MATH 220 Final Exam Fall 2011

1	2	3	4	5	6	7	8	9	10	11	12
(20)	(20)	(20)	(15)	(15)	(15)	(15)	(15)	(15)	(15)	(15)	(20)

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Name (print) ____________________ **Signature** ____________________

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Some potentially useful formulas:

$$\int x\sin(Ax)dx = -\frac{x}{A}\cos(Ax)+\frac{1}{A^2}\sin(Ax), \qquad \int x\cos(Ax)dx = \frac{x}{A}\sin(Ax)+\frac{1}{A^2}\cos(Ax)$$

$$y_2(t) = y_1(t)\int^t [y_1(u)]^{-2}\exp\left(-\int^u p(v)dv\right)du$$

$$y_p(t) = y_1(t)\int^t \frac{-g(u)y_2(u)}{W[y_1,y_2](u)}du + y_2(t)\int^t \frac{g(u)y_1(u)}{W[y_1,y_2](u)}du, \quad W[y_1,y_2](t) = y_1(y)y_2'(t) - y_1'(t)y_2(t)$$

$$\mathcal{L}\{1\} = \frac{1}{s}, \quad \mathcal{L}\left\{e^{at}\right\} = \frac{1}{s-a}, \quad \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad \mathcal{L}\{\sin(bt)\} = \frac{b}{s^2+b^2}, \quad \mathcal{L}\{\cos(bt)\} = \frac{s}{s^2+b^2},$$

$$\mathcal{L}\left\{e^{at}t^n\right\} = \frac{n!}{(s-a)^{n+1}}, \quad \mathcal{L}\left\{e^{at}f(t)\right\} = \mathcal{L}\{f\}(s-a) = F(s-a),$$

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0), \quad \mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0),$$

$$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s), \quad \mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}, \quad \mathcal{L}\{\delta(t-a)\} = e^{-as}$$

1. (8+12) Consider the equation

$$y'' + y = t^2 + e^t.$$

(a) Find the general solution to the homogeneous equation (with right hand side replaced by 0).

(b) Find a particular solution to the inhomogeneous equation.

2. (10+10) Solve

(a) $\displaystyle \frac{dy}{dx} = \frac{y}{x} + xe^x,$ (b) $\displaystyle \frac{dy}{dx} = \frac{\sin x - ye^{xy}}{xe^{xy} + 1}.$

3. (20) Use the Euler method with *two* steps to approximate $y(1)$, where

$$y' = (1 + x)y + x^2, \qquad y(0) = 1.$$

4. (3+12) Let $p(t)$ be the population of some species, which satisfies the ODE

$$p' = 3p - p^2.$$

(a) If $p(0) = 0$ what is $p(t)$ for $t > 0$?

(b) Solve for $p(t)$ for the initial condition $p(0) = 1$, and then evaluate the limit of $p(t)$ as $t \to \infty$.

5. (8+7) Solve

(a) $t^2y'' - ty' + y = 0,$ (b) $t^2y'' + ty' + 4y = 0.$

6. (3+12) Consider the ODE

$$y'' + 2ty' - 4y = 0.$$

(a) Is $y_1(t) = 2t^2 + 1$ a solution ?

(b) Find the general solution. (Hint: Consider the method of reduction of order. You may leave your answer as an integral.)

7. (15) Find the general solution to

$$y'' - 2y' + y = \frac{2e^t}{t^3}.$$

8. (15) Find the general solution to the system

$$x' = 2x + y, \qquad y' = x + 2y.$$

9. (15) Use Laplace transforms to solve

$$y'' + y' = e^t, \qquad y(0) = 1, \quad y'(0) = 0.$$

To receive any credit, you **must** use Laplace transforms.

10. (15) Use Laplace transforms to solve

$$y'' + 4y = \delta(t-1), \qquad y(0) = 0, \quad y'(0) = 1.$$

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11. (15) Solve the heat equation

$$u_t = \beta u_{xx}\ ; \qquad 0 < x < 1, \;\; t > 0$$

$$u(0,t) = 0, \qquad u(1,t) = 0$$

$$u(x,0) = 2\sin(\pi x) + \sin(4\pi x).$$

Start by writing the general solution that satisfies the PDE and boundary conditions, and then find the solution that also satisfies this initial condition.

12. (3+2+3+12) Let $f(x) = 2x$ for $0 < x < 1$.

(a) Sketch the *even* extension of $f(x)$ over the range $-1 < x < 1$.

(b) Does the Fourier series of (the extended) $f(x)$ involve sines, cosines or both ?

(c) What is the average value of (the extended) $f(x)$ over a period ?

(d) Compute explicitly the Fourier series of (the extended) $f(x)$.

(1.)

a.) $y'' + y = t^2 + e^t$

$r^2 + 1 = 0$

$r = \pm i$

$y_h(t) = c_1 \cos t + c_2 \sin t$

b.)

$y_{p_1}(t) = At^2 + Bt + C$

$y_{p_1}'(t) = 2At + B$

$y_{p_1}''(t) = 2A$

$2A + At^2 + Bt + C = t^2 \Rightarrow A = 1,\ B = 0,\ C = -2$

$y_{p_1}(t) = t^2 - 2$

$y_{p_2}(t) = De^t,\ y_{p_2}'(t) = De^t,\ y_{p_2}''(t) = De^t$

$De^t + De^t = e^t \Rightarrow D = \frac{1}{2}$

$y_{p_2}(t) = \frac{1}{2} e^t$

$$\boxed{y(t) = c_1 \cos t + c_2 \sin t + t^2 - 2 + \frac{1}{2} e^t}$$

2. a.) linear

$\frac{dy}{dx} - \frac{1}{x} y = x e^x$

$\mu(x) = \exp(\int -\frac{1}{x} dx)$

$= \exp(-\ln x)$

$= x^{-1}$

$\int Q(x)\mu(x)dx = \int e^x dx = e^x + c$

$y(x) = x e^x + cx$

b.) $(\sin x - y e^{xy}) dx + (-1 - x e^{xy}) dy = 0$

$\overset{\shortparallel}{M}$ $\overset{\shortparallel}{N}$

$\frac{\partial M}{\partial y} = -e^{xy} - xy e^{xy} = \frac{\partial N}{\partial x}$ ODE is exact, hence t.e. F s.t. $M = \frac{\partial F}{\partial x}$, $N = \frac{\partial F}{\partial y}$

$F(x,y) = \int M(x,y) dx + g(y)$

$= \int \sin x - y e^{xy} dx + g(y)$

$= -\cos x - e^{xy} + g(y)$

$-1 - x e^{xy} = N = \frac{\partial F}{\partial y} = -x e^{xy}\ g'(y)$

$g'(y) = -1$

$g(y) = -y + c$

$\boxed{F(x,y) = -\cos x - e^{xy} - y + c}$

3.

$n = 2,\ h = \frac{1-0}{2} = 0.5$

$y' = (1+x)y + x^2,\ y(0) = 1$

x	y	$f(x,y)$
0	1	1
0.5	$1 + 0.5 \cdot 1 = 1.5$	$1.5 \cdot 1.5 + 0.5^2 = 2.5$
1	$1.5 + 0.5 \cdot 2.5 = 2.75$	

$\boxed{y(1) \approx 2.75}$

4. a.) $p' = 3p - p^2$

$$\frac{dp}{3p - p^2} = dt$$

$$\frac{A}{p} + \frac{B}{p-3} = -\frac{1}{p(p-3)}$$

$$A(p-3) + Bp = -1$$

$$Ap - 3A + Bp = -1$$

$$A = \frac{1}{3}, \quad B = -\frac{1}{3}$$

$$\left(\frac{1}{3}\frac{1}{p} - \frac{1}{3}\frac{1}{p-3}\right) dp = dt$$

$$\frac{1}{3}\ln p - \frac{1}{3}\ln(p-3) = t + c$$

$$\ln\left(\frac{p}{p-3}\right) = 3t + c$$

$$\frac{p}{p-3} = K \cdot e^{3t}$$

$$p = Ke^{3t}p - 3Ke^{3t}$$

$$p(1 - Ke^{3t}) = -3Ke^{3t}$$

$$p = \frac{3Ke^{3t}}{Ke^{3t} - 1}$$

$$0 = p(0) = \frac{3K}{K-1} \Rightarrow K = 0$$

$$\boxed{p(t) = 0}$$

b.)

$$1 = p(0) = \frac{3K}{K-1}$$

$$3K - K + 1 = 0$$

$$2K = -1$$

$$K = -\frac{1}{2}$$

$$\boxed{p(t) = \frac{3e^{3t}}{e^{3t} + 2}}$$

$$p(t) = \frac{3}{1 + 2e^{-3t}}$$

$$\lim_{t \to \infty} p(t) = 3$$

5.

a.) $r^2 - 2r + 1 = 0$

$(r-1)^2 = 0$

$r = 1$

$y(t) = c_1 t + c_2 t \ln t$

b.) $r^2 + 4 = 0$

$r = \pm 2i$

$y(t) = c_1 \cos(2 \ln t) + c_2 \sin(2 \ln t)$

6.

a.) $y_1(t) = 2t^2+1$

$y_1'(t) = 4t$

$y_1''(t) = 4$

$4 + 8t^2 - 8t^2 - 4 = 0$, hence y_1 is a solution

b.)

$y_2(t) = v(t)\, y_1(t)$

$y_2' = v'y_1 + vy_1'$

$y_2'' = y_1''v + 2v'y_1' + v''y_1$

$$y_2'' + 2ty' - 4y = y_1''v + 2v'y_1' + v''y_1 + 2tv'y_1 + 2tvy_1' - 4vy_1$$

$$= y_1 v'' + (2y_1' + 2ty_1)v' + \underbrace{(y_1'' + 2t\cancel{y_1'} - 4y_1)}_{0}v$$

$-y_1 v'' = (2y_1' + 2ty_1)v'$

$$w' + 2t\, w = -2\frac{y_1'}{y_1}$$

$$w' + 2tw = -\frac{8t}{2t^2+1}$$

$\mu(t) = \exp\left(\int 2t\,dt\right) = \exp(t^2)$

$$y_2(t) = \frac{\int e^{t^2}\left(-\frac{8t}{t^2+1}\right)dt}{e^{t^2}}$$

$$\boxed{y(t) = c_1 y_1(t) + c_2 y_2(t)}$$

(7) variation of parameters

$$r^2 - 2r + 1 = 0$$
$$(r-1)^2 = 0$$
$$y(t) = c_1 e^t + c_2 t e^t$$

$$y_p(t) = v_1 y_1 + v_2 y_2$$

$$\begin{cases} y_1 v_1' + y_2 v_2' = 0 \\ y_1' v_1' + y_2' v_2' = 2e^t t^{-3} \end{cases}$$

$$\begin{cases} e^t v_1' + t e^t v_2' = 0 \\ e^t v_1' + (t+1) e^t v_2' = 2 e^t t^{-3} \end{cases}$$

$$\begin{cases} \cancel{e^t} v_2' = 2 \cancel{e^t} t^{-3} \\ v_1' = -t v_2' \end{cases}$$

$$v_2 = \int 2t^{-3}\,dt = -\frac{1}{t^2}$$
$$v_1 = -2\int t^{-2}\,dt = \frac{2}{t}$$

$$y_p(t) = \frac{2}{t}e^t - \frac{1}{t}e^t = t^{-1}e^t$$

$$\boxed{y(t) = c_1 e^t + c_2 t e^t + \frac{e^t}{t}}$$

8. $x' = 2x + y, \quad y' = x + 2y$

$$\begin{cases} (D-2)[x] - y = 0 \\ -x + (D-2)[y] = 0 \end{cases} \qquad \begin{cases} -(D^2-4D+4)[x] - (D-2)[y] = 0 \\ -x + (D-2)[y] = 0 \end{cases}$$

$$(D^2 - 4D + 3)[x] = 0$$

$$\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 3x = 0$$

$$r^2 - 4r + 3 = 0$$

$$(r-3)(r-1) = 0$$

$$x(t) = c_1 e^{3t} + c_2 e^t$$

$$y(t) = (D-2)[x]$$

$$= 3c_1 e^{3t} + c_2 e^t - 2c_1 e^{3t} - 2c_2 e^t$$

$$= c_1 e^{3t} - c_2 e^t$$

(9.) $y'' + y' = e^t, \quad y(0) = 1, \quad y'(0) = 0$

$$s^2 Y - s + sY - 1 = \frac{1}{s-1}$$

$$Y(s(s+1)) = s + 1 + \frac{1}{s-1}$$

$$Y = \frac{s+1}{s(s+1)} + \frac{1}{(s-1)s(s+1)}$$

$$= \frac{s^2 - 1 + 1}{(s-1)s(s+1)} = \frac{A}{s-1} + \frac{B}{s} + \frac{C}{s+1}$$

$$s^2 = As(s+1) + B(s-1)(s+1) + Cs(s-1)$$

$\underline{s=1}$ $\quad 1 = 2A$ $\qquad A = \frac{1}{2}, \; B = 0, \; C = \frac{1}{2}$

$\underline{s=0}$ $\quad 0 = -B$

$\underline{s=-1}$ $\quad 1 = 2C$

$$Y(s) = \frac{1}{2}\frac{1}{s-1} + \frac{1}{2}\frac{1}{s+1}$$

$$\boxed{y(t) = \frac{1}{2}e^t + \frac{1}{2}e^{-t}}$$

10. $y'' + 4y = \delta(t-1)$, $y(0) = 0$, $y'(0) = 1$

$$s^2 Y - 1 + 4Y = e^{-s}$$

$$Y(s^2 + 2^2) = 1 + e^{-s}$$

$$Y = \frac{1}{s^2+2^2} + e^{-s}\frac{1}{s^2+2^2}$$

$$y(t) = \frac{1}{2}\sin 2t + \left[\frac{1}{2}\sin 2t\right]_{t-1}$$

$$= \frac{1}{2}\sin 2t + \frac{1}{2}\sin(2t-2)$$

(11.)

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-(\beta n\pi/L)^2 t} \sin\left(\frac{n\pi x}{L}\right)$$

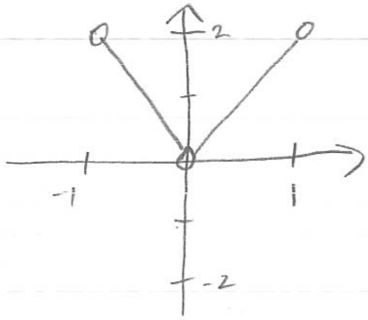
$$2\sin(\pi x) + \sin(4\pi x) = u(x,0) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$$

$b_1 = 2,\ b_4 = 1,\ b_n = 0$ for other n

$$\boxed{u(x,t) = 2e^{-(\beta\pi)^2 t}\sin(\pi x) + e^{-(4\beta\pi)^2 t}\sin(4\pi x)}$$

(12.)

a.)



b.) cosines only since it is even

c.)

$$\frac{1}{2}\left(\int_{-1}^{0} -2x\,dx + \int_{0}^{1} 2x\,dx\right)$$

$$= -\frac{x^2}{2}\Big|_{-1}^{0} + \frac{x^2}{2}\Big|_{0}^{1} = \frac{1}{2} + \frac{1}{2} = 1$$

d.)

$$a_n = 2\int_0^1 2x\cos(n\pi x) = 2\left(\frac{2x}{\pi n}\sin(n\pi x) + \frac{2}{\pi^2 n^2}\cos(n\pi x)\Big|_0^1\right)$$

$$= \frac{4}{\pi^2 n^2}\left((-1)^n - 1\right)$$

$$f(x) = 1 + \sum_{n=1}^{\infty} \frac{4}{\pi^2 n^2}\left((-1)^n - 1\right)\cos(n\pi x)$$