MATHEMATICS 220: FINAL EXAM
University of Illinois at Chicago
(Abramov, Heard, Nicholls)
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Please read the exam carefully and follow all instructions. **SHOW ALL OF YOUR WORK.** Please put a box around your final answer.

1. (25 points) Solve the initial value problem
$$x\frac{dy}{dx} + 3(y + x^3) = \frac{\sin x}{x^2}, \quad y(\pi) = -\pi^3/2.$$

2. (20 points) Use Euler's method to approximate the solution of the initial value problem
$$\frac{dy}{dx} = x - y^2, \quad y(0) = 0,$$
at the points $x = 1, 2, 3$.

3. (20 points) Solve the initial value problem
$$y'' + 2y' + 17y = 0, \quad y(0) = 1, \quad y'(0) = -1.$$

4. (25 points) Using the **Method of Undetermined Coefficients (any other method will receive no credit)** find the general solution to the equation
$$y''(t) + 9y(t) = -12\sin(3t).$$

5. (20 points) Find the solution to the initial value problem
$$t^2y''(t) + 3ty'(t) + 5y(t) = 0, \quad y(1) = 2, \quad y'(1) = -8.$$

6. (20 points) Solve the given initial value problem using the **Method of Laplace transforms (any other method will receive zero points)**:

$$y''(t) - 4y(t) = 0, \quad y(0) = 0, \quad y'(0) = 4\pi.$$

7. (25 points) Solve the initial value problem

$$y''(t) + y(t) = -\delta(t - \pi), \quad y(0) = 0, \quad y'(0) = 1.$$

8. (20 points) Compute the **Fourier Sine** series of

$$f(x) = x, \quad 0 < x < \pi.$$

9. (25 points) Solve the initial–boundary value problem

$$\begin{aligned}
&\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, && 0 < x < \pi, \quad t > 0,\\
&u(0, t) = 0 && t > 0,\\
&u(\pi, t) = 0 && t > 0,\\
&u(x, 0) = 1, && 0 < x < \pi.
\end{aligned}$$

List of PDE Formulae

1. The solution of the homogeneous heat equation $u_t = \beta^2 u_{xx}$ with Dirichlet boundary conditions is:

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-(\beta n\pi/L)^2 t} \sin\left(\frac{n\pi}{L}x\right).$$

2. The solution of the homogeneous heat equation $u_t = \beta^2 u_{xx}$ with Neumann boundary conditions is:

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-(\beta n\pi/L)^2 t} \cos\left(\frac{n\pi}{L}x\right).$$

3. The inhomogeneous heat equation has a solution of the form $u(x,t) = v(x) + w(x,t)$, where v is the steady–state solution and w solves a homogeneous heat equation.

4. The solution of the homogeneous wave equation $u_{tt} = \alpha^2 u_{xx}$ with Dirichlet boundary conditions is:

$$u(x,t) = \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\alpha\frac{n\pi}{L}t\right) + b_n \sin\left(\alpha\frac{n\pi}{L}t\right)\right\} \sin\left(\frac{n\pi}{L}x\right).$$

List of Laplace Transforms

1. $\mathcal{L}\{1\} = \dfrac{1}{s}, \quad s > 0$

2. $\mathcal{L}\{e^{at}\} = \dfrac{1}{s-a}, \quad s > a$

3. $\mathcal{L}\{t^n\} = \dfrac{n!}{s^{n+1}}, \quad s > 0$

4. $\mathcal{L}\{\sin(bt)\} = \dfrac{b}{s^2+b^2}, \quad s > 0$

5. $\mathcal{L}\{\cos(bt)\} = \dfrac{s}{s^2+b^2}, \quad s > 0$

6. $\mathcal{L}\{e^{at}t^n\} = \dfrac{n!}{(s-a)^{n+1}}, \quad s > a$

7. $\mathcal{L}\{e^{at}\sin(bt)\} = \dfrac{b}{(s-a)^2+b^2}, \quad s > a$

8. $\mathcal{L}\{e^{at}\cos(bt)\} = \dfrac{s-a}{(s-a)^2+b^2}, \quad s > a$

9. $\mathcal{L}\{e^{at}f(t)\}(s) = \mathcal{L}\{f\}(s-a)$

10. $\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0)$

11. $\mathcal{L}\{f''\}(s) = s^2\mathcal{L}\{f\}(s) - sf(0) - f'(0)$

12. $\mathcal{L}\left\{f^{(n)}\right\}(s) = s^n\mathcal{L}\{f\}(s) - s^{n-1}f(0) - \ldots - f^{(n-1)}(0)$

13. $\mathcal{L}\{t^n f(t)\}(s) = (-1)^n \dfrac{d^n}{ds^n}\mathcal{L}\{f\}(s)$

14. $\mathcal{L}\{f(t-a)u(t-a)\}(s) = e^{-as}F(s)$

15. $\mathcal{L}\{u(t-a)\}(s) = \dfrac{e^{-as}}{s}$

16. $\mathcal{L}\{g(t)u(t-a)\}(s) = e^{-as}\mathcal{L}\{g(t+a)\}(s)$

17. If f has period T then

$$\mathcal{L}\{f\}(s) = \frac{F_T(s)}{1-e^{-sT}} = \frac{\int_0^T e^{-st}f(t)\,dt}{1-e^{-sT}}$$

18. $\mathcal{L}\{\delta(t-a)\}(s) = e^{-as}$

① linear!

$$x\frac{dy}{dx} + 3(y + x^3) = \frac{\sin x}{x^2}, \quad y(\pi) = -\pi^3/2$$

$$x\frac{dy}{dx} + 3y = \frac{\sin x}{x^2} - 3x^3$$

$$\frac{dy}{dx} + \frac{3}{x}y = \frac{\sin x}{x^3} - 3x^2$$

$$\mu(x) = \exp\left(\int \frac{3}{x}dx\right) = \exp(3\ln x) = x^3$$

$$\int Q(x)\mu(x)\,dx = \int \sin x - 3x^5\,dx = -\cos x - \frac{1}{2}x^6 + C$$

$$y(x) = \frac{\int Q(x)\mu(x)\,dx}{\mu(x)} = -\frac{\cos x}{x^3} - \frac{1}{2}x^3 + Cx^{-3}$$

$$-\frac{\pi^3}{2} = y(\pi) = \frac{1}{\pi^3} - \frac{1}{2}\pi^3 + \frac{C}{\pi^3}$$

$$\frac{1+C}{\pi^3} = 0 \Rightarrow C = -1$$

$$\boxed{y(x) = -\frac{\cos x}{x^3} - \frac{1}{2}x^3 - \frac{1}{x^3}}$$

2. Problem doesn't specify, but let's take $h = 1$

x	y	$f(x,y)$
0	y_0	0
1	$0 + 1 \cdot 0 = 0$	1
2	$0 + 1 \cdot 1 = 1$	1
3	$1 + 1 \cdot 1 = 2$	/

(3.) easiest to use Laplace transform

$$y'' + 2y' + 17y = 0, \quad y(0) = 1, \; y'(0) = -1$$

$$s^2 Y - s y(0) - y'(0) + 2(sY - y(0)) + 17Y = 0$$

$$Y(s^2 + 2s + 17) = s - 1 + 2$$

$$s^2 + 2s + 17 = (s+1)^2 + 16$$

$$Y = \frac{s+1}{(s+1)^2 + 4^2}$$

$$\boxed{y(t) = e^{-t}\cos(4t)}$$

(4.)

$y'' + 9y = -12\sin(3t)$

$r^2 + 9 = 0$

$r = \pm 3i$

$y_h(t) = C_1 \cos 3t + C_2 \sin 3t$

$y_p(t) = At\cos 3t + Bt\sin 3t = t(A\cos 3t + B\sin 3t)$

$y_p'(t) = A\cos 3t + B\sin 3t - 3At\sin 3t + 3Bt\cos 3t$

$y_p''(t) = -3A\sin 3t + 3B\cos 3t - 3A\sin 3t - 9At\cos 3t - 9Bt\sin 3t + 3B\cos 3t$

$= -6A\sin 3t + 6B\cos 3t - 9At\cos 3t - 9Bt\sin 3t$

$-12\sin(3t) = y_p'' + 9y_p = -6A\sin 3t + 6B\cos 3t$

$B = 0, \; A = 2$

$\boxed{y(t) = C_1\cos 3t + C_2\sin 3t + 2t\cos 3t}$

(5.) $t^2 y''(t) + 3t\, y'(t) + 5y(t) = 0, \quad y(1) = 2, \; y'(1) = -8$

$y = t^r, \quad y' = r t^{r-1}, \quad y'' = (r^2 - r) t^{r-2}$

$(r^2 - r + 3r + 5)\, t^r = 0$

$r^2 + 2r + 5 = 0$

$r = \dfrac{-2 \pm \sqrt{4 - 20}}{2} = \dfrac{-2 \pm 4i}{2} = -1 \pm 2i$

$y(t) = C_1 t^{-1} \cos(2 \ln t) + C_2 t^{-1} \sin(2 \ln t)$

$y'(t) = -C_1 t^{-2} \cos(2 \ln t) - C_1 \frac{2}{t} t^{-1} \sin(2 \ln t)$

$\qquad - C_2 t^{-2} \sin(2 \ln t) + C_2 \frac{2}{t} t^{-1} \cos(2 \ln t)$

$2 = y(1) = C_1$

$-8 = y'(1) = -2 + 2C_2 \quad \Rightarrow \quad C_2 = 3$

$\boxed{y(t) = 2t^{-1} \cos(2 \ln t) + 3t^{-1} \sin(2 \ln t)}$

6. $y'' - 4y = 0, \ y(0) = 0, \ y'(0) = 4\pi$

$s^2 Y - sy(0) - y'(0) - 4Y = 0$

$Y(s^2 - 4) = 4\pi$

$Y = \frac{4\pi}{s^2-4} = \frac{A}{s-2} + \frac{B}{s+2}$

$4\pi = A(s+2) + B(s-2)$

$s = 2 \quad 4\pi = 4A$

$s = -2 \quad 4\pi = -4B$

$A = \pi, \ B = -\pi$

$Y = \frac{\pi}{s-2} - \frac{\pi}{s+2}$

$\boxed{y(t) = \pi e^{2t} - \pi e^{-2t}}$

(7.) $y''(t) + y(t) = -\delta(t-\pi), \; y(0) = 0, \; y'(0) = 1$

$$s^2 Y - s y(0) - y'(0) + Y = -e^{-\pi s}$$

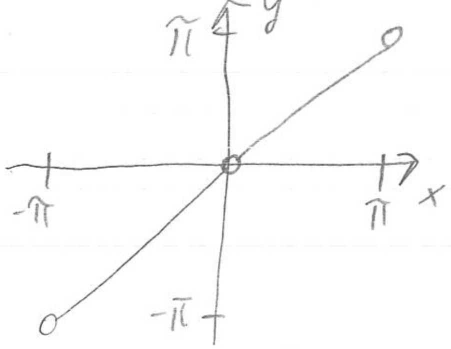
$$Y(s^2+1) = -e^{-\pi s} + 1$$

$$Y = -e^{-\pi s}\frac{1}{s^2+1} + \frac{1}{s^2+1}$$

$$y(t) = -u(t-\pi)\sin(t-\pi) + \sin(t)$$

$$= u(t-\pi)\sin t + \sin t$$

(8.) $f(x) = x, \quad 0 < x < \pi$



$$b_n = \frac{2}{\pi}\int_0^{\pi} x\sin(nx)\,dx$$

$$= \frac{2}{\pi}\left[-\frac{x}{n}\cos(nx) + \frac{1}{n^2}\sin(nx)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left(-\frac{\pi}{n}(-1)^n\right)$$

$$= (-1)^{n+1}\frac{2}{n}$$

$$\boxed{f(x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{n}\sin(nx)}$$

(9.) $u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2 t} \sin(nx)$

$0 = u(0,t) = 0$

$0 = u(\pi,t) = 0$

$1 = u(x,0) = \sum_{n=1}^{\infty} b_n \sin(nx)$

need to find Fourier sine series of $f(x)=1 \quad 0<x<\pi$

$b_n = \frac{2}{\pi}\int_0^{\pi} \sin(nx)\,dx = -\frac{2}{\pi}\frac{1}{n}(\cos nx)\Big|_0^{\pi} = \frac{2}{\pi n}\left((-1)^{n+1}+1\right)$

$u(x,t) = \frac{4}{\pi} e^{-t}\sin(x) + \frac{4}{3\pi} e^{-9t}\sin(3x) + \ldots$