Corollary 5.2.13 $X(\mathscr{M}_\infty) = X(\mathscr{M}_i)$ for any $i \in I$. □

5.3 A sufficient condition for constructing an $(\aleph_2, \aleph_0)$-Model

In this section we prove a number of results useful in our construction of an $(\aleph_2, \aleph_0)$-Model. At this point, many of the results we desire to prove require us to work in an AEC which contains ordered structures.

In the proof of Jensen's original gap-2 Transfer Theorem, one is able to assume that, without loss of generality, the language contains an ordering which behaves in a particular fashion by ordering a particular structure and expanding the language to include a symbol for that ordering. In the abstract case, one cannot simply do this, though we discuss some approaches to this difficulty in Section 5.7. For this reason, we must now assume we work in language which contains a binary relation interpreted as a linear order in elements of our AEC, **K**. In this chapter, we take the convention that "order" means linear order unless we specifically specify that we are talking about a partial order.

Assumption 5.3.1 *Though we exhibit this as an explicit hypothesis, we emphasize to the reader that throughout Section 5.3 we work in an AEC* **K** *satisfying AP and JEP in a language containing (amongst other symbols) a binary symbol* $<$ *which linearly orders elements of* **K**.[1] *In particular, we assume that our fixed monster model* $\mathfrak{C}$ *is an ordered structure.*

The following assumption is fixed as a hypothesis for the remainder of this section. It is a strong version of a X-Vaughtian pair; ideally, we would like to be able to deduce Assumption

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[1] *For example,* **K** *might be an AEC which consists of expansions of some unordered structures in another AEC consisting of structures in a language not containing "*$<$*".*

5.3.3 from the existence of a (κ^{++}, κ) model, though we are able to make only partial progress in this vein. It includes assumptions that the models involved behave in a certain manner with respect to the ordering and are at least to some degree galois homogeneous.

Notation 5.3.2

$$\mathrm{Pr}^{\mathscr{M}}(a) := \{b \in M : \mathscr{M} \models b < a\}$$

Assumption 5.3.3 *Let $\mathscr{B}^0, \mathscr{B}^1$ be models in $\mathbf{K}$ where $\mathbf{K}$ is an AEC in a language containing a binary symbol $<$ which linearly orders elements of $\mathbf{K}$ such that:*

(I) $\mathscr{B}^0 \prec_{\mathbf{K}} \mathscr{B}^1$

(II) There exists a strong embedding $\sigma : \mathscr{B}^0 \to \mathscr{B}^1$ and constants $e_i \in B^i$ for $i = 0, 1$ where $\sigma(e_0) = e_1$ such that:

  (a) $\sigma \restriction \mathrm{Pr}^{\mathscr{B}^0}(e_0) = \mathrm{Id}_{\mathrm{Pr}^{\mathscr{B}^0}(e_0)}$

  (b) $\mathscr{B}^0 \subseteq \mathrm{Pr}^{\mathscr{B}^1}(e_1)$

  (c) Suppose that $a, b \in B^0$ where $\mathrm{tp}^{\mathrm{ga}}(a) = \mathrm{tp}^{\mathrm{ga}}(b) = \mathrm{tp}^{\mathrm{ga}}(e_0)$ and there exists $\gamma \in \mathrm{aut}(\mathscr{B}^0)$ such that $\gamma(a) = b$ then there exists $\delta \in \mathrm{aut}(\mathscr{B}^1)$ such that $\delta(\sigma(a)) = b$.

(III) $\mathscr{B}^0$ is a local superlimit.

(IV) $(\mathscr{B}^0, e_0) \cong (\mathscr{B}^1, e_0)$.

(V) There is an invariant set[1] X such that $X(\mathscr{B}^0) = X(\mathscr{B}^1) \subseteq \mathrm{Pr}^{\mathscr{B}^0}(e_0)$.

---

[1] *It should be noted that X is necessarily disjoint from $\mathrm{tp}^{\mathrm{ga}}(e_0)$.*

A priori, there is some loss of generality in demanding that X be invariant over the empty set, however if X were invariant over a countable $\mathscr{M}$, one might add constants for $\mathscr{M}$ to the language and consider the class of structures for which $\mathscr{M}$ is a submodel instead. If we can find an $(\aleph_2, \aleph_0)$-model in this AEC, there will exist one in the original class as well.

Below is an easy consequence of Assumption 5.3.3:

Lemma 5.3.4

$$(\mathscr{B}^1, e_1) \cong (\mathscr{B}^0, e_0)$$

proof. Note that $\mathrm{Id}_{\mathscr{B}^0} \in \mathrm{aut}(\mathscr{B}^0)$ so by IIc there exists $\delta \in \mathrm{aut}(\mathscr{B}^1)$ such that $\delta(e_1) = \delta(\sigma(e_0)) = e_0$. It follows that: $(\mathscr{B}^1, e_1) \cong (\mathscr{B}^1, e_0)$ by IIc. By IV, $(\mathscr{B}^1, e_0) \cong (\mathscr{B}^0, e_0)$. So $(\mathscr{B}^1, e_1) \cong (\mathscr{B}^0, e_0)$ as desired. □

Because it fits as well here as anywhere, we show now, in Proposition 5.3.5, that the isomorphism type of a countable locally superlimit model is also closed under direct unions with respect to a countable directed set. This is almost a sufficient condition for us to be able to complete the limit stage of our construction of a $(\aleph_2, \aleph_0)$-model in Theorem 5.4.2. In the special case where $\mathscr{B}^0$ is galois-homogeneous this suffices; in general, it is still a useful exercise to prove this Proposition as a warm-up to proving Lemma 5.3.8.

Proposition 5.3.5 *Let I be a non-empty countable directed set, let $(\mathscr{M}_i, f_{i,j})_{i \leq j, i,j \in I}$ where for all $i, j \in I$, $\mathscr{M}_i \cong \mathscr{M}_j$ and for some $i \in I$ we suppose $\mathscr{M}_i$ is a local superlimit (in the sense of Definition 2.3.1), if $\mathscr{M}_\infty := \varinjlim_{i,j \in I} (\mathscr{M}_i, f_{i,j})$ then $\mathscr{M}_\infty \cong \mathscr{M}_i$.*

proof. If I has a maximal element M then $\mathscr{M}_\infty \cong \mathscr{B}_M$ so we may as well assume I has no maximal elements. In this case we can choose an infinite $\leq$-increasing and unbounded sequence $(i_n)_{n<\omega}$ consisting of elements of I. Since I is countable, there is no loss of generality in assuming this sequence has order type ω.

Let $a \sim b$ where $a \in M_i, b \in M_j$ if and only if there exists $k \in I$ such that $f_{i,k}(a) = f_{j,k}(b)$. By definition:

$$\mathscr{M}_\infty = \coprod_{i \in I} \mathscr{M}_i / \sim$$

Claim 5.3.6

$$\mathscr{M}_\infty = \coprod_{n<\omega} \mathscr{M}_{i_n} / \sim$$

Suppose that $a \in \coprod_{i \in I} \mathscr{M}_i$, then for some i, $a \in M_i$. Since the sequence $(i_n)_{n<\omega}$ is unbounded in I, we may choose $n < \omega$ large enough so that $i \leq i_n$. We note that $a \sim f_{i,i_n}(a)$. So for every $a \in \mathscr{M}_i$ there exists an $n < \omega$ and an $a' \in \mathscr{M}_{i_n}$ such that $a \sim a'$.

Suppose $a \sim b$, then for some i, j, k, $f_{i,k}(a) = f_{j,k}(b)$. Choose $n < \omega$ such that $k \leq i_n$. Let $a' := f_{i,i_n}(a)$, let $b' := f_{j,i_n}(b)$. Clearly $a \sim a' \sim b' \sim b$. But in particular note that $f_{i_n,i_{n+1}}(a') = f_{i_n,i_{n+1}}(b')$. It follows that:

$$\mathscr{M}_\infty = \coprod_{n<\omega} \mathscr{M}_{i_n} / \sim$$

Let $f_{i,\infty} : \mathscr{M}_i \to \mathscr{M}_\infty$ be the canonical embeddings. Since $\mathscr{M}_\infty = \coprod_{n<\omega} \mathscr{M}_{i_n} / \sim$ it follows that $\mathscr{M}_\infty = \bigcup_{n<\omega} f_{i_n,\infty}(\mathscr{M}_{i_n})$. Note that$f_{i_n,\infty}(\mathscr{M}_{i_n}) \cong \mathscr{M}_{i_n} \cong \mathscr{M}_0$. Since $\mathscr{M}_0$ is a local superlimit $\mathscr{M}_0 \cong f_{i_0,\infty}(\mathscr{M}_0) \cong \bigcup_{n<\omega} f_{i_n,\infty}(\mathscr{M}_{i_n}) = \mathscr{M}_\infty$. $\square$

We must add an additional assumption to deal with the limit stage of the construction, which essentially amounts to "the limit step works". While this may seem a strong assumption at this juncture, we will be able to demonstrate that this is satisfied by certain classes of structures which satisfy properties quite stronger than Assumption 5.3.3. In particular, countable galois homogeneous models of a first order theory satisfy Assumption 5.3.7. We explore such examples in section 5.5 and show that at least the original case of an elementary class where our invariant set defined by a first order formula is deducible from the work presented here in Corollary 5.7.17. That is, while we must apply work specific to the first-order case to complete steps 2 and 3 of proof outline, we can apply our proof of step 4, the actual construction of an $(\aleph_2, \aleph_0)$-model in the first order context. In other words, Assumptions 5.3.3 and 5.3.7 are deducible from the existence of a classical (κ^{++}, κ)-model.[1]

Assumption 5.3.7 *Suppose that $\mathscr{B}^0, \mathscr{B}^1$ are as in Assumption 5.3.3. Suppose further that $(\mathscr{B}^i)_{i<\gamma}$ is a countable $\prec_{\mathbf{K}}$-increasing chain of extensions of $\mathscr{B}^1$ with some distinguished set of constants $(e_i)_{i<\gamma}$ satisfying the following conditions:*

1. *for $i < j < \gamma$, $e_i \in B^j$ and*
2. *for $i < j < \gamma$, $(\mathscr{B}^i, e_j) \cong (\mathscr{B}^0, e_0)$.*

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[1]See Definition 5.7.13, for the classical definition of a (κ^{++}, κ)-model in first order model theory.

Then for all $j < \gamma$, $(\bigcup_{i<\gamma} \mathscr{B}^{\gamma}, e_j) \cong (\mathscr{B}^0, e_0)$.

We separate assumption 5.3.7 from assumption 5.3.3 for two main reasons. The first is that the assumptions made in Assumption 5.3.3 suffice for the successor steps of the construction of an $(\aleph_2, \aleph_0)$-model, while Assumption 5.3.7 is necessary only for limit steps. Additionally, 5.3.7 is not a "two-cardinal-type" assumption, it assumes properties hold of the models $\mathscr{B}^0, \mathscr{B}^1$ that do not directly relate to the invariant set which does not gain new realizations.

We now prove that if the above assumption holds, then an analogue holds for direct limits as well.

Lemma 5.3.8 *Suppose that* $\mathscr{B}^0$ *is as in Assumption 5.3.7 and assume that* $(\mathscr{B}^i, f_{i,j})_{i<j,i,j\in I}$ *is a directed system with respect to a countable, well-founded, directed partial order* $(I, <)$. *Let* $\mathscr{B}^{\infty}$ *be the direct limit of this directed system with canonical embeddings* $f_{i,\infty}$. *Suppose further that there are distinguished constants* $(e^i_{\alpha})_{\alpha\in\theta_i}$ *where* $e^i_{\alpha} \in \mathscr{B}^j$ *for some set of countable (possibly finite) ordinals* $(\theta_i)_{i\in I}$ *that satisfy if* $i \leq j$ *then* $\theta_i < \theta_j$. *Suppose yet further that for all* $j \geq i$ *and any* $\alpha \in \theta_i$, $(\mathscr{B}^j, f_{i,j}(e^i_{\alpha})) \cong (\mathscr{B}^0, e^0_0)$. *Then for any* $i \in I$ *and* $\alpha < \theta_i$, $(\mathscr{B}^{\infty}, f_{i,\infty}(e^i_{\alpha})) \cong (\mathscr{B}^0, e^0_0)$.

proof. The argument presented here is quite similar to the proof of Proposition 5.3.5. Suppose first that I has a maximal element M. Then without loss of generality $\mathscr{B}^{\infty} = B^M$, so the desired result holds by assumption. So we assume I has no maximal element. As in the proof of Proposition 5.3.5 we can fix a $\leq$-chain $(i_n)_{n<\omega}$ such that $\mathscr{B}^{\infty} = \bigcup_{n<\omega} f_{i_n,\infty}(\mathscr{B}_{i,i_n})$. There is no loss of generality in assuming that $i_0 = 0$. In fact, given any $j \in I$, we can find such a sequence with $j = i_n$ for some $n < \omega$. Define for each choice function $g : \omega \to \bigcup_{n<\omega} \theta_{i_n}$ $e^g_n := f_{i,\infty}(e^{i_n}_{g(n)})$. Observe that by assumption 5.3.7 that:

$$(\mathscr{B}^\infty, e^g_n) \cong (f_{0,\infty}(\mathscr{B}^0), f_{0,\infty}(e^0_0)) \cong (B^0, e^0_0)$$

Since we may choose $(i_n)_{n<\omega}$ to contain any $j \in I$ and our argument holds for any choice function $g : \omega \to \bigcup_{n<\omega} \theta_{i_n}$ this shows $(\mathscr{B}^\infty, f_{i,\infty}(e^i_\alpha)) \cong (\mathscr{B}^0, e^0_0)$ for any $i \in I$ where $\alpha < \theta_j$. $\square$

5.4 Constructing an $(\aleph_2, \aleph_0)$-model

Assumption 5.4.1 *Throughout this section members of* $\mathbf{K}$ *are linearly ordered by an ordering* $<$ *which is contained in the vocabulary* $\mathscr{L}$ *of* $\mathbf{K}$. *In particular,* $\mathfrak{C}$ *is an ordered structure. We note that an* $\aleph_3$*-saturated monster should suffice for this construction.*

The main theorem of this section isolates the set-theoretic combinitorics from the first order model theory necessary to perform the gap 2-transfer. The AEC axioms plus AP and JEP are all that are necessary to proceed through the construction. It is the other aspects of the gap 2-transfer theorem that utilize properties of first order logic that do not generalize painlessly to the AEC case.

We now prove the main theorem of the chapter, that is, we construct an $(\aleph_2, \aleph_0)$-model. It is worth noting that the only set theoretic hypothesis necessary for this result is the existence of a simplified morass.

Theorem 5.4.2 *Assume there exists a simplified morass (see Definition 5.1.1)* $(\theta_\alpha, \delta_\alpha, \mathscr{F}_{\alpha,\beta})$, *that Assumption 5.3.3 (which is assumed throughout this section) and Assumption 5.3.7 hold,* *then there exists an* $(\aleph_2, \aleph_0)$ *model. In particular, if* X *and* $\mathscr{B}^0$ *are as in Assumption 5.3.3, then there exists a model* $\mathscr{B}^{\omega_2} \in \mathbf{K}_{\aleph_2}$ *such that* $X(\mathscr{B}^{\omega_2}) = X(\mathscr{B}^0)$.

proof. We construct inductively a sequence of models $\mathscr{B}^i$ for $i \leq \omega_1$, strong embedding f^* for $f \in \mathscr{F}_{\alpha,\beta}$, and order embeddings $h_\alpha : \theta_\alpha \to \mathscr{B}^\alpha$ which satisfy the following properties for $\alpha < \beta \leq \omega_1$:

(c1) For $\alpha < \omega_1$, $\mathscr{B}^\alpha \cong \mathscr{B}^0$ is a local superlimit.

(c2) $h_\alpha : \theta_\alpha \to \theta_\beta$ is an order embedding (*not* a strong embedding).

(c3) $f^* : \mathscr{B}^\alpha \to \mathscr{B}^\beta$ for $f \in \mathscr{F}_{\alpha,\beta}$ is a strong embedding.

(c4) For $\eta \in \theta_\alpha$, $\alpha < \omega_1$, $(\mathscr{B}^\alpha, h_\alpha(\eta)) \cong (\mathscr{B}^0, e_0)$.

(c5) Suppose $\alpha < \beta < \gamma \leq \omega_2$, then $(fg)^* = f^* g^*$ for $f \in \mathscr{F}_{\beta,.\gamma}, g \in \mathscr{F}_{\alpha,\beta}$.

(c6) $h_\beta f = f^* h_\alpha$ for $f \in \mathscr{F}_{\alpha,\beta}$.

(c7) If $f \in \mathscr{F}_{\alpha,\beta}$ and $\mathrm{ran}(f) \subseteq \eta < \theta_\beta$ then $\mathrm{ran}(f^*) \subseteq \mathrm{Pr}^{\mathscr{B}^\beta}(h_\beta(\eta))$.

(c8) $\mathrm{Pr}^{\mathscr{B}^0}(e_0) = \mathrm{Pr}^{\mathscr{B}^\alpha}(e_0)$ and for $f \in \mathscr{F}_{\alpha,\beta}$ $f^* \restriction \mathrm{Pr}^{\mathscr{B}^\alpha}(e_0) = \mathrm{Id}_{\mathrm{Pr}^{\mathscr{B}^\alpha}(e_0)}$.

(c9) $X(\mathscr{B}^\alpha) = X(\mathscr{B}^0)$, in particular $X(\mathscr{B}^\alpha) = X(\mathscr{B}^0) \subseteq \mathrm{Pr}^{\mathscr{B}^\alpha}(e_0)$.

We begin with the $\mathscr{B}^0, \mathscr{B}^1$ given by II of Assumption 5.3.3. We let σ and e_0, e_1 be as in IIc as well. We let $h_0 : 0 \mapsto e_0$. We verify the construction for the base case now:

- c1 follows from III and IV.
- c2 follows since $|\mathrm{dom}(h_0)| = 1$.

- c3, c4, c5, c6, and c7 require no verification for the base case.
- c8 and c9 follow from V.

We move on now to the successor case. Suppose we have defined for $\beta_0 < \beta_1 \leq \alpha$ and $\beta \leq \alpha$ the objects $\mathscr{B}^\beta$, f^* for $f \in \mathscr{F}_{\beta_0,\beta_1}$, and $h_\beta : \theta_\beta \to \mathscr{B}^\beta$. For convenience of notation we define $e_\alpha := h_\alpha(\delta_\alpha)$.

Since $(\mathscr{B}^\alpha, e_\alpha) \cong (\mathscr{B}^0, e_0)$ by c4 we may fix $\phi \in \mathrm{aut}(\mathfrak{C})$ such that $\phi(\mathscr{B}^\alpha, e_\alpha) = (\mathscr{B}^0, e_0)$. We define $\mathscr{B}^{\alpha+1} := \phi^{-1}(\mathscr{B}^1)$. If $\mathscr{F}_{\alpha,\alpha+1} = \{\mathrm{Id}_{\theta_\alpha}, f_\alpha\}$ we set $\mathrm{Id}^*_{\theta_\alpha} := \mathrm{Id}_{\mathscr{B}^\alpha}$ and let $f^*_\alpha := \phi^{-1}\sigma\,\phi$. Define (again for notational convenience) $e_{\alpha+1} := f^*_\alpha(e_\alpha)$.

The following properties of $\mathscr{B}^\alpha$, $\mathscr{B}^{\alpha+1}$, and f^*_α are easily deduced from Assumption 5.3.3:

(o1) $f^*_\alpha : (\mathscr{B}^\alpha, e_\alpha) \to (\mathscr{B}^{\alpha+1}, e_{\alpha+1})$

(o2) $f^*_\alpha \restriction \mathrm{Pr}^{\mathscr{B}^\alpha}(e_\alpha) = \mathrm{Id}_{\mathrm{Pr}^{\mathscr{B}^\alpha}(e_\alpha)}$

(o3) $\mathscr{B}^\alpha \subseteq \mathrm{Pr}^{\mathscr{B}^{\alpha+1}}(e_{\alpha+1})$

(o4) If $\mathrm{tp}^{\mathrm{ga}}(b) = \mathrm{tp}^{\mathrm{ga}}(a) = \mathrm{tp}^{\mathrm{ga}}(e_\alpha) = \mathrm{tp}^{\mathrm{ga}}(e_0)$ and there exists $\gamma \in \mathrm{aut}(\mathscr{B}^\alpha)$ such that $\gamma(a) = b$ then there exists $\delta \in \mathrm{aut}(\mathscr{B}^{\alpha+1})$ such that $\delta(f^*_\alpha(a)) = b$.

(o5) $X(\mathscr{B}^{\alpha+1}) = X(\mathscr{B}^\alpha)$ (see Lemma 5.2.11)

(o6) $\mathrm{Pr}^{\mathscr{B}^{\alpha+1}}(e_\alpha) = \mathrm{Pr}^{\mathscr{B}^\alpha}(e_\alpha)$

M6 states that $\mathrm{ran}(\mathrm{Id}_{\theta_\alpha}) \cup \mathrm{ran}(f_\alpha) = \theta_{\alpha+1}$ and $\mathrm{ran}(\mathrm{Id}_{\theta_\alpha})$ is disjoint from $\theta_{\alpha+1} \setminus \theta_\alpha$ so every $\nu \in \theta_{\alpha+1} \setminus \theta_\alpha$ is equal to $f_\alpha(\eta)$ for some $\eta < \theta_\alpha$. Since f_α is order-preserving this η is unique. Thus we may define $h_{\alpha+1} : \theta_{\alpha+1} \to \mathscr{B}^{\alpha+1}$ by:

$$h_{\alpha+1}(\nu) := \begin{cases} h_\alpha(\nu) & \nu < \theta_\alpha \\ f^*_\alpha(h_\alpha(\eta)) \qquad \eta = f_\alpha(\nu), & \nu \geq \theta_\alpha \end{cases}$$

We have defined maps f^* for $f \in \mathscr{F}_{\alpha,\alpha+1}$ and we now define maps f^* for any $f \in \mathscr{F}_{\beta,\alpha+1}$. Note that by M2 we can write $f = gh$ for $g \in \mathscr{F}_{\alpha,\alpha+1}$ and $h \in \mathscr{F}_{\beta,\alpha}$. Let $f^* := g^*h^*$. We need to argue that there is no ambiguity in defining f^* in this manner.

First we argue that h is uniquely determined by the equation $f = gh$. Suppose we had maps $g' \in \mathscr{F}_{\alpha,\alpha+1}$ and $h' \in \mathscr{F}_{\beta,\alpha}$ such that $f = g'h'$. There are two cases to consider:

1. First suppose $g = g'$. Note that all maps in any $\mathscr{F}_{\beta,\gamma}$ are order-preserving and in particular are injective. So if $g(h(x)) = g'(h'(x))$ for all $x \in \theta_\beta$ then it must be the case that $h(x) = h'(x)$ for all $x \in \theta_\beta$.

2. Now suppose $g \neq g'$, we consider various sub-cases below:

   (a) Suppose that $\mathrm{ran}(h), \mathrm{ran}(h') \subseteq \delta_\alpha$, then since $f_\alpha \restriction \delta_\alpha = \mathrm{Id}_{\theta_\alpha} \restriction \delta_\alpha$, by the same argument as for the case $g = g'$, we see that $h = h'$.

   (b) Suppose that one of $\mathrm{ran}(h), \mathrm{ran}(h') \not\subseteq \delta_\alpha$, without loss of generality we may assume $\mathrm{ran}(h) \not\subseteq \delta_\alpha$. So we can find $x \in \theta_\alpha$ such that $h(x) \geq \delta_\alpha$.

We will argue that $h'(x) \geq \delta_\alpha$. If $h'(x) < \delta_\alpha$, then since $f_\alpha \restriction \delta_\alpha = \mathrm{Id}_{\theta_\alpha} \restriction \delta_\alpha$, no matter which of $f_\alpha, \mathrm{Id}_{\theta_\alpha}$ is equal to g', we see that $g'h'(x) = h'(x) < \delta_\alpha$. Meanwhile, since both possible choices for g (that is, f_α and $\mathrm{Id}_{\theta_\alpha}$) are non-decreasing, $gh(x) \geq h(x) \geq \delta_\alpha$. But then $f(x) = gh(x) \geq \delta_\alpha$ and $f(x) = g'h'(x) < \delta_\alpha$, which is a contradiction.

So we have both $h(x), h'(x) \geq \delta_\alpha$. Without loss of generality (since we now assume the same conditions on both h and h') we may assume that $g = \mathrm{Id}_{\theta_\alpha}$ and $g' = f_\alpha$. Thus we see that $g(h(x)) = \mathrm{Id}_{\theta_\alpha}(h(x)) \in \theta_\alpha$. On the other hand, since $h'(x) \geq \delta_\alpha$, $g'(h'(x)) = f_\alpha(h'(x)) \in \theta_{\alpha+1} \backslash \theta_\alpha$. But then $gh(x) \neq g'h'(x)$, which is a contradiction.

Thus we see there is only one choice possible for h, though there may still be more than one choice for g. We must argue that this does not effect the definition of f^*. There are two cases to consider:

- If $\mathrm{ran}(h) \not\subseteq \delta_\alpha$ then g *must* be f_α, thus there is no ambiguity in the definition of f^*.
- If $\mathrm{ran}(h) \subseteq \delta_\alpha$ then note that by o2 that $f^*_\alpha \restriction \mathrm{Pr}^{\mathscr{B}^\alpha}(e_\alpha) = \mathrm{Id}_{\mathrm{Pr}^{\mathscr{B}^\alpha}(e_\alpha)}$. By c7 $\mathrm{ran}(h^*) \subseteq \mathrm{Pr}^{\mathscr{B}^\beta}(e_\alpha)$. So again there is no ambiguity in the definition of f^*.

We now proceed with the verification of the inductive properties of the construction:

- c1 clearly holds.
- c2 follows since $f^*_\alpha(e_\alpha) \geq x$ for all $x \in B^\alpha$ (see o3) and $\mathscr{B}^\alpha \subseteq \mathrm{Pr}^{\mathscr{B}^{\alpha+1}}(e_{\alpha+1})$, $h_{\alpha+1}$ is order preserving. To elaborate on this point:

It is completely obvious from its definition that $h_{\alpha+1} \upharpoonright \theta_\alpha$ and $h_{\alpha+1} \upharpoonright \theta_{\alpha+1} \setminus \theta_\alpha$ are order preserving, since both f^*_α and h_α are order preserving maps. Since $h_{\alpha+1}$ sends elements of $\theta_{\alpha+1} \setminus \theta_\alpha$ to elements greater than any element in $\mathscr{B}^\alpha$ (which follows from that fact that f^*_α takes $\operatorname{ran}(h_\alpha \upharpoonright \theta \setminus \delta_\alpha)$ to elements greater than any element of $\mathscr{B}^\alpha$) it is clear that $h_{\alpha+1}$ is order-preserving on its entire domain.

- c3 has been verified in o1.
- It is fairly obvious that:

$$(\mathscr{B}^{\alpha+1}, e_{\alpha+1}) \cong (\mathscr{B}^1, e_1) \cong (\mathscr{B}^0, e_0) \tag{5.1}$$

since $\phi(\mathscr{B}^{\alpha+1}, e_{\alpha+1}) = (\mathscr{B}^1, e_1)$ and Lemma 5.3.4 states that $(\mathscr{B}^1, e_1) \cong (\mathscr{B}^0, e_0)$.

Verifying c4 in full requires some additional work. We consider two cases:

  - Suppose $\eta < \theta_\alpha$. So $h_{\alpha+1}(\eta) = h_\alpha(\eta)$. By applying the inductive assumption of c4 twice we see $(\mathscr{B}^\alpha, h_\alpha(\eta)) \cong (\mathscr{B}^0, e_0) \cong (\mathscr{B}^\alpha, h_\alpha(\delta_\alpha)) = (\mathscr{B}^\alpha, e_\alpha)$. By the definition of galois type, $\mathrm{tp}^{\mathrm{ga}}(h_\alpha(\eta)) = \mathrm{tp}^{\mathrm{ga}}(e_0) = \mathrm{tp}^{\mathrm{ga}}(e_\alpha)$ and there exists $\gamma \in \operatorname{aut}(\mathscr{B}^\alpha)$ such that $\gamma(h_\alpha(\eta)) = e_\alpha$. Thus, by o4 there exists $\delta \in \operatorname{aut}(\mathscr{B}^{\alpha+1})$ such that $\delta(h_\alpha(\eta)) = f^*_\alpha(e_\alpha) := e_{\alpha+1}$. It follows that:

$$\begin{aligned}
(\mathscr{B}^{\alpha+1}, h_{\alpha+1}(\eta)) &= (\mathscr{B}^{\alpha+1}, h_\alpha(\eta)) && \text{since } \eta < \theta_\alpha) \\
&\cong (\mathscr{B}^{\alpha+1}, e_{\alpha+1}) && \text{since } \delta(h_\alpha(\eta)) = e_{\alpha+1} \\
&\cong (\mathscr{B}^0, e_0) && \text{by Equation 5.1}
\end{aligned}$$

– Suppose $\eta \geq \theta_\alpha$. So $h_{\alpha+1}(\eta) = f^*_\alpha(h_\alpha(\nu))$ for some $\nu < \theta_\alpha$. By the inductive assumption of c4 we have:

$$(\mathscr{B}^\alpha, h_\alpha(\nu)) \cong (\mathscr{B}^0, e_0) \cong (\mathscr{B}^\alpha, e_\alpha) \tag{5.2}$$

So $\mathrm{tp}^{\mathrm{ga}}(h_\alpha(\nu)) = \mathrm{tp}^{\mathrm{ga}}(e_0) = \mathrm{tp}^{\mathrm{ga}}(e_\alpha)$ and there exists $\gamma \in \mathrm{aut}(\mathscr{B}^\alpha)$ such that $\gamma(h_\alpha(\nu)) = e_\alpha$. By o4 there exists $\delta \in \mathrm{aut}(\mathscr{B}^{\alpha+1})$ such that $\delta(h_\alpha(\nu)) = e_\alpha$ so:

$$(\mathscr{B}^{\alpha+1}, f^*_\alpha(h_\alpha(\nu))) \cong (\mathscr{B}^{\alpha+1}, e_\alpha) \tag{5.3}$$

Claim 5.4.3 $(\mathscr{B}^{\alpha+1}, e_\alpha) \cong (\mathscr{B}^{\alpha+1}, e_{\alpha+1})$

By Lemma 5.3.4 there exists $\Gamma \in \mathrm{aut}(\mathscr{B}^1)$ such that $\Gamma : e_1 \mapsto e_0$, set $\Delta := \phi^{-1}\Gamma\,\phi$. Note $\Delta \in \mathrm{aut}(\mathscr{B}^{\alpha+1})$. Observe:

$$\Delta : \qquad e_{\alpha+1} \xrightarrow{\phi} e_1 \xrightarrow{\Gamma} e_0 \xrightarrow{\phi^{-1}} e_\alpha$$

thus verifying the claim.

So:

$$\begin{aligned}
(\mathscr{B}^{\alpha+1}, h_{\alpha+1}(\eta)) &= (\mathscr{B}^{\alpha+1}, f^*_\alpha(h_\alpha(\nu))) && \text{by Equation 5.2}\\
&\cong (\mathscr{B}^{\alpha+1}, e_\alpha) && \text{by Equation 5.3}\\
&\cong (\mathscr{B}^{\alpha+1}, e_{\alpha+1}) && \text{by Claim 5.4.3}\\
&\cong (\mathscr{B}^{0}, e_1) && \text{by Equation 5.1}\\
&\cong (\mathscr{B}^{0}, e_0)
\end{aligned}$$

as desired.

- c5 is clear from the induction and our choice for $f \in \mathscr{F}_{\beta,\alpha+1}$ to have $f^* = g^*h^*$ for $g \in \mathscr{F}_{\alpha,\alpha+1}$ and $h \in \mathscr{F}_{\beta,\alpha}$.

- First we verify c6 for maps in $\mathscr{F}_{\alpha,\alpha+1}$. It is completely trivial to verify c6 for $\mathrm{Id}_{\theta_\alpha}$ since $\mathrm{Id}^*_{\theta_\alpha}$ was also chosen to be an identity map. So we verify the property for f_α. There are two cases to consider:

  – If $\eta < \delta_\alpha$ then $f_\alpha(\eta) = \eta$ and by o2 $f^*_\alpha \restriction \mathrm{Pr}^{\mathscr{B}^\alpha}(e_\alpha)$ is the identity map. Since h_α is order-preserving, $h(\eta) \leq h(\delta_\alpha) = e_\alpha$; it follows that $f^*_\alpha(h_{\alpha+1}(\eta)) = h_{\alpha+1}(\eta)$. Thus we have:

$$h_{\alpha+1}f_\alpha(\eta) = h_{\alpha+1}(\eta) = \mathrm{Id}_{\mathrm{Pr}^{\mathscr{B}^\alpha}(e_\alpha)}(h_{\alpha+1}(\eta)) = f^*_\alpha(h_{\alpha+1}(\eta))$$

  – If $\eta > \delta_\alpha$ then $f_\alpha(\eta) > \delta_\alpha$ so by the definition of $h_{\alpha+1}$, $h_{\alpha+1}(f(\eta)) = f^*_\alpha(h_\alpha(\eta))$.

For $g \in \mathscr{F}_{\beta,\alpha+1}$ where $\beta < \alpha$ we can write $g = f_0 f_1$ where $f_0 \in \mathscr{F}_{\alpha,\alpha+1}$ and $f_1 \in \mathscr{F}_{\beta,\alpha}$. We know by c5 that $g^* = f_0^* f_1^*$, so using induction and our observations above:

$$h_{\alpha+1} g = h_{\alpha+1} f_0 f_1 = f_0^* h_\alpha f_1 = f_0^* f_1^* h_\beta = g^* h_\beta$$

- We wish to verify c7. Let $f \in \mathscr{F}_{\alpha,\alpha+1}$ and suppose $\mathrm{ran}(f) \subseteq \eta < \theta_{\alpha+1}$. It follows that $f = \mathrm{Id}_{\theta_\alpha}$ since $\mathrm{ran}(f_\alpha)$ is unbounded in $\theta_{\alpha+1}$ by M5. For any $g \in \mathscr{F}_{\beta,\alpha+1}$ one may write $g = f_0 f_1$ for $f_0 \in \mathscr{F}_{\alpha,\alpha+1}$ and $f_1 \in \mathscr{F}_{\beta,\alpha}$. Suppose $\mathrm{ran}(g) \subseteq \eta < \theta_{\alpha+1}$. We consider two cases:
  - First suppose $f_0 = \mathrm{Id}_{\theta_\alpha}$. Then $\mathrm{ran}(g) = \mathrm{ran}(f_1) \subseteq \theta_\alpha$. So $\eta < \theta_\alpha$ and it follows from the definition of $h_{\alpha+1}$ that $h_{\alpha+1}(\eta) = h_\alpha(\eta)$. By induction $\mathrm{ran}(g) \subseteq \mathrm{Pr}^{\mathscr{B}^\alpha}(h_\alpha(\eta)) \subseteq \mathrm{Pr}^{\mathscr{B}^{\alpha+1}}(h_\alpha(\eta)) = \mathrm{Pr}^{\mathscr{B}^{\alpha+1}}(h_{\alpha+1}(\eta))$.
  - Now suppose that $f_0 = f_\alpha$. If $\eta < \delta_\alpha$, we could have assumed without loss of generality that $f_0 = \mathrm{Id}_{\theta_\alpha}$, so we may as well assume $\eta > \delta_\alpha$. By the definition of $h_{\alpha+1}$, we know that $h_{\alpha+1}(\eta) = f_\alpha^*(h_{\alpha+1}(\nu))$ for ν such that $f_\alpha(\nu) = \eta$. For $\mathrm{ran}(g) \subseteq \eta$ to hold, it must be the case that $\mathrm{ran}(f_1) \subseteq \nu < \theta_\alpha$. So by induction $\mathrm{ran}(f_1) \subseteq \mathrm{Pr}^{\mathscr{B}^\alpha}(h_\alpha(\nu))$. So $\mathrm{ran}(g^*) = f_\alpha^*(\mathrm{ran}(f_1^*)) \subseteq f_\alpha^*(\mathrm{Pr}^{\mathscr{B}^\alpha}(h_\alpha(\nu))) \subseteq \mathrm{Pr}^{\mathscr{B}^{\alpha+1}}(f_\alpha^*(h_\alpha(\nu))) = \mathrm{Pr}^{\mathscr{B}^{\alpha+1}}(h_{\alpha+1}(\eta))$.

- We now verify c8. Recall that $e_\alpha := h_\alpha(\delta_\alpha) = h_{\alpha+1}(\delta_\alpha)$[1]. By o6 $\Pr^{\mathscr{B}^\alpha}(e_\alpha) = \Pr^{\mathscr{B}^{\alpha+1}}(e_\alpha)$. By c8, c5 and the fact that $h_{\alpha+1}$ and h_α are order-preserving we can see that $h_{\alpha+1}(0) = h_\alpha(0) = e_0$. Furthermore, for the same reason, $e_0 \leq e_\alpha = h_\alpha(\delta_\alpha) = h_{\alpha+1}(\delta_\alpha)$. It follows that $\Pr^{\mathscr{B}^{\alpha+1}}(e_0) = \Pr^{\mathscr{B}^\alpha}(e_0)$, since both sets are initial segments of, respectively, $\Pr^{\mathscr{B}^{\alpha+1}}(e_\alpha) = \Pr^{\mathscr{B}^\alpha}(e_\alpha)$.[2] Inductively, we know $\Pr^{\mathscr{B}^\alpha}(e_0) = \Pr^{\mathscr{B}^0}(e_0)$, so we know $\Pr^{\mathscr{B}^{\alpha+1}}(e_0) = \Pr^{\mathscr{B}^0}(e_0)$. So we have verified the first clause of c8.

  Now suppose $f \in \mathscr{F}_{\beta,\alpha+1}$. Since both $\mathrm{Id}_{\mathscr{B}^{\alpha+1}}$ and f^*_α fix $\Pr^{\mathscr{B}^\alpha}(e_\alpha) \supseteq \Pr^{\mathscr{B}^\alpha}(e_0)$, it follows trivially from c5, M3, and induction that $f^* \restriction \Pr^{\mathscr{B}^\alpha}(e_0) = \mathrm{Id}_{\Pr^{\mathscr{B}^\alpha}(e_0)}$. Thus the second clause of c8 holds as well.

- Finally, we verify the last condition, c9. We know that $X(\mathscr{B}^{\alpha+1}) = X(\mathscr{B}^\alpha)$ by o5 and by induction $X(\mathscr{B}^\alpha) = X(\mathscr{B}^0)$. We have shown that c8 holds already, so since $X(\mathscr{B}^0) \subseteq \Pr^{\mathscr{B}^0}(e_0) = \Pr^{\mathscr{B}^{\alpha+1}}(e_0)$, $X(\mathscr{B}^{\alpha+1}) \subseteq \Pr^{\mathscr{B}^{\alpha+1}}(e_0)$. This completes our verification of c9.

We have now finished the successor step and move on to the limit step. Let γ be a limit ordinal and suppose the desired properties already hold of h_α, f^* for $f \in \mathscr{F}_{\alpha,\beta}$, and $\mathscr{B}^\alpha$ when $\alpha < \beta < \gamma$. We can proceed much as in Devlin. Set:

$$\mathscr{F}_\gamma := \bigcup_{\alpha<\gamma} \mathscr{F}_{\alpha<\gamma}$$

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[1]The second equality follows from the definition of $h_{\alpha+1}$ given earlier.

[2]Recall $\mathscr{B}^\alpha \prec_{\mathbf{K}} \mathscr{B}^{\alpha+1}$, in particular this means that $\mathscr{B}^\alpha$ and $\mathscr{B}^{\alpha+1}$ interpret "<" identically on their common initial segment $\Pr^{\mathscr{B}^\alpha}(e_0) = \Pr^{\mathscr{B}^{\alpha+1}}(e_0)$.

Define $d(f) = \alpha$ where $f \in \mathscr{F}_{\alpha,\gamma}$. We now define an ordering on $\mathscr{F}_\gamma$; let $f <^* f'$ if and only if the following two conditions hold:

1. $d(f) < d(f')$
2. There exists $g \in \mathscr{F}_{d(f),d(f')}$ such that $f = f'g$.

Note that if there exists a $g \in \mathscr{F}_{d(f),d(f')}$ such that $f = f'g$, there in fact exists a unique $g \in \mathscr{F}_{d(f),d(f')}$ such that $f = f'g$ (since all maps in $\mathscr{F}_\gamma$ are injective). It's clear that $(\mathscr{F}_\gamma, <^*)$ is a partial order. By properties M3 and M5 of Definition 5.1.1 $(\mathscr{F}_\gamma, <^*)$ is directed. For $f <^* f'$, define $g(f, f') = g$ such that $f = f'g$. Note $g(f, f') : \mathscr{B}^{d(f)} \to \mathscr{B}^{d(f')}$, so $(\mathscr{B}^{d(f)}, g(f, f'))_{f<^* f'\ ,\ f,f' \in \mathscr{F}_\gamma}$ is a directed system. Let $\mathscr{B}^\gamma$ be the direct limit of the previously mentioned directed system and for $f \in \mathscr{F}_\gamma$ let f^* be the canonical embedding with domain $\mathscr{B}^{d(f)}$.

By property c8 and Lemma 5.2.15, we may assume without loss of generality that $\mathrm{Pr}^{\mathscr{B}^0}(e_0) = \mathrm{Pr}^{\mathscr{B}^\gamma}(e_0)$ and that for $f \in \mathscr{F}_{\alpha,\gamma}$ that $f^* \restriction \mathrm{Pr}^{\mathscr{B}^\alpha}(e_0) = \mathrm{Id}_{\mathrm{Pr}^{\mathscr{B}^\alpha}(e_0)}$. It follows that c8 holds at a limit stage. We choose $h_\gamma : \theta_\gamma \to \mathscr{B}^\gamma$ by demanding that for all $\alpha < \gamma$, all $f \in \mathscr{F}_{\alpha,\gamma}$ that $h_\gamma f = f^* h_\alpha$.

As in Devlin, properties c2-c7 follow trivially except c4. c4 and c1 follow from Lemma 5.3.8 when $\gamma < \omega_2$. c8 was noted to be true in the previous paragraph. From Lemma 5.2.16 we see that c9 holds.

At the ω_1-step of our construction, we can maintain all properties except c1 and c4. In particular we have an order embedding of ω_2 into $\mathscr{B}^{\omega_1}$ (thus a structure of size $\aleph_2$), but $X(\mathscr{B}^{\omega_2}) = X(\mathscr{B}^0)$ (so in particular $|X(\mathscr{B}^{\omega_2})| = \aleph_0$). $\square$

5.5 1-transitive and galois homogeneous structures

In this section we explore sufficient conditions that imply Assumptions 5.3.3 and 5.3.7 (and thus allow us to construct an $(\aleph_2, \aleph_0)$-model, or in other words "complete step 4".). The first of these conditions, being globally 1-transitive with respect to a galois type over the empty set, is weaker than the second condition, which is a much stronger homogeneity condition. On the other hand, we must also assume that the 1-transitive structure is a local superlimit to satisfy Assumption 5.3.7, while the stronger homogeneity condition implies the structure is a local superlimit. We introduce the weaker notion first.

Again, we work with ordered structures, unless otherwise noted.

Assumption 5.5.1 *Throughout this section, unless otherwise noted, members of* $\mathbf{K}$ *are linearly ordered by an ordering* $<$ *which is contained in the vocabulary* $\mathscr{L}$ *of* $\mathbf{K}$*. In particular,* $\mathfrak{C}$ *is an ordered structure.*

Definition 5.5.2 *We say that a structure* $\mathscr{M} \in \mathbf{K}$ *is* 1-transitive with respect to p *where* p *is a Galois type over some (possibly empty) submodel of* $\mathscr{M}$ *if given an* $a, b \in M$ *where* $a, b \models p$ *then there exists* $\gamma \in \mathrm{aut}(\mathscr{M})$ *such that* $\gamma(a) = b$*.*

Lemma 5.5.3 *Suppose that* $\mathscr{M}_0 \cong \mathscr{M}_1$ *and that* $\mathscr{M}_0$ *is* 1*-transitive with respect to* $p := \mathrm{tp}^{\mathrm{ga}}(a)$*, where* $a \in \mathscr{M}_0$*, then* $\mathscr{M}_1$ *is also* 1*-transitive with respect to* p*.*

proof. Let $\phi : \mathscr{M}_1 \to \mathscr{M}_0$ be an isomorphism. Suppose that $b, c \models p$ where $b, c \in M_1$. Since p is a type over the empty set, $\phi(b), \phi(c) \models p$, hence there exists $\gamma \in \mathrm{aut}(\mathscr{M}_0)$ such that $\gamma(b) = c$. Let $\delta = \phi^{-1}\gamma\phi$. Note $\delta \in \mathrm{aut}(\mathscr{M}_1)$ and $\delta(b) = c$. $\square$

The following Proposition shows that you can reduce the sufficient conditions for existence of an $(\aleph_2, \aleph_0)$-model, that is, Assumptions 5.3.3 and 5.3.7 to a single, slightly shorter list of assumptions in the case that $\mathscr{B}^0$ is 1-transitive with respect to $\mathrm{tp}^{\mathrm{ga}}(e_0)$.

Proposition 5.5.4 *Suppose that $\mathscr{B}^0, \mathscr{B}^1 \in \mathbf{K}$ where $\mathbf{K}$ is an AEC in a language including a binary relation "$<$" which is interpreted as an ordering in an element of $\mathbf{K}$. Suppose that $\mathscr{B}^0$ and $\mathscr{B}^1$ satisfy the following conditions:*

(I) $\mathscr{B}^0 \prec_{\mathbf{K}} \mathscr{B}^1$

(II) There exists a strong embedding $\sigma : \mathscr{B}^0 \to \mathscr{B}^1$ and constants $e_i \in B^i$ for $i = 0, 1$ where $\sigma(e_0) = e_1$ such that:

  (a) $\sigma \restriction \mathrm{Pr}^{\mathscr{B}^0}(e_0) = \mathrm{Id}_{\mathrm{Pr}^{\mathscr{B}^0}(e_1)}$

  (b) $\mathscr{B}^0 \subseteq \mathrm{Pr}^{\mathscr{B}^1}(e_1)$

(III) $\mathscr{B}^0$ is a local superlimit.

(IV) $\mathscr{B}^0$ is 1-transitive with respect to $\mathrm{tp}^{\mathrm{ga}}(e_0)$.

(V) There is an invariant set X over the empty set such that $X(\mathscr{B}^0) = X(\mathscr{B}^1) \subseteq \mathrm{Pr}^{\mathscr{B}^0}(e_0)$.

then $\mathscr{B}^0$ and $\mathscr{B}^1$ satisfy both Assumption 5.3.3 and Assumption 5.3.7.

proof. Clearly, the necessary conditions to verify from Assumption 5.3.3 are (IIc) and (IV). Note Lemma 5.5.3 shows that $\mathscr{B}^1$ as well as $\mathscr{B}^0$ is 1-transitive with respect to $\mathrm{tp}^{\mathrm{ga}}(e_0)$. Note that since σ is a strong embedding for any $a \in M_0$, $\mathrm{tp}^{\mathrm{ga}}(a) = \mathrm{tp}^{\mathrm{ga}}(\sigma(a))$ (Since strong-embeddings preserve galois types over $\emptyset$). Thus, 5.3.3 (IIc) follows trivially from 1-transitivity with respect to $\mathrm{tp}^{\mathrm{ga}}(e_0)$. That is, if $\mathrm{tp}^{\mathrm{ga}}(a) = \mathrm{tp}^{\mathrm{ga}}(b) = \mathrm{tp}^{\mathrm{ga}}(e_0)$, then $\mathrm{tp}^{\mathrm{ga}}(\sigma(a)) = \mathrm{tp}^{\mathrm{ga}}(b)$ so there exists $\delta \in \mathrm{aut}(\mathscr{B}^1)$ such that $\delta(\sigma(a)) = b$. Likewise, (IVc) also follows trivially from 1-transitivity. That is, by definition there exists an isomorphism $\phi : \mathscr{B}^1 \to \mathscr{B}^0$, and clearly (see Proposition 5.2.6) $\mathrm{tp}^{\mathrm{ga}}(\phi(e_0)) = \mathrm{tp}^{\mathrm{ga}}(e_0)$ so there exists $\gamma \in \mathrm{aut}(\mathscr{B}^0)$ such that $\gamma(\phi(e_0)) = e_0$, thus $(\mathscr{B}^0, e_0) \cong (B^1, e_0)$. Thus Assumption 5.3.3 holds.

Now suppose that, as in Assumption 5.3.7 we have a countable sequence of models $(\mathscr{B}^i)_{i<\gamma}$ and constants $(e_i)_{i<\gamma}$ such that:

1. for $i < j < \gamma$, $e_i \in B^j$ and
2. for $i < j < \gamma$, $(\mathscr{B}^i, e_j) \cong (B^0, e_0)$.

Since $\mathscr{B}^0$ is a local superlimit, $\mathscr{B}^\gamma := \bigcup_{i<\gamma} \mathscr{B}^i \cong \mathscr{B}^0$. By Lemma 5.5.3, $\mathscr{B}^\gamma$ is 1-transitive with respect to $\mathrm{tp}^{\mathrm{ga}}(e_0) = \mathrm{tp}^{\mathrm{ga}}(\psi(e_0))$ where $\psi : \mathscr{B}^\gamma \to \mathscr{B}^0$ is any isomorphism. Note that condition (2) above implies that $\mathrm{tp}^{\mathrm{ga}}(e_j) = \mathrm{tp}^{\mathrm{ga}}(e_0)$ for any $j < \gamma$. So by 1-transitivity of $\mathscr{B}^\gamma$ with respect to $\mathrm{tp}^{\mathrm{ga}}(e_0)$, for any $j < \gamma$, $(\mathscr{B}^\gamma, e_j) \cong (\mathscr{B}^0, e_0)$. Thus Assumption 5.3.7 holds as well. $\square$

Notation 5.5.5 *For tuples* $\bar{a}, \bar{b} \in \mathfrak{C}$ *we write* $\bar{a} \sim_{\mathfrak{C}/\mathscr{M}} \bar{b}$ *if* $\mathrm{tp}^{\mathrm{ga}}(\bar{a}/\mathscr{M}) = \mathrm{tp}^{\mathrm{ga}}(\bar{b}/\mathscr{M})$, *that is if there exists an automorphism* $\gamma \in \mathrm{aut}(\mathfrak{C}/\mathscr{M})$ *such that* $\gamma(\bar{a}) = \bar{b}$. *If* $\mathscr{M} = \emptyset$ *then we write* $\bar{a} \sim_{\mathfrak{C}} \bar{b}$.

Definition 5.5.6 *We say that* $\mathscr{N} \in \mathbf{K}_\mu$ *is* μ-galois homogeneous over $\mathscr{M}$ *if given* $\bar{a}, \bar{b} \in N$ *where* $|\bar{a}| = |\bar{b}| < \mu$ *and* $\bar{a} \sim_{\mathfrak{C}/\mathscr{M}} \bar{b}$ *then there exists* $\gamma \in \mathrm{aut}(\mathscr{N}/\mathscr{M})$ *such that* $\gamma(a) = b$. *If*

$\mathscr{M} = \emptyset$ *then we say* $\mathscr{N}$ *is* μ-galois homogeneous*. In the case that* $\mu = |N|$ *we say* $\mathscr{N}$ is galois homogeneous*.*

Once one fixes a monster model $\mathfrak{C}$ for an elementary class, it should be clear that galois homogeneity is equivalent to sequence homogeneity[1] as it is classically defined. We provide an example where galois homogeneity differs from homogeneity.

Example 5.5.7 *Let* $\mathscr{L} = \{E, s\}$*, a single binary relation* E *and single unary function* s*. Let* $\mathbf{K}$ *consist of* $\mathscr{L}$*-structures that interpret* E *as an equivalence relation where each* E *class contains a model of* $\mathrm{Th}(\mathbb{Z}, s)$ *(*s *is interpreted as the successor function in each* E*-class). We demand further that, in any structure in* $\mathbf{K}$*, there be infinitely many* E*-classes containing one copy of* $(\mathbb{Z}, s)$ *and infinitely many* E*-classes contain two copies of* $(\mathbb{Z}, s)$*. Define* $\mathscr{N} \prec_{\mathbf{K}} \mathscr{M} \iff \mathscr{N} \subseteq \mathscr{M}$ *and for any* $x \in \mathscr{M}$ *if* $\exists y \in \mathscr{N}$ *such that* $E(x, y)$ *then* $x \in \mathscr{N}$*. (We are only permitted to add new equivalence classes in a strong extension, not add any elements to existing classes).*

$\mathbf{K}$ is an AEC with $\mathrm{LS}(\mathbf{K}) = \aleph_0$. Note that $\mathbf{K}$ satisfies AP and JEP. Any countable model is galois homogeneous, however if one fixes some a in an E-class containing two copies of $(\mathbb{Z}, s)$ and b in an E-class with a single copy of $(\mathbb{Z}, s)$ then the map (a, b) is partial elementary. However, if c is in the same E-class as a but not in the same copy of $(\mathbb{Z}, s)$ then there is no way to extend the map (a, b) to include c in its domain. Thus one cannot have a homogeneous (in the classical sense) model in $\mathbf{K}$.

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[1]That is $\mathscr{M}$ is μ-homogeneous if given an two tuples $\bar{a}, \bar{b}$, where $|\bar{a}| = |\bar{b}| < \mu$ if $\mathrm{tp}^{\mathrm{syntactic}}(\bar{a}) = \mathrm{tp}^{\mathrm{syntactic}}(\bar{b})$ for any $a \in \mathscr{M}$ there exists $b \in \mathscr{M}$ such that $\mathrm{tp}^{\mathrm{syntactic}}(\bar{a}a) = \mathrm{tp}^{\mathrm{syntactic}}(\bar{b}b)$.

There is a useful equivalent formulation of homogeneity for countable models given below. The proof is a typical back-and-forth argument.

Proposition 5.5.8 *$\mathscr{M} \in \mathbf{K}_{\aleph_0}$ is galois homogeneous if and only if given any $\overline{a}, \overline{b} \in M$ such that $\overline{a} \sim_{\mathfrak{C}} \overline{b}$ and $|\overline{a}| = |\overline{b}| < \omega$ then for all $a \in M$ there exists $b \in M$ such that $\overline{a}a \sim_{\mathfrak{C}} \overline{b}b$.*

proof. If $\mathscr{M} \in \mathbf{K}_{\aleph_0}$ is galois homogeneous then if we are given $\overline{a}, \overline{b} \in M$ with $|\overline{a}| = |\overline{b}| < \omega$ there is $\gamma \in \mathrm{aut}(\mathscr{M})$ such that $\gamma(\overline{a}) = \overline{b}$. Given any $a \in M$, $\overline{a}a \sim_{\mathfrak{C}} \overline{b}\gamma(a)$.

On the other hand, suppose that given any $\overline{a}, \overline{b} \in M$ such that $\overline{a} \sim_{\mathfrak{C}} \overline{b}$ and $|\overline{a}| = |\overline{b}| < \omega$ then for all $a \in M$ there exists $b \in M$ such that $\overline{a}a \sim_{\mathfrak{C}} \overline{b}b$. Now suppose that $\overline{a}, \overline{b} \in M$ satisfy $|\overline{a}| = |\overline{b}| < \omega$ and $\overline{a} \sim_{\mathfrak{C}} \overline{b}$. Fix enumerations $(a_i)_{i<\omega}$ and $(b_i)_{i<\omega}$ of, respectively $M \setminus \overline{a}, M \setminus \overline{b}$. We will construct a sequence of automorphisms of $\mathfrak{C}$, $(\gamma_i)_{i<\mathfrak{C}}$ and tuples $\overline{a}_i$, $\overline{b}_i$ such that for $i < \omega$:

1. $\overline{a}_i \sqsubseteq \overline{a}_{i+1}$
2. $\overline{b}_i \sqsubseteq \overline{b}_{i+1}$
3. $a_i \subset \overline{a}_{i+1}$
4. $b_i \subset \overline{b}_{i+1}$
5. $\gamma_{i+1} \restriction \overline{a}_{i+1} \supseteq \gamma_i \restriction \overline{a}_i$
6. $\overline{b}_i = \mathrm{ran}(\gamma_i \restriction \overline{a}_i)$
7. $|\overline{a}_i| = |\overline{b}_i| < \omega$
8. $\overline{a}_i \sim_{\mathfrak{C}} \overline{b}_i$.

We proceed by induction on $i < \omega$. For the base case, note that since $\bar{a} \sim_{\mathfrak{C}} \bar{b}$ there exists $\gamma_0 \in \mathrm{aut}(\mathfrak{C})$ such that $\gamma_0(\bar{a}) = \bar{b}$. Let $\bar{a}_0 = \bar{a}$ and $\bar{b}_0\bar{b}$.

Suppose that we have defined γ_i,$\bar{a}_i$, and $\bar{b}_i$, we proceed to define γ_{i+1}, $\bar{a}_{i+1}$, and $\bar{b}_{i+1}$. We note that, by induction, $\bar{a}_i \sim_{\mathfrak{C}} \bar{b}_i$, so, if $a_i \not\subset \bar{a}_i$ then there exists $b \in M$ such that $\bar{a}a_i \sim_{\mathfrak{C}} \bar{b}b$ (and if $a_i \subset \bar{a}_i$ let $b = \emptyset$). If $b_i \notin \bar{b}b$ then there exists $a \in M$ such that $\bar{a}a_ia \sim_{\mathfrak{C}} \bar{b}b_ib$ (and if $b_i \subset \bar{a}_ia_i$ let $a = \emptyset$). Let $\bar{a}_{i+1} = \bar{a}a_ia$ and $\bar{b}_{i+1} = \bar{b}_ibb_i$. Let $\gamma_{i+1} \in \mathrm{aut}(\mathfrak{C})$ witness that $\bar{a}_{i+1} \sim_{\mathfrak{C}} \bar{b}_{i+1}$.

It is clear that $\bigcup_{i<\omega} \gamma_i \restriction \bar{a}_i \in \mathrm{aut}(\mathscr{M})$ which takes $\bar{a}$ to $\bar{b}$ by construction. $\square$

If a structure is of cardinality greater $\lambda > \aleph_0$, then it becomes combinatorially difficult to prove an analogous result for λ-galois homogeneous structures of size λ. (one must deal with infinite tuples); it is however certainly true that if $\mathscr{M} \in \mathbf{K}_\lambda$ is $\aleph_0$-galois homogeneous then the "if" clause of the above proposition still holds, by the same proof as above.

Proposition 5.5.9 *If $\mathscr{M} \in \mathbf{K}_\lambda$ is $\aleph_0$-galois homogeneous then given any $\bar{a}, \bar{b} \in M$ such that $\bar{a} \sim_{\mathfrak{C}} \bar{b}$ and $|\bar{a}| = |\bar{b}| < \omega$ then for all $a \in M$ there exists $b \in M$ such that $\bar{a}a \sim_{\mathfrak{C}} \bar{b}b$.*

The following obvious fact holds:

Fact 5.5.10 *Suppose that $\mathscr{M}$ is galois homogeneous, then $\mathscr{M}$ is 1-transitive with respect to $\mathrm{tp}^{\mathrm{ga}}(a)$ for any $a \in M$.*

We can show a countable galois homogeneous model is a local superlimit by using a back-and-forth argument, the proof is identical to the proof that shows countable homogeneous models (in the classic sense of the definition) that realize the same syntactic types over the empty set are isomorphic.

Proposition 5.5.11 [1] *If $\mathscr{M}$ is countable and galois homogeneous then $\mathscr{M}$ is a local superlimit.*

proof. Suppose that $(\mathscr{M}_i)_{i<\omega}$ is a countable increasing sequence of models, $\mathscr{M}_0 := \mathscr{M}$ and $\mathscr{M}_i \cong \mathscr{M}_0$. Let $\mathscr{M}_\omega := \bigcup_{i<\omega} \mathscr{M}_i$. We claim $\mathscr{M}_\omega \cong \mathscr{M}_0$.

We show $\mathscr{M}_\omega \cong \mathscr{M}_0$ by a typical back-and-forth argument. Let $(a_i)_{i<\omega}$ enumerate M_ω and let $(b_i)_{i<\omega}$ enumerate M_0. Similarly to the proof of Proposition 5.5.8 we will construct a sequence of automorphisms of $\mathfrak{C}$, $(\gamma_i)_{i<\mathfrak{C}}$ and tuples $\overline{a}_i$, $\overline{b}_i$ such that for $i < \omega$:

1. $\overline{a}_i \sqsubseteq \overline{a}_{i+1}$
2. $\overline{b}_i \sqsubseteq \overline{b}_{i+1}$
3. $a_i \subset \overline{a}_{i+1}$
4. $b_i \subset \overline{b}_{i+1}$
5. $\gamma_{i+1} \upharpoonright \overline{a}_{i+1} \supseteq \gamma_i \upharpoonright \overline{a}_i$
6. $\overline{b}_i = \mathrm{ran}(\gamma_i \upharpoonright \overline{a}_i)$
7. $|\overline{a}_i| = |\overline{b}_i| < \omega$
8. $\overline{a}_i \sim_{\mathfrak{C}} \overline{b}_i$.

We proceed by induction. Note for some $j < \omega$, $a_0 \in M_j \cong M_0$. Let $\phi : \mathscr{M}_j \to \mathscr{M}_0$ be an isomorphism. $a_0 \sim_{\mathfrak{C}} \phi(a_0)$. By Proposition 5.5.8 there's some $a \in A_0$ such that $a_0 a \sim_{\mathfrak{C}} bb_0$. Let $\overline{a}_0 = a_0 a$ and $\overline{b}_0 bb_0$. Let $\gamma_0 \in \mathrm{aut}(\mathfrak{C})$ take $\overline{a}_0$ to $\overline{b}_0$.

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[1] *It is not required* **K** *be ordered for this result.*

Suppose we have defined $\bar{a}_0$, $\bar{b}_0$, and γ_i. We note that for some $j < \omega$, $\bar{a}_i a_i \subseteq M_j \cong \mathscr{M}_0$. Let $\phi : \mathscr{M}_j \to \mathscr{M}_0$ be an isomorphism (possibly different then ϕ fixed in the base case). Note that $\phi(\bar{a}_i a_i) \sim_{\mathfrak{C}} \bar{a}_i a_i$, by homogeneity of $\mathscr{M}_0$ we know that there is an automorphism $\gamma \in \operatorname{aut}(\mathscr{M}_0)$ where $\gamma(\phi(\bar{a} a_i)) = \bar{a}_i a_i$ so without loss of generality $\phi(\bar{a}_i a_i) = \bar{a}_i a_i$. Since $\bar{a}_i \sim_{\mathfrak{C}} \bar{b}_i$ by induction, we can find some $b \in M_0$ such that $\bar{a}_i a_i \sim_{\mathfrak{C}} \bar{b}_i b$, similarly there is some $a \in M_0$ such that $\bar{a}_i a_i a \sim_{\mathfrak{C}} \bar{b} b b_i$. Let $\bar{a}_{i+1} = \bar{a}_i a_i a$, $\bar{b}_{i+1} = \bar{b} b b_i$, and let $\gamma_{i+1} \in \operatorname{aut}(\mathfrak{C})$ take $\bar{a}_{i+1}$ to $\bar{b}_{i+1}$.

Let $\gamma := \bigcup_{i<\omega} \gamma_i \restriction \bar{a}_i \in \operatorname{aut}(\mathscr{M})$, it should be clear that γ is an isomorphism from $\mathscr{M}_\omega$ to $\mathscr{M}_0$. $\square$

As with the assumption that we work with 1-transitive structures, the assumption that we work with countable galois homogeneous structures reduces the number of additional assumptions necessary to complete the inductive construction of an $(\aleph_2, \aleph_0)$-model.

Proposition 5.5.12 *Suppose that $\mathscr{B}^0, \mathscr{B}^1 \in \mathbf{K}_{\aleph_0}$ where $\mathbf{K}$ is an AEC in a language including a binary relation "$<$" which is interpreted as a linear ordering in any element of $\mathbf{K}$. Suppose that $\mathscr{B}^0$, $\mathscr{B}^1$ satisfy the following conditions:*

(I) $\mathscr{B}^0 \prec_{\mathbf{K}} \mathscr{B}^1$

(II) There exists a strong embedding $\sigma : \mathscr{B}^0 \to \mathscr{B}^1$ and constants $e_i \in B^i$ for $i = 0, 1$ where $\sigma(e_0) = e_1$ such that:

- *(a)* $\sigma \restriction \operatorname{Pr}^{\mathscr{B}^0}(e_0) = \operatorname{Id}_{\operatorname{Pr}^{\mathscr{B}^0}(e_1)}$
- *(b)* $\mathscr{B}^0 \subseteq \operatorname{Pr}^{\mathscr{B}^1}(e_1)$

(III) $\mathscr{B}^0 \cong \mathscr{B}^1$ is galois homogeneous

(IV) There is an invariant set X over the empty set such that $X(\mathscr{B}^0) = X(\mathscr{B}^1) \subseteq \mathrm{Pr}^{\mathscr{B}^0}(e_0)$.

Then $\mathscr{B}^0$ and $\mathscr{B}^1$ satisfy both Assumption 5.3.3 and Assumption 5.3.7. In particular, an $(\aleph_2, \aleph_0)$-model $\mathscr{B}^{\omega_2}$ exists where $X(\mathscr{B}^{\omega_2}) = X(\mathscr{B}^0)$.

proof. By Proposition 5.5.11 $\mathscr{B}^0$ is a local superlimit. Note also that galois homogeneous structures are 1-transitive with respect to any galois type over the empty set (see Fact 5.5.10). So we may apply Proposition 5.5.4 to see that Assumption 5.3.3 and 5.3.7 hold.

Since Assumption 5.3.3 and 5.3.7 hold, we may apply Theorem 5.4.2 to build an $(\aleph_2, \aleph_0)$-model $\mathscr{B}^{\omega_2}$ such that $X(\mathscr{B}^{\omega_2}) = X(\mathscr{B}^0)$. $\square$

Finally we provide a construction of a countable homogeneous structure extending accountable structure in an AEC $\mathbf{K}$; it's worth noting that our ambient assumptions of joint embedding and amalgamation (we give an example that shows amalgamation at least is necessary) are necessary, while it seems *unnecessary* to assume that we have amalgamation over sets.

Theorem 5.5.13 *[AP, JEP][1] Suppose that $\mathscr{M} \in \mathbf{K}_{\aleph_0}$, there exists $\mathscr{N} \in \mathbf{K}_{\aleph_0}$ such that $\mathscr{M} \prec_{\mathbf{K}} \mathscr{N}$ and $\mathscr{N}$ is galois homogeneous.*

proof. Fix $\mathscr{M} \in \mathbf{K}_{\aleph_0}$. We will construct a countable $\prec_{\mathbf{K}}$-increasing sequence of countable models $\mathscr{M}_0$ with the following properties:

1. $\mathscr{M}_0 = \mathscr{M}$.

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[1] *It is not required $\mathbf{K}$ be ordered for this result.*

2. For any finite tuples $\overline{a}, \overline{b} \in M_i$ where $\overline{a} \sim_{\mathfrak{C}} \overline{b}$, $\mathscr{M}_{i+1}$ and singleton $a \in M_i$ there exists $b \in M_{i+1}$ such that $\overline{a}a \sim_{\mathfrak{C}} \overline{b}b$.

Condition 1. is trivial, to see Condition 2. can be satisfied consider the following:

Since $\mathscr{M}_i$ is countable, there are only countably many pairs of finite tuples $(\overline{a}, \overline{b})$ of elements of $\mathscr{M}_i$ such that $\overline{a} \sim \overline{b}$; clearly there are only countably many choices for $a \in M_i$. If $\overline{a} \sim \overline{b}$ there is $\gamma_{\overline{a},\overline{b}} \in \mathrm{aut}(\mathfrak{C})$ such that $\gamma_{\overline{a},\overline{b}}(\overline{a}) = \overline{b}$; note $\overline{a}a \sim_{\mathfrak{C}} \overline{b}\gamma_{\overline{a},\overline{b}}(a)$. The set $\{\gamma_{\overline{a},\overline{b}}(a) : \overline{a} \sim_{\mathfrak{C}} \overline{b}, \overline{a}, \overline{b} \in M_i, a \in M_i\}$ is countable, hence contained in some countable model $\mathscr{M}_{i+1}$.

Let $\mathscr{N} = \bigcup_{i<\omega} \mathscr{M}_i$. Given any $\overline{a}, a \in \mathscr{M}$, there exists $i < \omega$ such that $\overline{a}, a \in M_i$. So there exists, by construction, $b \in M_{i+1}$ such that $\overline{a}a \sim_{\mathfrak{C}} \overline{b}b$. So by Proposition 5.5.8 $\mathscr{N}$ is galois homogeneous. $\square$

5.6 Non-existence of galois homogeneous extensions in PCΓ classes

Assumption 5.6.1 *In this section* **K** *is not ordered and doesn't satisfy AP.*

In the previous section, Theorem 5.5.13 shows that in an AEC **K** which has the amalgamation property and joint embedding property, any countable model can be extended to a galois homogeneous structure, much as any structure has an elementary extension which is countably homogeneous in the classical model theoretic sense. However, this is not true for AECs which do not satisfy the the amalgamation property. We give an example below of an AEC which is a PCΓ class in which certain models have no galois homogeneous extension; indeed, the notion of "galois homogeneous" is not even well-defined in the absence of either the joint embedding or amalgamation properties since there is no suitable "monster model" to define the $\sim_{\mathfrak{C}}$-notion with respect to. Below we give an example where a countable structure cannot be extended to a model which is homogeneous in the traditional model theoretic sense.

The example is produced in the language $\{P_n : n \in \omega\} \cup \{E\}$ where the P_n are unary and E is binary. The P_n are interpreted as and infinite family of refining predicates and E will be interpreted as an equivalence relation dividing each P_n into two infinite classes. However, we will force there to be a different finite number of elements in the two equivalence classes in the intersection $\bigcup_{n\in\omega} P_n(\mathscr{M})$ for certain models $\mathscr{M}$.

Example 5.6.2 *1. Let T be the first order theory in $\mathscr{L} := (P_n)_{n<\omega} \cup \{E\}$ of infinitely many unary predicates and one binary relation E, such that*

(a) $\forall x P_0(x)$.

(b) $\forall x(P_{n+1}(x) \implies P_n(x))$

(c) $\exists^\infty x(P_n(x) \wedge \neg P_{n+1}(x))$

(d) E partitions the realizations of P_n in a model into two infinite equivalence classes.

2. $p(x_1, x_2, y_1, y_2, y_3)$ is a partial type including the following (collections) of formulas:

- *For $n < \omega$ $\bigwedge_{i=1,2\ j=1,2,3} P_n(x_i) \wedge P_n(y_i)$*
- *$x_1 \neq x_2 \wedge \bigwedge_{i\neq j, j=1,2,3} y_i \neq y_j$*
- *$x_1 E x_2 \wedge \bigwedge_{i,j\in\{1,2,3\}} y_i E y_j$*
- *For $n < \omega$ $\bigwedge_{i=1,2\ j=1,2,3} \neg x_i E y_j$*

3. $q(x_1, x_2, x_3, x_4)$ is a partial type including the following (collections) of formulas:

- *For $n < \omega$ $\bigwedge_{i=1}^{4} P_n(x_i)$*
- $\bigwedge_{i \neq j} x_i \neq x_j$

Note that in Example 5.6.2.2 if p is realized in a $\mathscr{M} \models T$ then E divides $\bigcap_{n\in\omega} P_n(\mathscr{M})$ into one equivalence class of cardinality at least two and another class of cardinality at least three. If $\mathscr{M} \models T$ realizes q then $\left|\bigcap_{n\in\omega} P_n(\mathscr{M})\right| \geq 4$. On the other hand if $\mathscr{M}$ omits q then $\left|\bigcap_{n\in\omega} P_n(\mathscr{M})\right| \leq 3$. If $\mathscr{M}$ also omits p there may be an element in one E class in $\bigcap_{n\in\omega} P_n(\mathscr{M})$ while the other equivalence class is allowed to contain at most two elements.

In particular if $\left|\bigcap_{n\in\omega} P_n(\mathscr{M})\right| = 3$ (which is the maximal possible size) then there are two non-empty E-classes of $\bigcap_{n\in\omega} P_n(\mathscr{M})$. One E-class has a unique element a_0 while the other E-class has two elements b_0, b_1. Since a_0 and b_0 have the same type over $\emptyset$ the map $b_0 \mapsto a_0$ is elementary, however there is no way to extend this map to have domain $\{b_0, b_1\}$. Furthermore, there is no way to extend this map to have domain $\{b_0, b_1\}$ in any elementary extension of $\mathscr{M}$ which also omits p. So in the $\mathrm{PC}(T, \{p, q\})$ class there is no elementary extension of $\mathscr{M}$ that is ω-galois homogeneous in the traditional, model-theoretic sense of the notion.

It should be clear to see that we may modify this example to allow one of the E-classes of $\bigcap_{n\in\omega} P_n$ to be infinite while the other has some fixed finite maximal size or to fix both classes to be of at most some different maximal size.

We emphasize that the above example occurs in a context where the amalgamation property fails. In our argument, dependence on the existence of $\mathfrak{C}$ is omnipresent, but beyond this we are also assuming the existence of a simplified morass, which follows from **V=L**. In particular, there is little reason not to assume weak GCH. We note the following theorem of Shelah (from (Shelah, 1987)):

Theorem 5.6.3 (WGCH) *Suppose that* $\mathbf{K}$ *is* λ*-categorical for some* $\lambda \geq \mathrm{LS}(K)$*. If* $\mathbf{K}_{\lambda^+}$ *has less than* 2^λ *isomorphism types then* $\mathbf{K}_\lambda$ *satisfies the amalgamation property in* λ*.*

Corollary 5.6.4 *If* $\mathbf{K}$ *is categorical in all* $\lambda > \kappa$ *for some* κ*, then* κ *satisfies the amalgamation property in* μ *for all* $\mu \geq \kappa$*.*

So "few models", in the appropriate sense implies amalgamation. Of course, we do not necessarily work with classes where we know there are few models in $\aleph_0$, or for that matter, in any cardinal.

5.7 Steps 2, 3, and 4

Assumption 5.7.1 *Throughout this section* $\mathbf{K}$ *is assumed to satisfy AP, JEP and have a monster* $\mathfrak{C}$*, but we do not assume the vocabulary* $\mathscr{L}$ *of* $\mathbf{K}$ *contains a binary symbol interpreted as an ordering in members of* $\mathbf{K}$*. We will, however, also work with various expanded languages including, in particular, a language* $\mathscr{L}''$ *that does contain a symbol for the ordering.*

Up until now, we have talked about conditions on countable structures which allow us to construct and $(\aleph_2, \aleph_0)$-model, in this section we explore the progress we have made toward finding conditions on uncountable models which will allow us to construct an $(\aleph)2, \aleph_0)$-model. More precisely, we look for conditions on uncountable models which allows us to derive some conditions on countable models which we already have shown we can use to build an $(\aleph_2, \aleph_0)$-model. In particular, we attempt to avoid the initial assumption that elements of $\mathbf{K}$ are linearly ordered. As noted in the introduction to the chapter, we make only partial progress in this direction.

First, §5.7.1, we will discuss issues relating to adding an ordering to a non-ordered AEC. In order to do so, we make use of Shelah's Presentation Theorem for AECs. A key sufficient condition, upon which all the progress we have been able to make in working with an AEC that does not already have a suitable ordering is the assumption that the monster model admits an expansion to an ordered structure that is at least $\aleph_1$-saturated.

We recall that in the first order case gap-2 transfer is accomplished through the following steps:

1. Build an isomorphic pair of structures in κ.
2. Obtain a Vaughtian pair (in κ) and an elementary embedding that codes some combinatorial information; expand the language to code the Vaughtian pair and combinatorial information into an expanded first order theory.
3. Find a "nice" pair of countable homogeneous models of the expanded theory.
4. Construct an $(\aleph_2, \aleph_0)$-gap using properties of a simplified morass.

We will first prove an analogue of step 1. for AECs. Then, in §5.7.2, we will discuss the proof of step 2. in the first order context and the difficulties we encounter in attempting to prove a similar result for AECs. We have, ultimately, not been able to prove any sort of analogue of step 2. for AECs. However, in §5.7.3, we introduce a condition stronger than the conclusion of 2., a sort of "2+." condition on models of size κ from which we can complete step 3 and then step 4, thus yielding the existence of an $(\aleph_2, \aleph_0)$-model. After achieving this, we discuss how, by slightly modifying the proof of Theorem 5.4.2, we can also weaken the strength of the "2+" assumption.

5.7.1 the presentation theorem and step 1.

One of the tools we will use in working toward this goal is representing an AEC as a collection of models of a first order theory omitting some collection of types.

Definition 5.7.2 *Let* Γ *be a collection of partial first order syntactic types in finitely many variables over* $\emptyset$ *in some language* $\mathscr{L}'$. *A* $\mathrm{PC}_{\mathscr{L}}(\Gamma, T')$-class *is the class of reducts to some* $\mathscr{L} \subseteq \mathscr{L}'$ *of models of the* $\mathscr{L}'$*-theory* T' *that omit the collection of types* Γ.

Notation 5.7.3 *We write* $\mathrm{PC}\Gamma$ *to denote that a class is* $\mathrm{PC}_{\mathscr{L}}(\Gamma, T')$ *for some* T', $\mathscr{L}'$, $\mathscr{L}$, *and* Γ.

Definition 5.7.4 *We say that* $\mathbf{K}$ *is* $\mathrm{PC}(\lambda, \mu)$ *if* $\mathbf{K}$ *is a* $\mathrm{PC}_{\mathscr{L}}(\Gamma, T')$ *for* $|\Gamma| \leq \lambda$, *and* $|T'| < \mu$.

Notation 5.7.5 *If* $\mathscr{N}'$ *is an* $\mathscr{L}'$ *structure, we write* $\mathscr{N}$ *to denote* $\mathscr{N}' \restriction \mathscr{L}$ *when we believe this is unlikely to cause confusion.*

If any result in the study of AECs can be considered "classical" we believe Shelah's Presentation Theorem qualifies as such. A proof of the theorem is available in (Baldwin, 2009) as Theorem 4.15, amongst other sources.

Theorem 5.7.6 (Shelah's Presentation Theorem) *If* $\mathbf{K}$ *is an AEC in a language* $\mathscr{L}$ *with* $|\mathscr{L}| \leq \mathrm{LS}(\mathbf{K})$ *there is a language* $\mathscr{L}' \supseteq \mathscr{L}$*, an* $\mathscr{L}'$ *first order theory* T'*, and a collection* Γ *of* $\mathscr{L}'$*-types over* $\emptyset$ *such that*

$$\mathbf{K} = \{\mathscr{M} : \mathscr{M}' \models T', \mathscr{M}' \textit{ omits } \Gamma\}$$

In particular, the following hold:

1. $|\Gamma| \leq 2^{\mathrm{LS}(\mathbf{K})}, |T'| \leq \mathrm{LS}(\mathbf{K})$.
2. *If* $\mathscr{M}', \mathscr{N}' \models T'$*,* $\mathscr{M}', \mathscr{N}'$ *omit* Γ*, and* $\mathscr{M}' \prec \mathscr{N}'$ *then* $\mathscr{M} \prec_{\mathbf{K}} \mathscr{N}$.
3. *If* $\mathscr{M} \prec_{\mathbf{K}} \mathscr{N}$ *then there are expansions* $\mathscr{M}', \mathscr{N}'$ *of respectively* $\mathscr{M}, \mathscr{N}$*, such that* $\mathscr{M}', \mathscr{N}' \models T'$*,* $\mathscr{M}', \mathscr{N}'$ *omit* Γ*, and* $\mathscr{M}' \prec \mathscr{N}'$.

In other words, Shelah's presentation theorem says that an AEC $\mathbf{K}$ can be represented as a $\mathrm{PC}(\mathrm{LS}(\mathbf{K}), 2^{\mathrm{LS}(\mathbf{K})})$ class.

As we discussed at the beginning of this chapter we can use the presentation theorem to make partial progress toward an analogue of Jensen's gap-2 transfer theorem.

Assumption 5.7.7 *For the remainder of the section we fix* $\mathbf{K}$ *to be an AEC satisfying AP and JEP in cardinals up to and including* κ^{++} *with* $\mathrm{LS}(\mathbf{K}) = \omega$*. Fix a* (κ^{++}, κ)*-model* $\mathscr{C}$*; that is* $|C| = \kappa^{++}$*,* $\mathscr{B} \prec_{\mathbf{K}} \mathscr{C}$*,* $|B| = \kappa$*, and an invariant set* X *(see section 5.2) such that* $X(\mathscr{C}) = X(\mathscr{B})$.

Observation 5.7.8 *Note that if* X *is invariant instead over a countable* $\mathscr{M}$*, we may add constants to the language for* $\mathscr{M}$ *and consider only structures for which* $\mathscr{M}$ *is a strong substructure.*

Now it's worth pointing out that, as long as the monster model $\mathfrak{C}$ admits an expansion to a sufficiently saturated structure $\mathfrak{C}''$ then any set $X \subset \mathfrak{C}$ which is invariant remains invariant when considered as a subset of the expanded structure; this should be trivial to see since any automorphism of a structure in some language will remain an automorphism of any reduct of that structure. What is *not* the case in general is that an invariant set X is invariant for a reduct (indeed, consider the reduct to the language of pure equality, if any bijection $b : \mathfrak{C} \to \mathfrak{C}$ is *not* an automorphism of $\mathfrak{C}$ then $\mathfrak{C} \upharpoonright \emptyset$ will not have the same invariant sets as $\mathfrak{C}$). On the other hand, if we start with an invariant set X with respect to $\mathscr{L}$, then move to the expanded language $\mathscr{L}''$ to apply Theorem 5.4.2, since X *was* invariant with respect to $\mathscr{L}$, if one takes a reduct to $\mathscr{L}$ after applying Theorem 5.4.2 then X remains invariant.

Now, a priori, one might be able to move from our PCΓ class-$\mathbf{K}'$ to a class of AECs in an expanded language $\mathbf{K}''$, satisfying any of the sufficient conditions we have explored (that is, 5.3.3 plus 5.3.7, the hypotheses of Proposition 5.5.4, and the hypotheses of Proposition 5.5.12) for the existence of an $(\aleph_2, \aleph_0)$-model *except for* being able to expand the monster model $\mathfrak{C}'$ to a monster model for $\mathbf{K}''$. This, however, cannot be the case, observe:

Observation 5.7.9 *If $\mathbf{K}''$ is an AEC whose members reduct to models in the* PCΓ *class $\mathbf{K}'$ (or equivalently members of the original AEC $\mathbf{K}$) then if there exists any monster model $\mathfrak{C}''$ (in the sense that $\mathfrak{C}''$ is a large-enough model-homogeneous and every $\mathscr{N} \in \mathbf{K}''$ where $|N| < |\mathfrak{C}|$ can be strongly embedded in $\mathfrak{C}''$) for $\mathbf{K}''$, $\mathfrak{C}'$ admits an expansion isomorphic to $\mathfrak{C}''$.*

The proof of the above observation is essentially a proof-by-definition. Since we've defined a monster model is a large model-homogeneous structure, a monster models in some vocabulary

τ must be unique up to isomorphism. Thus, if any monster model $\mathfrak{C}''$ exists, it's reduct to $\mathscr{L}'$ is isomorphic to $\mathscr{C}'$. Assuming **GCH**, since given that a Morass exists if **V=L**, it is somewhat natural to do so, the only obstacle to the existence of a monster model is failure of JEP and/or AP. In fact the key problem we face in expanding the language, if we wish to apply the results of this chapter, is making the class $\mathbf{K}''$ satisfy AP and JEP.

Notation 5.7.10 *By the Theorem 5.7.6 we can find some $\mathscr{L}', T', \Gamma$ such that $\mathbf{K}$ is represented as a $\mathrm{PC}_{\mathscr{L}}(T', \Gamma)$ class. As in the first order proof of Jensen's Gap-2 transfer theorem, we can expand $\mathscr{L}'$ to $\mathscr{L}'' = \mathscr{L}' \cup \{<\}$ and the pair of models $(\mathscr{B}, \mathscr{C})$ can be expanded to a pair of $\mathscr{L}''$-structures $(\mathscr{B}'', \mathscr{C}'')$ which are ordered by "$<$" so that $\mathscr{B}''$ has order type κ and $\mathscr{C}''$ is an end extension of $\mathscr{B}''$ with order type κ^{++}. (Note that we do not assume that $\mathscr{B}'' \equiv_{\mathscr{L}''} \mathscr{C}''$, merely that $\mathscr{B}''$ is an $\mathscr{L}''$-substructure of $\mathscr{C}''$).*

Note that, by our previous remarks, there is no serious loss of generality in Theorem 5.4.2 in assuming the language contains an ordering with regard to the behaviour of the invariant set manipulated in the construction. However, the assumption (Assumption 5.7.18) that our class of structures satisfies AP and JEP does potentially limit the generality of the scope of our results in this section.

In (Devlin, 1984) step 1. is essentially Lemma 3.2. The proof of Lemma 3.2 in (Devlin, 1984) can be applied with minimal modification in the context of AEC to yield a version of step 1. for AEC. The key to adapting the first order proof to the PCΓ case is the ability to work in $\mathscr{C}''$ as an ambient structure which omits Γ. We provide a proof below, following closely the outline of the proof in (Devlin, 1984), but being careful to check that the steps remain valid in the PCΓ context. Note that for this particular proof, we don't actually need AP or JEP in the expanded language because we can work entirely within the structure $\mathscr{C}''$.

Proposition 5.7.11 ($\mathbf{2}^\kappa = \kappa^+$) *Let $\mathscr{C}'', \mathscr{L}'', X$ be as in Notation 5.7.10. There is an X-Vaughtian pair $\mathscr{N}'', \mathscr{M}'' \equiv_{\mathscr{L}''} \mathscr{C}''$, an elementary map $\sigma : \mathscr{N}'' \to \mathscr{M}''$, and some $e \in M''$ such that:*

1. *$\mathscr{N}'', \mathscr{M}''$ omit Γ.*

2. *$\mathscr{N}'' \prec_{\mathscr{L}''} \mathscr{M}''$*

3. *$X(\mathscr{N}'') = X(\mathscr{M}'')$*[1]

4. *$\sigma \restriction \mathrm{Pr}(e) = \mathrm{Id}_{\mathscr{N}''}$*

5. *$N'' \subseteq \mathrm{Pr}(\sigma(e))$ (In particular, σ is not onto)*

6. *$|N''| = |M''| = \kappa$*

proof. For $\alpha < \kappa^{++}$ define $\mathscr{A}_\alpha$ to be a $\prec_{\mathscr{L}''}$-minimal strong sub-structure of $\mathscr{C}''$ containing $\kappa \cup \alpha$ (that is, $\mathscr{A}_\alpha$ is minimal containing $B'' \cup \alpha$, where $\mathscr{B}''$ is as in Notation 5.7.10). That is, there is no proper strong substructure of $\mathscr{A}_\alpha$ which contains α. Since a subset of an ordinal is well-ordered, we can choose such an $\mathscr{A}_\alpha$. Fix S to be a cofinal subset of κ^{++} such that $\mathscr{A}_\alpha \neq \mathscr{A}_\beta$ for distinct $\alpha, \beta \in S$. By downward Löwenheim-Skolem, for all $\alpha < \kappa^{++}$ $|A_\alpha| = \kappa$. Since there are at most κ^+ distinct well orders of cardinality κ (and a subset of an ordinal is a well order)

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[1] *Note that X is invariant in $\mathscr{L}$', hence also $\mathscr{L}$".*

we may assume without loss of generality that there is some ordinal $\theta < \kappa^+$ such that for all $\alpha \in S$, $\mathscr{A}_\alpha$ is of order type θ.

For $\alpha \in S$ we can write $A_\alpha = \{a^\alpha_\nu : \nu < \theta\}$ where for fixed α and $\nu < \eta < \theta j \implies \mathscr{A}_\alpha \models a^\alpha_\nu < a^\alpha_\eta$. Since $\alpha \in \mathscr{A}_\alpha$ we find a least $\varrho < \theta$ such that $(a^\alpha_\varrho)_{\alpha \in S}$ is cofinal in κ^{++}.

Claim 5.7.12 *We may assume that $(a^\alpha_\nu)_{\nu<\varrho} = (a^\beta_\nu)_{\nu<\varrho}$ for $\nu < \theta$. In addition, we may assume without loss of generality for $\alpha, \beta \in S$ that $\alpha < \beta \implies a^\alpha_\nu < a^\beta_\varrho$.*

Let $\gamma := \sup\{a^\alpha_\eta : \alpha \in S, \eta < \varrho\}$. Since ϱ is the least ordinal such that $(a^\alpha_\varrho)_{\alpha \in S}$ is cofinal in κ^{++}, it follows that $\gamma < \kappa^{++}$, i.e. $|\gamma| = \kappa^+$. Since $(\kappa^+)^\kappa = \kappa^+$, there are only κ^+ maps from ϱ ($|\varrho| \leq \kappa$) into γ ($|\gamma| \leq \kappa^+$), while there are κ^{++} elements of S. So we can find a subset $S' \subseteq S$ where for $\alpha, \beta \in S'$, $(a^\alpha_\nu)_{\nu<\varrho} = (a^\beta_\nu)_{\nu<\varrho}$ for $\nu < \theta$ and $\alpha < \beta \implies a^\alpha_\nu < a^\beta_\varrho$. We replace S with S' if necessary. This completes the proof of the claim.

Set:

$$Y = \{a^\alpha_\nu : \nu < \varrho\}$$

$$Z_\alpha = \{a^\alpha_\nu : \varrho \leq \nu < \theta\}$$

So Y and Z_α partition each $\mathscr{A}_\alpha$ into "the part before ϱ" and "the part after ϱ". In particular for all $\alpha, \beta \in S$ where $\alpha < \beta$:

$$A_\alpha = Y \cup Z_\alpha$$

$$Y \cap Z_\alpha = \emptyset$$

$$Y < Z_\alpha < Z_\beta$$

By $2^\kappa = \kappa^+$ there are at most $\kappa^\kappa = \kappa^+$ non-isomorphic structures of size κ, so we can find $\alpha, \beta \in S$, $\alpha < \beta$ where $\mathscr{A}_\alpha \cong \mathscr{A}_\beta$. Thus, we can find an isomorphism $\sigma : \mathscr{A}_\alpha \to \mathscr{A}_\beta$., in particular σ must respect the ordering on the sets A_α, so σ must be the unique order isomorphism from A_α onto A_β. Let $e = a^\alpha_\varrho$. σ will fix $\mathrm{Pr}(e) \cap \mathscr{A}_\alpha$. Since $\alpha < \beta$, for all $\nu < \varrho$, $a^\alpha_\nu < a^\beta_\varrho$, it follows that $\mathscr{A}_\alpha \subseteq \mathrm{Pr}^{\mathscr{C}''}(\sigma(e))$. Since for any $\gamma \in S$, $X(\mathscr{C}'') = X(\mathscr{B}'') \subseteq B'' \subseteq A_\gamma$, $X(\mathscr{A}_\alpha) = X(\mathscr{A}_\beta)$. Furthermore, since $B'' \subseteq Y$, it follows that $X(\mathscr{A}_\alpha) \subseteq \mathrm{Pr}^{\mathscr{A}_\alpha}(e)$.

By downward Löwenheim-Skolem we can choose $\mathscr{M}'' \prec_{\mathbf{K}} \mathscr{C}''$ to be of size κ such that $A_\alpha \cup \sigma(A_\alpha) \subseteq M''$. Note that $\sigma : \mathscr{A}_\alpha \to \mathscr{M}''$ cannot be surjective, because if $b \in A_\alpha \setminus \mathrm{Pr}(e)$ then $b \notin \sigma(A_\alpha)$. We let $\mathscr{N}'' := \mathscr{A}_\alpha$, with e and σ as defined above, we satisfy the conclusion of the hypotheses. Notice in particular that all structures were chosen as submodels of $\mathscr{C}''$, hence they omit Γ. $\square$

5.7.2 Step 2

In Section 5.7 we noted that the proof of the gap-2 transfer theorem for first order logic can be broken down into 4 steps. We proved an analogue of the first step in Section 5.7 and we proved an analogue of the fourth step in Section 5.3.[1] In this section we discuss one of the difficulties inherent in providing a substitute for Step 2 in the context of AECs.

In Proposition 5.7.15 we state the result which allows one to complete step 2 in the first order context. In Corollary 5.7.17, we complete the argument that Jensen's result is a special case of our analysis. It's worth noting one must use certain first order results (namely Proposition 5.7.15) to obtain the full gap-two transfer theorem and not just the sufficient condition (Assumption 5.3.3 plus Assumption 5.3.7) on a pair countable models for the existence of an $(\aleph_2, \aleph_0)$-model. In Proposition 5.7.19, we provide an analogue for step 3 in the AEC case, which we remind the reader, is provable from a slightly stronger condition than the conclusion of step 2 in the outline of the first order proof.

We begin with a discussion of the proof in the first order case. So we define:

Definition 5.7.13 *A* classical (κ, λ)-model *is a model $\mathscr{M}$ where $|M| = \kappa$ and there is some definable predicate ϕ such that $|\phi(\mathscr{M})| = \lambda$.*

One may prove an analogue, Proposition 5.7.14, of Proposition 5.7.11 for first order (κ, λ)-models in the essentially the same way as we proved 5.7.11. We omit the proof (which can be found in (Devlin, 1984), as 3.2), but give a statement of the proposition:

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[1] Of course, in Section 5.3, it was necessary to assume Assumption 5.3.3 and 5.3.7 as a substitute for the conclusion of Step 3.

Proposition 5.7.14 ($\mathbf{2}^\kappa = \kappa^+$) *Suppose $\mathscr{C}''$ is a classical (κ^{++}, κ)-model, witnessed by the definable predicate ϕ. There are $\mathscr{N}'', \mathscr{M}'' \equiv_{\mathscr{L}''} \mathscr{C}''$, an elementary map $\sigma : \mathscr{N}'' \to \mathscr{M}''$, and some $e \in N''$ such that:*

1. $\mathscr{N}'' \prec_{\mathscr{L}''} M''$
2. $\phi(\mathscr{N}'') = \phi(\mathscr{M}'')$
3. $N'' \subseteq \Pr(\sigma(e))$ *(in particular σ is not onto)*
4. $\sigma \upharpoonright \Pr(e) = \mathrm{Id}_{\mathscr{N}''}$
5. $\mathscr{N}'' \subseteq \Pr(\sigma(e))$
6. $|N''| = |M''| = \kappa$

We reproduce the statement and sketch the proof of Lemma 3.3 from (Devlin, 1984) as Proposition 5.7.15 (This Lemma achieves step 2 and step 3 of the proof outline of the first order gap two transfer theorem.) below to illustrate its dependencies on properties of first order logic. We diverge somewhat from the proof in (Devlin, 1984) to emphasize a subtle point of the argument is omitted from the proof which appears in (Devlin, 1984).[1]

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[1] It seems likely that Devlin was under the impression that a reduct of a homogeneous structures is also homogeneous, which is one of the very few false assertions made in (Chang and Keisler, 1977). The proof of Lemma 3.3 in (Devlin, 1984) references the earlier result Lemma 1.5's proof. In this proof, he states it's clear that the reduct of certain countable homogeneous structure is also a countable homogeneous structure, which is not true in general. Thus, this argument requires just a little bit of extra care be taken to verify the proposition, which remains true.

Proposition 5.7.15 ($2^\kappa = \kappa^+$) *We assume that $\mathscr{C}''$ is a classical (κ^{++}, κ)-model, witnessed by the definable predicate ϕ. There are countable homogeneous models $\mathscr{B}_0, \mathscr{B}_1 \equiv_{\mathscr{L}''} \mathscr{C}''$, $e_0 \in B_0$, and an elementary embedding $\sigma_0 : \mathscr{B}_0 \to \mathscr{B}_1$ such that:*

1. $\mathscr{B}_0 \prec_{\mathscr{L}''} \mathscr{B}_1$
2. $\phi(\mathscr{B}_0) = \phi(\mathscr{B}_1)$
3. $(\mathscr{B}_0, e_0) \cong (\mathscr{B}_1, \sigma(e_0))$
4. $\sigma_0 \upharpoonright \mathrm{Pr}^{\mathscr{B}_0}(e_0) = \mathrm{Id}_{\mathrm{Pr}^{\mathscr{B}_0}(e_0)}$
5. $B_0 \subseteq \mathrm{Pr}^{\mathscr{B}_1}(\sigma_0(e_0))$ *(in particular, σ_0 is not surjective)*

proof. Because we work in a large number of different languages, we take some time to list them all here:

- $\mathscr{L}$ is some vocabulary for an AEC $\mathbf{K}$. (In this case, an elementary class)
- $\mathscr{L}'$ is the language given by the Presentation Theorem. (In this case, there is no need to apply the Presentation Theorem and we may assume $\mathscr{L}' = \mathscr{L}''$)
- $\mathscr{L}''$ is $\mathscr{L}'$ expanded by adding a linear order "$<$"
- $\mathscr{L}'''$ will be an expansion of $\mathscr{L}''$ defined below to contain a predicate σ for an $\mathscr{L}''$-elementary embedding and a constant symbol as well as a predicate for an $\mathscr{L}''$-elementary submodel.
- $\mathscr{L}''''$ will be an expansion of $\mathscr{L}'''$ defined below to include a function symbol for an $\mathscr{L}''$-isomorphism.

Let $\mathscr{M}'', \mathscr{N}'', \sigma, e$ be as in Proposition 5.7.14, expand $\mathscr{L}''$ to $\mathscr{L}'''$ by adding a function symbol for σ, constant symbol for e, and predicate for N''. Let T''' be the $\mathscr{L}'''$ theory of $(M'', N'', \sigma, e, ...)$.

If κ is singular, then **GCH** allows us to find a special model $(M''', N''', \sigma''', e''')$ of T''' in the classical model theoretic sense, that is, a model with a filtration $(\mathscr{M}_i)_{i<\kappa}$ such that $\mathscr{M}_{i+1}$ is $|M_i|$-saturated with respect to syntactic types. (If κ is regular this is simply a model saturated with respect to syntactic types). Reducts of special models are special, so there is an isomorphism $h : \mathscr{N}''' \restriction \mathscr{L}'' \to \mathscr{M}''' \restriction \mathscr{L}''$. Let T'''' be the $\mathscr{L}''''$ theory of $(M'''', N'''', \sigma'''', e'''', h, ...)$ (where $\mathscr{L}''''$ is an expansion of $\mathscr{L}'''$ that contains a function symbol for the map h).

By compactness, we can find a countable homogeneous model $(B_1, B_0, \sigma_0, e_0, h_0...) \models T''''$. However, reducts of homogeneous models are not necessarily homogeneous and we desire to have $\mathscr{B}_1$ and $\mathscr{B}_0$ be homogeneous as $\mathscr{L}''$ structures (as well as realize the same $\mathscr{L}''$ types), so some additional work is necessary. Of course, since the theory T'''' knows that h_0 is an $\mathscr{L}''$ isomorphism, then we already know that $\mathscr{B}_1$ and $\mathscr{B}_0$ realize the same $\mathscr{L}''$ types and, furthermore, if one $\mathscr{B}_i$ is homogeneous with respect to $\mathscr{L}''$ then so is the other structure; thus it is enough to show:

Claim 5.7.16 $(B_1, B_0, \sigma_0, e_0, h_0...)$ *can be extended to a structure* $(B_1^H, \mathscr{B}_0^H, \sigma_0^H, e_0^H, h_0^H, ...)$ *such that* $\mathscr{B}_1^H \restriction \mathscr{L}''$ *is homogeneous.*

We proceed by an inductive construction, constructing models $\mathscr{B}^n := (B_1^n, B_0^n, \sigma_0^n, e_0^n, h_0^n...)$ where given any finite tuples of the same length $\overline{a}, \overline{b} \in B^n$ and a singleton $a \in B^n$ such that $\overline{a} \equiv_{\mathscr{L}''} \overline{b}$ there is $b \in B^{n+1}$ such that $\overline{a}a \equiv_{\mathscr{L}''} \overline{b}b$. We start with $\mathscr{B}^0 := (B_1, B_0, \sigma_0, e_0, h_0...)$.

Consider, for each singleton $a \in B^n$ and tuples $\overline{a}, \overline{b} \in B^n$ where $\overline{a} \equiv_{\mathscr{L}'} \overline{b}$ the type $p_{\overline{a},a,\overline{b}}(x,y)$ which contains for every $\mathscr{L}''$ formula $\phi(x,\overline{x})$ the formula $\phi(x,\overline{a}) \leftrightarrow \phi(y,\overline{b})$. If $p_{\overline{a},a,\overline{b}}(x,y)$ is inconsistent then there is an $\mathscr{L}''$-formula ϕ such that $\phi(x,\overline{a}) \leftrightarrow \phi(x,\overline{b})$ is inconsistent with T''''. So $T'''' \models \exists x\phi(x,\overline{a}) \wedge \neg\exists y\phi(y,\overline{b})$, but this contradicts that $\overline{a} \equiv_{\mathscr{L}''} \overline{b}$, because $\phi(x,\overline{x})$ is an $\mathscr{L}''$-formula.

Since $\mathscr{L}''$ is countable and there are only countably many finite tuples in $\mathscr{B}^n$ we can extend $\mathscr{B}^n$ to a model $\mathscr{B}^{n+1}$ which is countable and realizes $p_{\overline{a},a,\overline{b}}$ for each $\overline{a}, a, \overline{b} \in B^n$ thus satisfying the desired property. That is, given any finite tuples of the same length $\overline{a}, \overline{b} \in B^n$ and a singleton $a \in B^n$ such that $\overline{a} \equiv_{\mathscr{L}''} \overline{b}$ there is $b \in B^{n+1}$ such that $\overline{a}a \equiv_{\mathscr{L}''} \overline{b}b$. If $\mathscr{B}^H := \bigcup_{n<\omega} \mathscr{B}^n$ then $\mathscr{B}^H_1 \restriction \mathscr{L}''$ is countable homogeneous.

So, without loss of generality $\mathscr{B}_1$ and $\mathscr{B}_0$ are countably homogeneous as $\mathscr{L}''$-structures, hence $(\mathscr{B}_0, e_0) \cong_{\mathscr{L}''} (\mathscr{B}_1, \sigma(e_0))$. $\square$

While the result is, of course, already known, we can also now deduce the first order Gap-2 Transfer Theorem from Proposition 5.7.15 and Theorem 5.4.2.

Corollary 5.7.17 (V=L)(Jensen) *Suppose T is a first order theory with a classical (κ^{++}, κ)-model $\mathscr{C}$, witnessed by the definable predicate ϕ. Then there is an $(\aleph_2, \aleph_0)$-model (witnessed by ϕ).*

proof. We apply Lemma 5.7.15 above, There are countable homogeneous models $\mathscr{B}_0, \mathscr{B}_1 \equiv_{\mathscr{L}''} \mathscr{C}''$, $e_0 \in B_0$, and an elementary embedding $\sigma_0 : \mathscr{B}_0 \to \mathscr{B}_1$ such that:

A. $\mathscr{B}_0 \prec_{\mathscr{L}"} \mathscr{B}_1$

B. $\phi(\mathscr{B}_0) = \phi(\mathscr{B}_1)$

C. $(\mathscr{B}_0, e_0) \cong (\mathscr{B}_1, \sigma(e_0))$

D. $\phi(\mathscr{B}_0) \subseteq B_0$

E. $\sigma_0 \restriction \mathrm{Pr}^{\mathscr{B}_0}(e_0) = \mathrm{Id}_{\mathrm{Pr}^{\mathscr{B}_0}(e_0)}$

F. $B_0 \subseteq \mathrm{Pr}^{\mathscr{B}_1}(\sigma_0(e_0))$ (in particular, σ_0 is not surjective)

(I) We verify that the properties of Assumption 5.3.3 and 5.3.7 hold. (I) is, of course, immediate from condition A.

(II) Property (IIa) of 5.3.3 holds by condition E, (IIb) holds by condition [C.]. Note that since we assume the existence of a large homogeneous model, $\mathfrak{C}$, syntactic types of finite tuples over the empty set are the same as the galois type over the empty set of the same tuple. So homogeneity of $\mathscr{B}_0, \mathscr{B}_1$ implies that if $\mathrm{tp}^{\mathrm{syntactic}}(a) = \mathrm{tp}^{\mathrm{syntactic}}(b)$ then there exists $\delta \in \mathrm{aut}(\mathscr{B}^1$ such that $\delta(\sigma(a)) = b$, since σ, being a strong (which in this case means elementary) embedding must preserve types over the empty set. Indeed, this condition is stronger than condition (IIc) of 5.3.3.

(III) Since any two countable homogeneous models realizing the same syntactic types are isomorphic, it follows that $\mathscr{B}_0 \cong \mathscr{B}_1$ is a local superlimit, so condition (III) of 5.3.3 is satisfied.

(IV) Property (IV) of 5.3.3 follows immediately from condition D.

(V) We note that the set $X := \phi(\mathfrak{C})$ is invariant. So by condition B. condition (V) of 5.3.3 is satisfied.

Since homogeneity is preserved under countable limits 5.3.7 holds as well. To elaborate: if we have a sequence $(\mathscr{B}^i)_{i<\omega}$ such that for each $i < \omega$ $(\mathscr{B}^i, e_i) \cong (\mathscr{B}^0, e_0)$ then in particular $\mathrm{tp}^{\mathrm{syntactic}}(e_i) = \mathrm{tp}^{\mathrm{syntactic}}(e_0)$. $\bigcup_{i<\omega} \mathscr{B}^i \cong \mathscr{B}^0$, since $\mathscr{B}^0$ is a superlimit, but more strongly, it's clear that for any i and isomorphism $f : \bigcup_{i<\omega} \mathscr{B}^i \to \mathscr{B}^0$, $\mathrm{tp}^{\mathrm{syntactic}}(f^{-1}(e_i)) = \mathrm{tp}^{\mathrm{syntactic}}(e_i) = \mathrm{tp}^{\mathrm{syntactic}}(e_0)$. So by homogeneity of $\bigcup_{i<\omega} \mathscr{B}^i$ there is an automorphism $g \in \mathrm{aut}(\bigcup_{i<\omega} \mathscr{B}^i)$ such that $g(e_i) = f^{-1}(e_0)$. The map fg witnesses that $(\bigcup_{j<\omega} \mathscr{B}^j, e_i) \cong (\mathscr{B}^0, e_0)$. $\square$

Note that it's very important, in this case, that the particular set we're looking at in Proposition 5.7.15 is not just invariant, but actually first order definable. The proof above uses a kind-of "Löwenheim-Skolem for pairs" argument that uses the definability of the invariant set. In the AEC case for a non-first-order-definable invariant set, one can replace this argument with the same sort of ω-chain argument Lessmann used to prove Lemma 3.4.9, however, this only produces a X-Vaughtian pair and the structures thus constructed might not satisfy Assumption 5.3.3. In Proposition 5.7.19, assuming stronger conditions on the behavior of the class and the X-Vaughtian pair of models in κ we can deduce the existence of sufficiently nice X-Vaughtian pair of countable structures, however we cannot deduce these conditions from the conclusion of Proposition 5.7.11.

It should be clear, at this point, that the method of expanding the language used to prove Proposition 5.7.15 to code properties of embeddings and pairs of structures is problematic to apply in the AEC context. The only means of extending the first order proof to the AEC context that seems at all promising is to apply the Presentation Theorem and then hope that we can choose all structures in the proof of Proposition 5.7.15 to also omit the collection of types Γ given by the Presentation Theorem. This, however, seems likely to be quite difficult.

5.7.3 step 3

Since we have been unable to find an analogue for step 2 of the first order proof of the gap-2 transfer theorem we turn now toward finding replacements for the first two steps of the proof. Indeed, in our proof of Theorem 5.4.2 Assumptions 5.3.3 and 5.3.7 act as a substitute for the conclusion of Proposition 5.7.15, in our own construction of an $(\aleph_2, \aleph_0)$-model. We would, however, like to be able to find reasonable model theoretic assumptions on an AEC $\mathbf{K}$ from which we could *prove* 5.3.3.

In 5.7.1 we noted that the Presentation Theorem seems helpful in dealing with AECs that do not already have a suitable ordering in their signature. However, this alone is insufficient. We must make the further assumption that $\mathfrak{C}$ expands to a sufficiently saturated $\mathscr{L}''$-structure.

Assumption 5.7.18 *We assume that $\mathfrak{C}$ expands to a $\mathfrak{C}''$, an $\aleph_1$-saturated model. Let $\mathbf{K}''$ be the AEC defined by $\mathbf{K}'' := \{\mathscr{M} : \exists \mathscr{N} (\mathscr{M} \cong \mathscr{N} \wedge \mathscr{N} \prec_{\mathscr{L}''} \mathfrak{C}'')\}$.*

Assuming the hypotheses of Proposition 5.7.19 (put imprecisely, that a nice X-Vaughtian pair of models exist of size κ), we are able to satisfy the hypotheses of Proposition 5.5.12 (put imprecisely, a nice X-Vaughtian pair of countable models exist). As such Proposition 5.7.19 is something of a replacement for Proposition 5.7.15 in the AEC case. After proving 5.7.19 we can, in Corollary 5.7.20, deduce the existence of an $(\aleph_2, \aleph_0)$-model.

Earlier, we described the hypotheses of Proposition 5.7.19 as a sort of "2+"; that is to say, the hypotheses of Proposition 5.7.19 are stronger than the conclusion of step 2 in the first order gap-2 transfer theorem. So, what we achieve is to provide analogues of steps 3 and 4 in the AEC case starting from the hypotheses of Proposition 5.7.19. Indeed, we can slightly weaken these hypotheses and discuss this in more depth after proving Corollary 5.7.20.

Note there is some similarity in the ω-chain argument below which replaces the downward Löwenheim-Skolem for pairs argument used in the first order case (Proposition 5.7.15) to the ω-chain argument used by Lessmann to prove Lemma 3.4.9.

Proposition 5.7.19 ($2^\kappa = \kappa^+$) *Assume for some $\Gamma'' \supseteq \Gamma$ and some expansion of $\mathfrak{C}$ to a saturated $\mathscr{L}''$-structure $\mathfrak{C}''$ there is a $\mathrm{PC}\Gamma''$-class $\mathbf{K}''$ which is an AEC (with $\prec_{\mathscr{L}''}$ as strong embedding). Suppose that we have found $\mathscr{B}^0_\kappa, \mathscr{B}^1_\kappa \in \mathbf{K}''_\kappa$ such that there is an $e \in B^0_\kappa$ satisfying:*

(i) $\mathscr{B}^0_\kappa \prec_{\mathscr{L}''} \mathscr{B}^1_\kappa \prec_{\mathscr{L}''} \mathscr{C}''$.

(ii) $\mathscr{B}^0_\kappa \prec_{\mathbf{K}} \mathscr{B}^1_\kappa$

(iii) There exists an $\mathscr{L}''$ elementary embedding $\sigma_\kappa : \mathscr{B}^0_\kappa \to \mathscr{B}^1_\kappa$ and constants $e_i \in B^i_\kappa$ for $i = 0, 1$ where $\sigma(e_0) = e_1$ such that:

(a) $\sigma_\kappa \restriction \mathrm{Pr}^{\mathscr{B}^0_\kappa}(e_0) = \mathrm{Id}_{\mathrm{Pr}^{\mathscr{B}^0_\kappa}}(e_0)$

(b) $\mathscr{B}^0_\kappa \subseteq \mathrm{Pr}^{\mathscr{B}^1_\kappa}(e_1)$

(iv) $\mathscr{B}^0_\kappa \cong \mathscr{B}^1_\kappa$ are $\aleph_0$-galois homogeneous, with respect to $\mathscr{L}''$ (which includes the ordering), and realize the same galois types (of finite tuples of any arity) over the empty set.

(v) $X(\mathscr{B}^0_\kappa) = X(\mathscr{B}^1_\kappa) \subseteq \mathrm{Pr}^{\mathscr{B}^0_\kappa}(e_0)$, where X is invariant with respect to $\mathscr{L}'$-automorphisms.

Then there are countable models $\mathscr{B}^0, \mathscr{B}^1 \prec_{\mathscr{L}''} \mathscr{C}''$, where $e_0 \in B_0$,as in Proposition 5.5.12, that is:

(I) $\mathscr{B}^0 \prec_{\mathscr{L}''} \mathscr{B}^1 \prec_{\mathscr{L}''} \mathscr{C}''$.

(II) $\mathscr{B}^0 \prec_{\mathbf{K}} \mathscr{B}^1$

(III) *There exists an* $\mathscr{L}''$ *elementary embedding* $\sigma : \mathscr{B}^0 \to \mathscr{B}^1$ *such that:*

(a) $\sigma \restriction \mathrm{Pr}^{\mathscr{B}^0}(e_0) = \mathrm{Id}_{\mathrm{Pr}^{\mathscr{B}^0}}(e_0)$

(b) $\mathscr{B}^0 \subseteq \mathrm{Pr}^{\mathscr{B}^1}(e_1)$

(IV) $\mathscr{B}^0 \cong \mathscr{B}^1$ *is galois homogeneous.*

(V) $X(\mathscr{B}^0) = X(\mathscr{B}^1) \subseteq \mathrm{Pr}^{\mathscr{B}^0}(e_0)$.

proof. We construct a sequence of pairs of countable $\mathscr{L}''$-structures $(\mathscr{B}^0_n, \mathscr{B}^1_n)_{n<\omega}$ which satisfy the following properties:

1. For $i = 0, 1$ and any finite tuples $\bar{a}, \bar{b} \in B^i_n$ where $\bar{a} \sim_{\mathfrak{C}} \bar{b}$ and singleton $a \in B^i_n$ there exists $b \in B^i_{n+1}$ such that $\bar{a}a \sim_{\mathfrak{C}} \bar{b}b$.
2. $\mathscr{B}^i_n \prec_{\mathscr{L}''} \mathscr{B}^i_\kappa$, $\mathscr{B}^0_n \prec_{\mathscr{L}''} \mathscr{B}^1_{n+1}$, and $\mathscr{B}^0_n \prec_{\mathscr{L}''} \mathscr{B}^1_n$.
3. $\sigma_\kappa(\mathscr{B}^0_n) \subseteq \mathscr{B}^0_{n+1}$.
4. For any finite tuple $\bar{a} \in B^0_n$ there is a $\bar{b} \in B^1_{n+1}$ such that $\bar{b} \models \mathrm{tp}^{\mathrm{ga}}(\bar{a})$, similarly if for any finite tuple $\bar{b} \in B^1_n$ there is an $\bar{a} \in B^0_{n+1}$ such that $\bar{b} \models \mathrm{tp}^{\mathrm{ga}}(\bar{b})$.
5. $e_0 \in B^0$.

We observe the above construction is possible. $e \in B^0_\kappa$ so by downward Löwenheim-Skolem there exists a countable model $\mathscr{B}^0_0 \prec_{\mathscr{L}''} \mathscr{B}^0_\kappa$ where $e \in B^0_0$. Similarly, since $\sigma_0(e_0) = e_1 \in B^1_\kappa$, there is a countable $\mathscr{B}^1_0 \prec_{\mathscr{L}''} \mathscr{B}^1_\kappa$.

Note that there are only countably many galois types over the empty set realized in $\mathscr{B}^i_n$. Furthermore, these are all galois types realized by both $\mathscr{B}^0_\kappa$ and $\mathscr{B}^1_\kappa$ (since we have assumed $\mathscr{B}^i_n \prec_{\mathscr{L}''} \mathscr{B}^i_\kappa$ and $\mathscr{B}^0_\kappa$ and $\mathscr{B}^1_\kappa$ realize the same galois types over the empty set). So there are countable models ${\mathscr{B}^0_n}'$ and ${\mathscr{B}^1_n}'$ such that for any finite tuple $\overline{a} \in {B^0_n}'$ there is a $\overline{b} \in {B^1_n}'$ such that $\overline{b} \models \mathrm{tp}^{\mathrm{ga}}(\overline{a})$, similarly if for any finite tuple $\overline{b} \in B^1_n$ there is an $\overline{a} \in {B^0_n}'$ such that $\overline{b} \models \mathrm{tp}^{\mathrm{ga}}(\overline{b})$. We may further demand, by another application of downward Löwenheim-Skolem that ${\mathscr{B}^1_n}'$ contains the set $\sigma(B^0_n)$.

Note that there are only countably many finite tuples in B^i_n. Since $\mathscr{B}^i_\kappa$ is countably galois homogeneous, by applying Proposition 5.5.9, we may extend ${\mathscr{B}^i_n}'$ to a countable model $\mathscr{B}^i_{n+1} \prec_{\mathscr{L}''} \mathscr{B}^i_\kappa$ that satisfies condition (1). Namely, for any finite tuples $\overline{a}, \overline{b} \in B^i_n$ where $\overline{a} \sim_{\mathfrak{C}} \overline{b}$ and a singleton $a \in B^i_n$ there exists $b \in B^i_{n+1}$ such that $\overline{a}a \sim_{\mathfrak{C}} \overline{b}b$. We may further demand, by another application of downward Löwenheim-Skolem that $\mathscr{B}^0_n \prec_{\mathscr{L}''} \mathscr{B}^1_n$

Let $\mathscr{B}^i := \bigcup_{i<\omega} \mathscr{B}^i_n$. By Proposition 5.5.8 $\mathscr{B}^i$ is galois homogeneous, by condition (4), $\mathscr{B}^0$ and $\mathscr{B}^1$ realize the same galois types over the empty set. We note $e_0 \in B^0$ and $\sigma_\kappa(e_0) \in B^1$. Let $\sigma := \sigma_\kappa \restriction \mathscr{B}^0$. It follows that $\sigma \restriction \mathrm{Pr}^{\mathscr{B}^0}(e_0)$ is the identity map. We note that, by construction, $\mathscr{B}^0 \prec_{\mathbf{K}} \mathscr{B}^1$ (applying the second clause of condition (3)). It follows then, from the fact that $\mathscr{B}^0_\kappa \subseteq \mathrm{Pr}^{\mathscr{B}^1_\kappa}(e_1)$ that $\mathscr{B}^0 \subseteq \mathrm{Pr}^{\mathscr{B}^1}(e_1)$. Note that $X(\mathscr{B}^0_\kappa) \subseteq \mathrm{Pr}^{\mathscr{B}^0_\kappa}(e)$, so it follows that $X(\mathscr{B}^1) \subseteq \mathrm{Pr}^{\mathscr{B}^0}(e)$. Since $\sigma \restriction \mathrm{Pr}^{\mathscr{B}^0}(e) = \mathrm{Id}_{\mathrm{Pr}^{\mathscr{B}^0}(e)}$, it follows that $X(\mathscr{B}^1) = X(\mathscr{B}^0)$, so condition (V) holds as well. $\square$

The next corollary follows trivially from the theorem above.

Corollary 5.7.20 *Assume for some* $\Gamma'' \supseteq \Gamma$ *and some expansion of* $\mathfrak{C}$ *to a saturated*[1] $\mathscr{L}''$*-structure* $\mathfrak{C}''$ *there is a* $\mathrm{PC}\Gamma''$*-class* $\mathbf{K}''$ *which is an AEC (with* $\prec_{\mathscr{L}''}$ *as strong embedding). Suppose that we have found* $\mathscr{B}^0_\kappa, \mathscr{B}^1_\kappa \in \mathbf{K}_\kappa$ *such that there is an* $e \in B^0_\kappa$ *satisfying:*

(i) $\mathscr{B}^0_\kappa \prec_{\mathscr{L}''} \mathscr{B}^1_\kappa \prec_{\mathscr{L}''} \mathscr{C}''$.

(ii) $\mathscr{B}^0_\kappa \prec_{\mathbf{K}} \mathscr{B}^1_\kappa$

(iii) There exists an $\mathscr{L}''$ *elementary embedding* $\sigma_\kappa : \mathscr{B}^0_\kappa \to \mathscr{B}^1_\kappa$ *and constants* $e_i \in B^i_\kappa$ *for* $i = 0, 1$ *where* $\sigma(e_0) = e_1$ *such that:*

(a) $\sigma_\kappa \restriction \mathrm{Pr}^{\mathscr{B}^0_\kappa}(e_0) = \mathrm{Id}_{\mathrm{Pr}^{\mathscr{B}^0_\kappa}}(e_0)$

(b) $\mathscr{B}^0_\kappa \subseteq \mathrm{Pr}^{\mathscr{B}^1_\kappa}(e_1)$

(iv) $\mathscr{B}^0_\kappa \cong \mathscr{B}^1_\kappa$ *are* $\aleph_0$*-galois homogeneous, with respect to* $\mathscr{L}''$ *(which includes the ordering), and realize the same galois types (of finite tuples of any arity) over the empty set.*

(v) $X(\mathscr{B}^0_\kappa) = X(\mathscr{B}^1_\kappa) \subseteq \mathrm{Pr}^{\mathscr{B}^0_\kappa}(e_0)$*, where* X *is invariant with respect to* $\mathfrak{C}'' \restriction \mathscr{L}'$*.*

then there exists an $(\aleph_2, \aleph_0)$*-model.*

proof. By Proposition 5.7.19 the hypotheses of Proposition 5.5.12 hold, so the conclusion of Proposition 5.5.12 holds. □

We can in fact do slightly better than the $\aleph_3$-saturation required by Assumption 5.4.1, in terms of the saturation required in the expanded language:

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[1] *In actuality,* $\aleph_1$*-saturation should suffice. See 5.4.1.*

Observation 5.7.21 *It suffices for $\mathfrak{C}''$ to be $\aleph_1$-saturated and for AP and JEP to hold only up to $\aleph_1$ in $\mathbf{K}''$. (Though $\mathfrak{C}$ must be at least $\aleph_3$ saturated and $\mathbf{K}$ must admit AP and JEP at least up to $\aleph_3$.*

The key observation is that utilizing $\aleph_1$-AP, JEP, and AP for $\mathbf{K}''$ we can complete every step of the construction which proves Theorem 5.4.2 except that without $\aleph_3$-saturation, it may not be possible to embed $\mathscr{B}^{\omega_1}$ into $\mathfrak{C}''$. $\mathscr{B}^{\omega} \restriction \mathscr{L}$, however, *can* be be embedded into $\mathfrak{C}$ over $\mathscr{B}^0 \restriction \mathscr{L}$ as long as our original monster model is $\aleph_3$-saturated (and the original class satisfies JEP and AP). Let $F : \mathscr{B}^{\omega_1} \to \mathfrak{C}$ over $\mathscr{B}^0 \restriction \mathscr{L}$.

We must argue that $X(\mathscr{B}^{\omega_1}) = X(\mathscr{B}^0)$. Suppose that $x \in X(\mathscr{B}^{\omega_1})$. Then there is some $i < \omega_i$ such that, in the proof of 5.4.2 $y := f^{-1}_{i,\omega_2}(F^{-1}(x)) \in B^i$. By construction $X(\mathscr{B}^i) = X(\mathscr{B}^0)$ so $y \in X(\mathscr{B}^0)$. By c9, $f_{i,\omega_2} \restriction X(\mathscr{B}^0) = \mathrm{Id}_{X(\mathscr{B}^0)}$. F was also chosen to fix $\mathscr{B}^0$, so it must be the case that $x = y \in X(\mathscr{B}^0)$.

5.8 Closing Remarks

Note, however, that there is still somewhat of gap here from having a true "gap-2 transfer" result for AECs; what we have shown is that if we have a *sufficiently nice* pair of models in some infinite cardinal then an $(\aleph_2, \aleph_0)$-model exists. Furthermore, we make the assumption (Assumption 5.7.18) that $\mathfrak{C}$ expands to an $\aleph_1$-saturated $\mathscr{L}''$ structure. This assumption is key, otherwise, as previously discussed in the paragraphs following Observation 5.7.8, the set X may not be invariant in the language $\mathscr{L}''$. Of course, we have observed (Observation 5.7.9) that as long as $\mathbf{K}''$ satisfies JEP and AP, there is no loss of generality in assuming that $\mathfrak{C}$ expands to an $\aleph_1$-saturated $\mathscr{L}''$ structure, so the real obstacle is that we need $\mathbf{K}''$ to satisfy at least some

JEP and AP. As we note in Observation 5.7.21, we can, in fact, get by with an $\aleph_1$-saturated expansion $\mathfrak{C}''$ of $\mathfrak{C}$ and JEP and AP in $\mathbf{K}''$ only for structures of size up to $\aleph_1$. We see no way to guarantee, however, that $\mathbf{K}''$ will satisfy the necessary amount of JEP and AP $\mathbf{K}$ satisfies full AP and JEP.

Another way to look at our work in this chapter, phrased in terms of the four step outline of the first order gap-two theorem is that we've done step 1, and we can do steps 3 and 4 starting with "2+" but there remains a gap between finishing step 1 and completing step 2 (or deriving "2+").

Since $\aleph_1$-saturated structures are clearly $\aleph_0$-galois homogeneous, it seems one might want to move from having an arbitrary (κ^{++}, κ)-model to having an $\aleph_1$-saturated (κ^{++}, κ)-model (assuming there is no conflict between the saturation hypothesis and the existence of an invariant set which is small). Since the existence of a Morass follows from $\mathbf{V} = \mathbf{L}$, it is not unreasonable to assume **GCH**, under which there is a saturated model in every successor cardinal (at, least, if the class $\mathbf{K}$ satisfies AP and JEP, which we must assume regardless). If we can construct such an $\aleph_1$-saturated (κ^{++}, κ)-model, it seems plausible that we could also construct a pair of $\aleph_0$-galois homogeneous structures which satisfy the hypotheses of Proposition 5.7.19 by modifying the proof Proposition 5.7.11 to consider only $\aleph_0$-galois homogeneous strong structures of $\mathscr{C}$.

One cannot, however, easily prove the existence of an $\aleph_1$-saturated X-Vaughtian pair merely from the existence of some arbitrary (κ^{++}, κ)-gap, $\mathscr{C}$.[1] If attempting to construct an $\aleph_1$-

[1]This is the case even if the class $\mathbf{K}$ is stable.

saturated model $\mathscr{C}_{\text{sat}}$ from $\mathscr{C}$ one must realize a type over every countable subset of C; this is κ^{++} many types, so we may also add κ^{++} realizations to $\mathscr{C}_{\text{sat}}$ of the p for which $|p(\mathscr{C})| = \kappa$. This still raises the question over whether it is even possible for the larger structure in the (κ^{++}, κ)-model to be $\aleph_1$-galois saturated, in particular:

Question 5.8.1 *Does $\aleph_1$-galois saturation of a (κ^{++}, κ)-model $\mathscr{M}$ in a language $\mathscr{L}''$ containing a symbol "$<$" interpreted as linear ordering of $\mathscr{M}$ force $(M, <)$ to be non-well-founded?*

CHAPTER 6

CONCLUSION

We review the general outline of the work done in this dissertation:

(I) In the introduction and Chapter 2 we introduce various conditions which could potentially define "superstability" for abstract elementary classes. In Chapter 2 we also give other basic definitions and results that we make use of the in the later chapters.

(II) In Chapter 3 we examine Lessmann's analogue of Vaught's Theorem for abstract elementary classes. We provide a sufficient condition for the construction of an $(\mathrm{LS}(\mathbf{K})^+, \mathrm{LS}(\mathbf{K}))$-model.[1] (a (p_0, λ)-superlimit class). In Section 3.4 we deduce Lessmann's Theorem as a special case of our "abstract" theorem.

(III) In Chapter 4 we discuss progress that has been made in proving uniqueness of limit models from various "superstability" assumptions. We contribute one small result in Proposition 4.1.5, eliminating the hypothesis of "disjoint amalgamation over limit models" appearing in previous proofs.[2]

(IV) In Chapter 5 we give a sufficient condition, or perhaps more accurately, various sufficient conditions for the existence of an $(\aleph_2, \aleph_0)$-model to exist, using the existence of a sim-

[1] In Theorem 3.3.1 we perform this construction.

[2] This theorem is proved assuming weak disjoint amalgamation in (VanDieren, 2006).

plified morass. Our work here is guided by the presentation of Jensen's work in proving the classical gap-2 transfer result for first order logic in (Devlin, 1984).

We feel our work in IV. offers the most significant new insight into how abstract elementary classes behave. One point which has become apparent in the course of researching Chapter 5 is where key differences lie between the elementary and non-elementary cases. In particular we have isolated at least two parts of the theorem which are problematic to transfer to the non-elementary context, namely:

(1) Beginning with an unordered class and adding an ordering

(2) Moving from a Vaughtian pair in κ to a sufficiently nice pair of countable models.

While we have made very small progress in dealing with the first of these issues, we feel we've made some significant progress in dealing with the second. In particular, in Proposition 5.7.19 we are able to give a sufficient condition on models in κ that allows us to deduce that a pair of countable models suitable for building an $(\aleph_2, \aleph_0)$-model exist.

Both Vaught's Theorem, going from Vaughtian pair to $(\aleph_1, \aleph_0)$-model, and Jensen's theorem there is a common high-level sketch of the argument:

(i) Start with an arbitrary two cardinal model or pair.

(ii) Push down to a countable pair of models.

(iii) Push back up to a two cardinal model.

What have been able to do here is offer two results which complete the step iii. of this outline. In both cases, it seems much harder to complete step ii., at the very least; we've made only partial progress in doing so in Chapter 5, namely in Proposition 5.7.19 and Proposition 5.7.11. In Chapter 3 the only time we are able to go from "Vaughtian pair" to "countable pair of models" is in the context where Lessmann already directly proved the result without the use of our framework of (p_0, λ)-superlimit classes.

It is a natural question to ask, under what conditions could we complete steps i.-iii. and construct either an $(\mathrm{LS}(\mathbf{K}^+), \mathrm{LS}(\mathbf{K}))$ or $(\aleph_2, \aleph_0)$-model starting above the Löwenheim Number for a class. It is also a fair question to ask what non-elementary ordered AECs satisfy Assumptions 5.3.3 and 5.3.7, since we have not yet provided interesting examples.

In Chapter 5, as a tool for our construction, we introduced the notions of "'1-transitive" and "galois homogeneous" structures. We think it might be an interesting question to investigate some further properties of these structures and whether or not they might be useful for generalizing other theorems of first order logic whose proofs involve the use of countably homogeneous models.

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VITA

Degrees:

- **PhD in Mathematics**

  University of Illinois at Chicago, May 2013.

- **MS in Mathematics**

  University of Illinois at Chicago, August 2008.

- **BS in Mathematics**

  University of Illinois at Chicago, May 2005. Magna Cum Laude with College Honors and High Distinction in Mathematics.

A more detailed and current vita should be available on request from the author.