MATHEMATICS 220: FINAL EXAM
University of Illinois at Chicago
(Abramov, Awanou, Nicholls)
December 11, 2014

Please read the exam carefully and follow all instructions. **SHOW ALL OF YOUR WORK.** Please put a box around your final answer.

1. (20 points) Solve the initial value problem
$$\frac{dy}{dx} = x^2(1+y), \qquad y(0) = 1.$$

2. (20 points) Solve the initial value problem
$$(2xy+3)\,dx + (x^2-1)\,dy = 0, \qquad y(0) = 0.$$

3. (25 points) A large 100L tank is initially filled with fresh water. At time $t = 0$, a brine solution begins to enter the tank at the rate of 5 L/min with concentration of 0.2 kg/L. The well-stirred solution is removed from the tank at the same rate of 5 L/min. Denote the amount (mass in kg) of salt in the tank as x, and then find the formula for $x(t)$ for $t \geq 0$.

4. (20 points) Find the general solution to the equation
$$y'' + 2y' + 2y = e^{-t}\cos(2t).$$
Use the **Method of Undetermined Coefficients** to find a particular solution for the non–homogeneous equation. **(Any other method will receive no credit.)**

5. (25 points) Solve the initial value problem using differential operators
$$\begin{aligned} x' &= 4x + y, && x(0) = 1 \\ y' &= -2x + y, && y(0) = 0. \end{aligned}$$

6. (20 points) Compute the inverse Laplace transform of

$$F(s) = \frac{s}{s^2 - 4s + 5}.$$

7. (20 points) Using the **Method of Laplace transforms** solve the initial value problem

$$y''(t) + 14y'(t) + 58y(t) = \delta(t - 8), \quad y(0) = 0, \quad y'(0) = 0.$$

(Any other method will receive no credit.)

8. (25 points) Find the values of λ for which the given problem has a nontrivial solution. Also determine the corresponding nontrivial solutions.

$$y'' + \lambda y = 0, \quad 0 < x < 1, \quad y'(0) = 0, \quad y(1) = 0.$$

9. (25 points) Consider the function

$$f(x) = 2, \quad 0 < x < \pi.$$

   (a) (19 points) Compute the **Fourier sine series** of this function.

   (b) (2 points) To what value does this series converge at $x = 0$? Why?

   (c) (2 points) To what value does this series converge at $x = \pi/2$? Why?

   (d) (2 points) To what value does this series converge at $x = \pi$? Why?

List of Laplace Transforms

1. $\mathcal{L}\{1\} = \dfrac{1}{s}, \quad s > 0$

2. $\mathcal{L}\{e^{at}\} = \dfrac{1}{s-a}, \quad s > a$

3. $\mathcal{L}\{t^n\} = \dfrac{n!}{s^{n+1}}, \quad s > 0$

4. $\mathcal{L}\{\sin(bt)\} = \dfrac{b}{s^2+b^2}, \quad s > 0$

5. $\mathcal{L}\{\cos(bt)\} = \dfrac{s}{s^2+b^2}, \quad s > 0$

6. $\mathcal{L}\{e^{at}t^n\} = \dfrac{n!}{(s-a)^{n+1}}, \quad s > a$

7. $\mathcal{L}\{e^{at}\sin(bt)\} = \dfrac{b}{(s-a)^2+b^2}, \quad s > a$

8. $\mathcal{L}\{e^{at}\cos(bt)\} = \dfrac{s-a}{(s-a)^2+b^2}, \quad s > a$

9. $\mathcal{L}\{e^{at}f(t)\}(s) = \mathcal{L}\{f\}(s-a)$

10. $\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0)$

11. $\mathcal{L}\{f''\}(s) = s^2\mathcal{L}\{f\}(s) - sf(0) - f'(0)$

12. $\mathcal{L}\left\{f^{(n)}\right\}(s) = s^n\mathcal{L}\{f\}(s) - s^{n-1}f(0) - \ldots - f^{(n-1)}(0)$

13. $\mathcal{L}\{t^n f(t)\}(s) = (-1)^n \dfrac{d^n}{ds^n}\mathcal{L}\{f\}(s)$

14. $\mathcal{L}\{f(t-a)u(t-a)\}(s) = e^{-as}F(s)$

15. $\mathcal{L}\{u(t-a)\}(s) = \dfrac{e^{-as}}{s}$

16. $\mathcal{L}\{g(t)u(t-a)\}(s) = e^{-as}\mathcal{L}\{g(t+a)\}(s)$

17. If f has period T then

$$\mathcal{L}\{f\}(s) = \frac{F_T(s)}{1-e^{-sT}} = \frac{\int_0^T e^{-st}f(t)\,dt}{1-e^{-sT}}$$

18. $\mathcal{L}\{\delta(t-a)\}(s) = e^{-as}$

List of PDE Formulae

1. The solution of the homogeneous heat equation $u_t = \beta^2 u_{xx}$ with Dirichlet boundary conditions is:

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-(\beta n\pi/L)^2 t} \sin\left(\frac{n\pi}{L}x\right).$$

2. The solution of the homogeneous heat equation $u_t = \beta^2 u_{xx}$ with Neumann boundary conditions is:

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-(\beta n\pi/L)^2 t} \cos\left(\frac{n\pi}{L}x\right).$$

3. The inhomogeneous heat equation has a solution of the form $u(x,t) = v(x) + w(x,t)$, where v is the steady–state solution and w solves a homogeneous heat equation.

4. The solution of the homogeneous wave equation $u_{tt} = \alpha^2 u_{xx}$ with Dirichlet boundary conditions is:

$$u(x,t) = \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\alpha\frac{n\pi}{L}t\right) + b_n \sin\left(\alpha\frac{n\pi}{L}t\right)\right\} \sin\left(\frac{n\pi}{L}x\right).$$

① separable

$$\int \frac{dy}{1+y} = \int x^2\,dx, \quad y(0) = 1$$

$$\ln(1+y) = \frac{x^3}{3} + c$$

$$\ln 2 = c$$

$$\ln(1+y) = \frac{x^3}{3} + \ln 2$$

$$1+y = 2e^{x^3/3}$$

$$\boxed{y = 2e^{x^3/3} - 1}$$

2.) $(2xy+3)\,dx+(x^2-1)\,dy=0, \quad y(0)=0$

$$\underbrace{2xy+3}_{M} \qquad \underbrace{x^2-1}_{N}$$

$\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}$ hence ODE is <u>exact</u>

and $M=\frac{\partial F}{\partial x}$, $N=\frac{\partial F}{\partial y}$

$$F(x,y) = \int M\,dx + g(y)$$
$$= \int 2xy+3\,dx + g(y)$$
$$= x^2y+3x+g(y)$$

$x^2-1 = \frac{\partial F}{\partial y} = x^2+g'(y) \implies g'(y)=-1$

$g(y)=-y+c$

$$F(x,y)=x^2y+3x-y+C$$

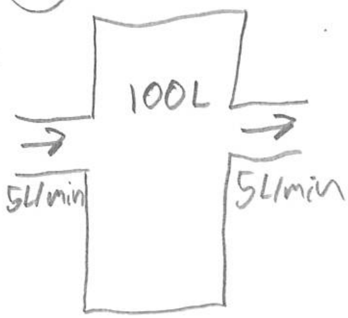
$F(0,0)=0 \implies C=0$

$$\boxed{F(x,y)=x^2y+3x-y}$$

we can write this as

$$x^2y+3x-y=0$$
$$y(x^2-1)=-3x$$
$$y=\frac{3x}{1-x^2}$$

3.



$$\frac{dx}{dt} = \text{input} - \text{output}$$
$$= 6 \cdot 0.2 - 6 \cdot \frac{x(t)}{100}$$

$$\frac{dx}{dt} + \frac{3}{50} x = 1.2$$

$$\mu(t) = \exp\left(\int +\tfrac{3}{50}\, dt\right) = e^{+0.06t}$$

$$\int Q(t)\, \mu(t)\, dt = 1.2 \int e^{+0.06t}\, dt = +20\, e^{+0.06t} + C$$

$$x(t) = C e^{0.06t} + 20$$

$$x(0) = 0 \implies C = -20$$

$$\boxed{x(t) = -20\, e^{0.06t} + 20}$$

4. $y'' + 2y' + 2y = e^{-t}\cos(2t)$

$r^2 + 2r + 2 = 0$

$r = \dfrac{-2 \pm \sqrt{4-8}}{2} = \dfrac{-2 \pm 2i}{2} = -1 \pm i$

$y_h(t) = c_1 e^{-t}\cos t + c_2 e^{-t}\sin t$

$y_p(t) = A e^{-t}\cos(2t)$

$y_p'(t) = -Ae^{-t}\cos(2t) - 2Ae^{-t}\sin(2t)$

$y_p''(t) = Ae^{-t}\cos(2t) + 2Ae^{-t}\sin(2t) + 2Ae^{-t}\sin(2t) - 4Ae^{-t}\cos(2t)$

$= 4Ae^{-t}\sin(2t) - 3Ae^{-t}\cos(2t)$

$e^{-t}\cos(2t) = -5Ae^{-t}\cos(2t) + 2Ae^{-t}\cos(2t) = -3Ae^{-t}\cos(2t)$

$A = -\frac{1}{3}$

$\boxed{y(t) = c_1 e^{-t}\cos t + c_2 e^{-t}\sin t - \frac{1}{3}e^{-t}\cos(2t)}$

5. $x(0)=1,\ y(0)=0$

$$\begin{cases} x' = 4x + y \\ y' = -2x + y \end{cases} \qquad \begin{cases} \frac{dx}{dt} + 4x - y = 0 \\ 2x + \frac{dy}{dt} - y = 0 \end{cases} \qquad \begin{cases} (D+4)[x] - y = 0 \\ 2x + (D-1)[y] = 0 \end{cases}$$

$$\begin{cases} (D+4)(D+1)[x] - (D-1)[y] = 0 \\ 2x + (D-1)[y] = 0 \end{cases}$$

$$(D^2 + 5D + 6)[x] = 0$$

$$x'' + 5x' + 6x = 0$$

$$(r+3)(r+2) = 0$$

$$r = -3, -2$$

$$x(t) = c_1 e^{-3t} + c_2 e^{-2t}$$

$$\begin{aligned} y &= (D+4)[x] \\ &= -3c_1 e^{-3t} - 2c_2 e^{-2t} \\ &\quad + 4c_1 e^{-3t} + 4c_2 e^{-2t} \\ &= c_1 e^{-3t} + 2c_2 e^{-2t} \end{aligned}$$

$$\begin{aligned} 1 &= x(0) = c_1 + c_2 \\ 0 &= y(0) = c_1 + 2c_2 \end{aligned} \quad \Rightarrow \quad \begin{aligned} c_2 &= -1 \\ c_1 &= 2 \end{aligned}$$

$$\boxed{\begin{aligned} x(t) &= 2e^{-3t} - e^{-2t} \\ y(t) &= 2e^{-3t} - 2e^{-2t} \end{aligned}}$$

6.

$$F(s) = \frac{s}{s^2-4s+5} = \frac{s}{(s-2)^2+1}$$

$$= \frac{s-2}{(s-2)^2+1} + 2\,\frac{1}{(s-2)^2+1}$$

$$\boxed{f(t) = e^{2t}\cos t + 2e^{2t}\sin t}$$

7.

$y''(t) + 14y'(t) + 58y(t) = \delta(t-8),\ y(0)=0,\ y'(0)=0$

$s^2 Y + 14sY + 58Y = e^{-8s}$

$Y = e^{-8s}\frac{1}{3}\frac{3}{(s+7)^2+9}$

$y(t) = u(t-8)\left[\frac{1}{3}e^{-7t}\sin 3t\right]_{t-8}$

$= \boxed{u(t-8)\,\frac{1}{3}e^{-7(t-8)}\sin(3(t-8))}$

8. $y'' + \lambda y = 0, \quad 0 < x < 1, \quad y'(0) = 0, \quad y(1) = 0$

$r^2 = -\lambda$

$r = \pm\sqrt{-\lambda}$

$\underline{\lambda = 0}$

$y(x) = c_1 + c_2 x$

$y'(x) = c_2 \quad 0 = y'(0) = c_2$

$0 = y(1) = c_1 \quad y(x) = 0$

no nontrivial solution

$\underline{\lambda < 0}$

$y(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}$

$y'(x) = c_1\sqrt{-\lambda}e^{\sqrt{-\lambda}x} - c_2\sqrt{-\lambda}e^{-\sqrt{-\lambda}x}$

$0 = y'(0) = c_1\sqrt{-\lambda} - c_2\sqrt{-\lambda} \Rightarrow c_1 = c_2$

$0 = y(1) = c_1 e^{\sqrt{-\lambda}} + c_1 e^{-\sqrt{-\lambda}} \Rightarrow c_1 = 0$

$y(x) = 0$

no nontrivial solution

$\underline{\lambda > 0}$

$y(x) = c_1\cos(\sqrt{\lambda}x) + c_2\sin(\sqrt{\lambda}x)$

$y'(x) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}x) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}x)$

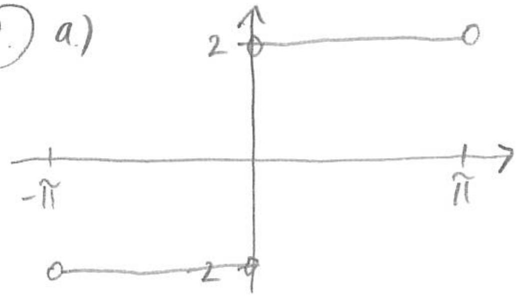
$0 = y'(0) = c_2\sqrt{\lambda} \Rightarrow c_2 = 0$

$0 = y(1) = c_1\cos(\sqrt{\lambda}) \Rightarrow \sqrt{\lambda} = \frac{2n+1}{2}\pi$

$\lambda = (2n+1)^2\frac{\pi^2}{4}$

$$\boxed{y(x) = c_1\cos\left(\frac{(2n+1)\pi x}{2}\right)}$$

(9.) a)



$$b_n = \frac{2}{\pi}\int_0^{\pi} 2\sin(nx)\,dx$$
$$= -\frac{4}{\pi}\frac{1}{n}\cos nx\Big|_{x=0}^{\pi}$$
$$= \left((-1)^{n+1}+1\right)\frac{4}{\pi n}$$

$$f(x) = \frac{8}{\pi}\sin(x) + \frac{8}{3\pi}\sin(3x) + \frac{8}{5\pi}\sin(5x) + \dots$$

b.)
$f(0) = 0$ since $\sin(nx) = 0$ when $x = 0$

c.)
$f(\pi/2) = 2$ (can just plug in original function since $0 < \frac{\pi}{2} < \pi$)

d.)
$f(\pi) = 0$ since $\sin(n\pi) = 0$