Math 220 Final – December 10, 2015

Name: ______________________________ **UIN:** ____________________

Instructor Name: ________________________ **Lecture Time:** ____________

Instructions:

- During the exam, you may **not** use your books, notes, reference materials, or **any electronic devices**, including calculators and cell phones. Violating this rule will result in expulsion from the exam and a score of zero (0)!
- **No form of reproduction or provision of this exam**, or any part thereof, including, but not limited to, copying for personal use, sharing with current or prospective students, or posting on the Internet in open access or restricted selective spaces **is permitted without the written permission of the course coordinator**, Prof. Alexey Cheskidov, Math 220, Fall 2015.
- You are required to show your work on each problem on this exam. **Mysterious or unsupported answers will not receive full credit.** A correct answer, unsupported by calculations, explanations, and/or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

Scores:

1. __________/15 points

2. __________/10 points

3. __________/10 points

4. __________/15 points

5. __________/15 points

6. __________/10 points

7. __________/15 points

8. __________/15 points

9. __________/15 points

Total: __________/120 points

$$\mathcal{L}\{e^{at}t^n\}(s) = \frac{n!}{(s-a)^{n+1}}\,, \qquad \mathcal{L}\{e^{at}\sin(bt)\}(s) = \frac{b}{(s-a)^2+b^2}\,, \qquad \mathcal{L}\{e^{at}\cos(bt)\}(s) = \frac{s-a}{(s-a)^2+b^2}\,,$$

$$\mathcal{L}\{f'(t)\}(s) = s\mathcal{L}\{f(t)\}(s) - f(0)\,, \qquad \mathcal{L}\{f(t-a)u(t-a)\}(s) = e^{-as}\mathcal{L}\{f(t)\}(s)\,, \qquad \mathcal{L}\{\delta(t-a)\}(s) = e^{-as}\,.$$

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1. **(15 points).** Solve the initial value problem Laplace transform

$$y'' + 2y' + y = \sin(t) - \delta(t-1), \qquad y(0) = 0, \quad y'(0) = 1.$$

$$\mathcal{L}\{y'' + 2y' + y\}(s) = \mathcal{L}\{\sin t - \delta(t-1)\}$$

$$\mathcal{L}\{y''\}(s) + 2\mathcal{L}\{y'\}(s) + \mathcal{L}\{y\}(s) = \mathcal{L}\{\sin t\}(s) - \mathcal{L}\{\delta(t-1)\}(s)$$

$$s^2 Y - sy'(0) - y(0) + 2sY - 2y(0) + Y = \frac{1}{s^2+1} - e^{-s}$$

$$Y(s^2+2s+1) - s = \frac{1}{s^2+1} - e^{-s}$$

$$Y = \frac{s^3+s+1}{(s^2+1)(s+1)^2} - e^{-s}\frac{1}{(s+1)^2}$$

$$\frac{s^3+s+1}{(s^2+1)(s+1)^2} = \frac{As+B}{s^2+1} + \frac{C}{s+1} + \frac{D}{(s+1)^2} \quad / \cdot (s^2+1)(s+1)^2$$

$$s^3+s+1 = (As+B)(s+1)^2 + C(s^2+1)(s+1) + D(s^2+1)$$

$$s^3+s+1 = (A+C)s^3 + (B+2A+C+D)s^2 + (A+2B+C)s + (B+C+D)$$

$$\begin{cases} A+C=1 \\ 2A+B+C+D=0 \\ A+2B+C=1 \\ B+C+D=1 \end{cases} \quad \begin{cases} C=1-A \\ A+B+D=-1 \\ B=0 \\ D=A \end{cases} \quad \begin{cases} 2A=-1 \\ B=0 \\ C=1-A \\ D=A \end{cases} \quad \begin{cases} A=-\frac{1}{2} \\ B=0 \\ C=3/2 \\ D=-1/2 \end{cases}$$

$$Y = -\frac{1}{2}\frac{s}{s^2+1} + \frac{3}{2}\frac{1}{s+1} - \frac{1}{2}\frac{1}{(s+1)^2} - e^{-s}\frac{1}{(s+1)^2}$$

$$\boxed{y(t) = -\frac{1}{2}\cos t + \frac{3}{2}e^{-t} - \frac{1}{2}e^{-t}t - u(t-1)\,e^{-(t-1)}(t-1)}$$

2. **(10 points).** Compute the Laplace transform of the function

$$f(t) = te^{-2t}\sin\left(t + \frac{\pi}{2}\right).$$

$\sin(t+\frac{\pi}{2}) = \cos t$

$f(t) = te^{-2t}\cos t$

$$F(s) = (-1)\left(\frac{s+2}{(s+2)^2+1}\right)'$$

$$= -\frac{(s+2)^2+1-2(s+2)^2}{(s^2+4s+5)^2}$$

$$= \frac{(s+2)^2-1}{(s^2+4s+5)^2}$$

$$= \frac{s^2+4s+3}{(s^2+4s+5)^2}$$

$$= \boxed{\frac{(s+3)(s+1)}{(s^2+4s+5)^2}}$$

Linear 1st order ODE

3. **(10 points)** Solve the following initial value problem:

$$y' = e^{x^3} - \frac{2y}{x}, \qquad y(1) = \frac{2e}{3}.$$

$$y' + \frac{2}{x} y = e^{x^3}$$

$$\mu(x) = \exp\left(\int \frac{2}{x} dx\right)$$
$$= \exp(2 \ln x)$$
$$= x^2$$

$$y(x) = \frac{\int \mu(x)\, Q(x)\, dx}{\mu(x)\, Q(x)}$$

$$= \frac{\int x^2 e^{x^3} dx}{x^2} \qquad u = x^3, \quad du = 3x^2 dx$$

$$= \frac{1}{3} \frac{\int e^u du}{x^2}$$

$$= \frac{1}{3} \frac{e^{x^3} + C}{x^2}$$

$$= \frac{1}{3} x^{-2} e^{x^3} + C x^{-2}$$

$$\frac{2e}{3} = y(1) = \frac{1}{3} e + C \implies C = \frac{e}{3}$$

$$\boxed{y(x) = \frac{1}{3} x^{-2} e^{x^3} + \frac{e}{3} x^{-2}}$$

exact differential equation

4. **(15 points).** Find the (implicit) solution to

$$\underbrace{\left(\frac{y}{x}+x^3\right)}_{M} dx + \underbrace{(y^2+\ln x)}_{N}\, dy = 0, \qquad y(1) = 1.$$

$\frac{\partial M}{\partial y} = \frac{1}{x} = \frac{\partial N}{\partial x} = \frac{1}{x} \Rightarrow$ i.e. an F s.t. $\frac{\partial F}{\partial x} = M$, $\frac{\partial F}{\partial y} = N$

$$F(x,y) = \int M(x,y)\,dx + g(y)$$
$$= \int \frac{y}{x} + x^3\,dx + g(y)$$
$$= y\ln x + \frac{x^4}{4} + g(y)$$

$$y^2 + \ln x = N(x,y) = \frac{\partial F}{\partial y} = \ln x + g'(y)$$

$$\Rightarrow g'(y) = y^2$$
$$g(y) = \frac{y^3}{3} + C$$

$$F(x,y) = y\ln x + \frac{x^4}{4} + \frac{y^3}{3} + C$$

$$F(1,1) = 0 \Rightarrow \frac{1}{4} + \frac{1}{3} + C = 0 \Rightarrow C = -\frac{1}{12}$$

$$\boxed{y\ln x + \frac{x^4}{4} + \frac{y^3}{3} = \frac{1}{12}}$$

5. **(15 points).** A nitric acid solution flows at a constant rate of 6 L/min into a large tank that initially held 200 L of a 0.5% nitric acid solution. The solution inside the tank is kept well stirred and flows out of the tank at a rate of 8 L/min. If the solution entering the tank is 20% nitric acid, determine the volume of nitric acid in the tank after t min.

$x(t)$ - volume of nitric acid at t minutes

$V(t)$ - volume of solution at t minutes

$V(0) = 200$, $V(t) = 200 + (6-8)t$
$= 200 - 2t$

$$\frac{dx}{dt} = \text{input} - \text{output}$$
$$= 6 \cdot \frac{20}{100} - 8 \cdot \frac{x(t)}{200-2t}$$
$$= 1.2 - \frac{4}{100-t} x$$

$$\frac{dx}{dt} + \frac{4}{100-t} x = 1.2$$

Linear 1st order ODE

$$\mu(t) = \exp\left(\int \frac{4}{100-t} dt\right)$$
$$= \exp\left(-4 \int \frac{du}{u}\right)$$
$$= \exp(-4 \ln u)$$
$$= u^{-4}$$
$$= (100-t)^{-4}$$

$u = 100 - t$

$du = -dt$

$$x(t) = \frac{\int (100-t)^{-4}(1.2)dt}{(100-t)^{-4}}$$
$$= \frac{1.2(100-t)^{-3}\left(\frac{1}{3}\right) + c}{(100-t)^{-4}}$$
$$= 0.4(100-t) + c(100-t)^4$$

$1 = x(0) = 40 - 10^8 c$

$c = 3.9 \cdot 10^{-7}$

$$\boxed{x(t) = 0.4(100-t) + 3.9 \cdot 10^{-7}(100-t)^4}$$

6. **(10 points).** Consider the following initial value problem:

$$\frac{dy}{dx} = x^2 - xy, \qquad y(0) = 1.$$

Use the Euler method with two steps to approximate $y(1)$.

$n = 2$, $h = \frac{1-0}{2} = 0.5$

$X_n = X_{n-1} + nh$

$y_n = y_{n-1} + h f(x_{n-1}, y_{n-1})$

x	y	f(x,y)
0	1	0
0.5	1	0.25 − 0.5 = −0.25
1	$1 + 0.5(-0.25) = 0.875$	

$\boxed{y(1) \approx 0.875}$

7. **(15 points).** Find a general solution to

principle of superposition + method of undetermined coefficients

$$y'' - y = t + e^t + \sin t.$$

$r^2 - 1 = 0$

$\Rightarrow r = \pm 1$

$y_h = C_1 e^t + C_2 e^{-t}$

$y_{p_1} = At$

$y_{p_1}' = A$

$y_{p_1}'' = 0$

$-At = t \Rightarrow A = -1$

$y_{p_2} = Bte^t$

$y_{p_2}' = B(e^t + te^t)$

$y_{p_2}'' = Be^t + B(t+1)e^t = B(t+2)e^t$

$B(t+2)e^t - Bte^t = e^t$

$\Rightarrow 2B = 1, \quad B = \frac{1}{2}$

$y_{p_3} = C\cos t + D\sin t$

$y_{p_3}' = -C\sin t + D\cos t$

$y_{p_3}'' = -C\cos t - D\sin t$

$-2(C\cos t + D\sin t) = \sin t$

$\Rightarrow C = 0, \quad D = -\frac{1}{2}$

$$y(t) = y_h(t) + y_{p_1}(t) + y_{p_2}(t) + y_{p_3}(t)$$

$$= C_1 e^t + C_2 e^{-t} - t + \frac{1}{2}te^t - \frac{1}{2}\sin t$$

8. **(15 points).** Consider the wave equation

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(x,t) = \alpha^2 \dfrac{\partial^2 u}{\partial x^2}(x,t) & 0 < x < L, \quad t > 0, \\ u(0,t) = u(L,t) = 0 & t > 0. \end{cases}$$

(a) Derive a formula for the general solution $u(x,t)$ of this wave equation. (*Hint:* use separation of variables or Fourier series.)

separation of variables

$u(x,t) = X(x)\,T(t)$, $\frac{\partial^2 u}{\partial x^2} = X''(x)\,T(t)$, $\frac{\partial^2 u}{\partial t^2} = X(x)\,T''(t)$

$X(x)\,T''(t) = \alpha^2 X''(x)\,T(t)$

$\frac{X''(x)}{X(x)} = \frac{T''(t)}{\alpha^2 T(t)}$

set $\frac{X''(x)}{X(x)} = -\lambda$

$X''(x) + \lambda X(x) = 0$

$r = \pm\sqrt{-\lambda}$

$\lambda = 0$

$X(x) = c_1 + c_2 t$

$u(0,t) = X(0)\,T(t)$

$\Rightarrow X(0) = 0$

for nontrivial soln

$c_1 + 0 = 0 = X(0) \Rightarrow c_1 = 0$

$c_2 L = 0 = X(L) \Rightarrow c_2 = 0$

hence no nontrivial soln

$\lambda < 0$

$X(x) = c_1 e^{\sqrt{\lambda} t} + c_2 e^{-\sqrt{\lambda} t}$

$X(0) = c_1 + c_2 = 0 \Rightarrow c_2 = -c_1$

$X(L) = 0 = c_1 e^{\sqrt{\lambda} L} - c_1 e^{-\sqrt{\lambda} L} \Rightarrow c_1 = 0$

hence no nontrivial solution

$\lambda > 0$

$X(x) = c_1 \cos(\sqrt{\lambda}\,x) + c_2 \sin(\sqrt{\lambda}\,x)$

$X(0) = 0 \Rightarrow c_1 = 0$

$X(L) = 0 \Rightarrow \lambda = (n\pi)^2/L^2$

$X_n(x) = C_n \sin\left(\frac{n\pi x}{L}\right)$

$T''(t) + \lambda \alpha^2 T(t) = 0$

$\lambda\alpha^2 > 0$, so

$T_n(t) = d_n \cos\left(\frac{\alpha n\pi t}{L}\right) + e_n \sin\left(\frac{\alpha n\pi t}{L}\right)$

$u(x,t) = \sum_{n=1}^{\infty} \left(d_n \cos\left(\frac{\alpha n\pi t}{L}\right) + e_n \sin\left(\frac{\alpha n\pi t}{L}\right)\right) \sin\left(\frac{n\pi x}{L}\right)$

(b) Find the solution $u(x,t)$ satisfying the initial values:

$$u(x,0) = \sin\left(\frac{\pi x}{L}\right), \qquad \frac{\partial u}{\partial t}(x,0) = \sin\left(\frac{2\pi x}{L}\right), \qquad 0 < x < L\,.$$

$$U(x,0) = \sin\left(\frac{\pi x}{L}\right) \Rightarrow d_1 = 1,\ d_n = 0 \text{ for } n > 1$$

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty}\left[\left(\frac{\alpha n \pi}{L}\right) d_n\left(-\sin\left(\frac{\alpha n \pi t}{L}\right)\right) + \left(\frac{\alpha \pi n}{L}\right) e_n \cos\left(\frac{\alpha n \pi t}{2}\right)\right]\sin\left(\frac{n\pi x}{L}\right)$$

$$\frac{\partial u}{\partial t}(x,0) = \sin\left(\frac{2\pi x}{L}\right) \Rightarrow e_2 = \frac{L}{2\pi\alpha},\ e_n = 0 \text{ for } n \neq 2$$

$$\boxed{u(x,t) = \cos\left(\frac{\alpha\pi t}{L}\right)\sin\left(\frac{\pi x}{L}\right) + \frac{L}{2\pi\alpha}\sin\left(\frac{2\alpha\pi t}{L}\right)\sin\left(\frac{2\pi x}{L}\right)}$$

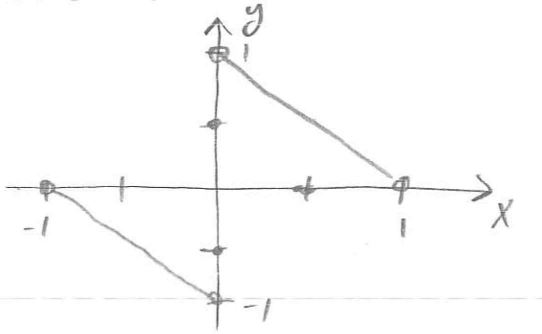
9. **(15 points).** Consider the function

$$f(x) = 1 - x, \qquad 0 < x < 1.$$

(a) Sketch the odd extension of $f(x)$ over the range $-1 < x < 1$.

$f_{odd}(-1) = -f(1) = 0$

$f_{odd}(-0.5) = -f(0.5) = -0.5$



(b) Does the Fourier series of the extended $f(x)$ involve only sines, only cosines, or both sines and cosines?

only sines since it is odd

(c) Compute explicitly the Fourier series of the extended $f(x)$.

$$b_n = \frac{2}{1}\int_0^1 (1-x)\sin\left(\frac{n\pi x}{1}\right)dx$$

$$= 2\left[-\frac{1-x}{n\pi}\cos(n\pi x) - \frac{1}{(n\pi)^2}\sin(n\pi x)\right]_0^1$$

$$= \frac{2}{(n\pi)^2}$$

$$\boxed{f_{odd}(x) = \sum_{n=1}^{\infty} \frac{2}{(n\pi)^2}\sin(n\pi x)}$$

$$\begin{array}{ll} 1-x & + \quad \sin(n\pi x) \\ -1 & - \quad -(n\pi)^{-1}\cos(n\pi x) \\ 0 & \quad -(n\pi)^{-2}\sin(n\pi x) \end{array}$$

EXTRA SPACE TO WORK – IF YOU USE THIS PAGE TO SOLVE SOME OF THE PROBLEMS, PLEASE **MARK THIS CLEARLY** BOTH HERE AND IN THE SPACE ASSIGNED TO THE PROBLEM YOU ARE SOLVING!