Computing the Extreme Core of Siegel Modular Forms

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Abstract

In this thesis we consider the problem of computing the extreme core of Siegel modular forms. Siegel modular forms are a generalization of the classical modular forms as holomorphic functions on the space of symmetric complex matrices with positive definite imaginary part. The extreme core intrinsically characterises a nonvanishing set for the coefficients of the Fourier series of Siegel cusp forms, generalizing the Valence Inequality from the theory of classical modular forms. We give an algorithm for computing the extreme core when $n = 2$ for a fixed weight k and show that an estimate for the extreme core due to Cris Poor and David S. Yuen is optimal for cusp forms of even weight between 10 and 44.

Introduction

A *modular form* is a complex analytic function on the upper half-plane invariant up to a factor of automorphy under the action of the group $SL_2(\mathbf{Z})$. The *weight* k of a modular form f is the power of the factor of automorphy that appears when acting on the input of f with an element of $SL_2(\mathbf{Z})$. All modular forms have Fourier series of the form

$$f(\tau) = \sum_{j=0}^{\infty} a(j;f)q^j \text{ where } q = e^{2\pi i\tau} \text{ and } a(j;f) \in \mathbf{C}.$$

The *Valence Inequality* relates the structure of the q-expansion of a modular form with its weight. In particular, if f is a nonzero weight k modular form, then at least one of the $a(j;f)$ is nonzero for $0 \leq j \leq \lfloor \frac{k}{12} \rfloor$, i.e. a weight k modular form is identically zero if and only if the first $\lfloor \frac{k}{12} \rfloor + 1$ terms of its Fourier series are zero. Since Δ, the modular discriminant, is a weight 12 modular form such that $a(0;\Delta) = 0$, but $a(1;\Delta) \neq 0$, this bound is optimal. While we state this result in terms of determining whether a modular form is zero, since linear combinations of weight k modular forms are also weight k modular forms, it is mostly used when trying to determine whether two modular forms are equal by comparing their Fourier coefficients.

Siegel modular forms, introduced by Carl Ludwig Siegel in the 1930s, generalize modular forms to complex analytic functions from the space of $n \times n$ symmetric complex matrices with positive imaginary part (the so-called *Siegel upper-half space*)[1], invariant up to a factor of automorphy under the action of the group $Sp_n(\mathbf{Z})$. We call the integer n the *degree* of the Siegel modular form. In order to differentiate the modular forms from the original definition from Siegel modular forms, from this point forward we will refer to the former as *classical* modular forms or *elliptic* modular forms (due to their connection with the theory of elliptic curves). Siegel modular forms also have Fourier series, but of the form

$$f(\Omega) = \sum_{t \in \mathcal{X}_n^{\text{semi}}} a(t;f)e^{2\pi i \operatorname{tr}(t\Omega)},$$

indexed over $\mathcal{X}_n^{\text{semi}}$, the set of $n \times n$ positive semidefinite matrices with integer entries along the diagonal and semi-integer entries elsewhere.

It is natural to ask whether there exists a generalization of the Valence Inequality to Siegel modular forms. Note that since the Fourier coefficients are indexed by spaces of

[1]It is possible to further generalize Siegel modular forms to vector-valued functions, although these are beyond the scope of this thesis. For an introduction to the theory of the vector-valued Siegel modular forms see [vdG07].

matrices, these nonvanishing regions will have a richer geometry—we replace the nonvanishing ray $\left[\lfloor \frac{k}{12} \rfloor, \infty\right)$ from the classical elliptic forms with *kernels*, convex bodies invariant under $\mathbf{R}_{\geq 1}$-dilation. One way to obtain a Valence Inequality-like result for Siegel modular forms is to introduce a linear ordering on the space of the index matrices by using *height functions*, maps which provide a linear ordering on the space of index matrices. The earliest result of this kind is due to Siegel and states that if f is a weight k degree n Siegel modular form such that $a(t; f) = 0$ for all $t \in \mathcal{X}_n^{\text{semi}}$ whose trace is less than a certain constant dependent on n and k, then f is identically zero. This approach, however, is dependent on our choice of height function—to find an optimal intrinsic nonvanishing set for Siegel modular forms we need to look at the *extreme core*.

We will later give estimates for the extreme core for $2 \leq n \leq 5$ based on [PY05] and show that the estimate for $n = 2$ is optimal for even cusp forms up to weight 44.

Throughout this thesis we assume that the reader is familiar with undergraduate linear algebra, abstract algebra and complex analysis. Previous exposure to classical modular forms is helpful, but not required. We assume no familiarity with the theory of Siegel modular forms.

In Chapter 1 we introduce some concepts and results from the geometry of numbers which will later prove useful. In Chapter 2 we begin our discussion of Siegel modular forms. The first two chapters are based on the presentation in [PSY]. We introduce the problem of finding the extreme core, our main goal, in Chapter 3, following [PY05]. Finally, in Chapter 4, we give an algorithm to find the best approximation for the extreme core for a fixed weight k, showing that the $\mathcal{K}\left[\frac{1}{10}\begin{pmatrix} 1 & 17/16 \\ 17/36 & 1 \end{pmatrix}\right]$ outer bound from [PY05] is optimal for cusp forms of even weight between 10 and 44. The `sage` code for this computation is given in the appendix.

I

Kernels and Height Functions

1.1 Notation

Definition 1.1.1 Let $\mathcal{V}_n$ be the set of symmetric matrices with real-valued entries. The space $\mathcal{V}_n$ with the inner product $\langle s, v\rangle := \operatorname{tr}(sv)$ is a Euclidean space. There exists a bijection between $\mathcal{V}_n$ and the space of quadratic forms[1], with $s \in \mathcal{V}_n$ corresponding to the quadratic form $Q_s(v) = v^\mathsf{T} s v$.

Definition 1.1.2 The subset of positive definite matrices in $\mathcal{V}_n$ is denoted $\mathcal{P}_n$. The closure of $\mathcal{P}_n$ with respect to the metric on $\mathcal{V}_n$ is $\overline{\mathcal{P}}_n$, the set of positive semidefinite symmetric matrices.

The set $\mathcal{P}_n$ has a partial ordering. Given $a, b \in \mathcal{P}_n$, we say $a < b$ if there exists a $v \in \mathcal{P}_n$ such that $a + v = b$.

Definition 1.1.3 A matrix $s \in \mathcal{V}_n$ is *dyadic* if it can be written as

$$s = \sum_{z \in \mathbf{Z}^n \setminus \{0\}} \alpha_z z z^\mathsf{T}, \text{ all } \alpha_z \in \mathbf{R}_{\geq 0}, \text{ finitely many } \alpha_z \neq 0.$$

We refer to the choice of vectors and positive coefficients as a *dyadic representation* of s. The subset of dyadic matrices in $\overline{\mathcal{P}}_n$ is $\mathcal{P}_n^*$, the rational closure of $\mathcal{P}_n$, and we thus have

$$\mathcal{P}_n \subseteq \mathcal{P}_n^* \subseteq \overline{\mathcal{P}}_n.$$

The name dyadic is a reference to the zz^T term which is called a *dyad* in tensor theory.

Definition 1.1.4 Let $\mathrm{GL}_n(\mathbf{Z})$ be the group of invertible $n \times n$ integer matrices whose inverse also has integer entries. We let

$$G_\mathbf{Z} := \mathrm{GL}_n(\mathbf{Z})$$

act from the right on $\mathcal{V}_n$ by

$$\begin{aligned} \mathcal{V}_n \times G_\mathbf{Z} &\to \mathcal{V}_n \\ (s, g) &\mapsto g^\mathsf{T} s g. \end{aligned}$$

[1]Because of this, we will often call symmetric matrices *forms*. To avoid confusion with modular forms, we will always refer to the latter using the full names *modular forms* or *cusp forms*.

From this point forward we will use the shorthand

$$s[g] := g^T s g$$

for suitably-sized matrices s and g. Thus $s, t \in \mathcal{V}_n$ are in the same G_Z-equivalence class if and only if there exists a $g \in G_Z$ such that $s = t[g]$.

For m an invertible matrix, we will use m^* to denote the inverse transpose of m.

1.2 Cones and Semihulls

Definition 1.2.1 (Cone and Dual Cone) A subset $\mathcal{C}$ of $\mathcal{V}_n$ is a *cone* if

- $\mathcal{C}$ is convex (i.e. for all $a, b \in \mathcal{C}$ and $\lambda \in [0, 1]$, $\lambda a + (1 - \lambda) b \in \mathcal{C}$ as well), and
- $\mathcal{C}$ is closed under scaling by elements of $\mathbf{R}_{>0}$.

The dual cone of a cone $\mathcal{C}$, $\mathcal{C}^\vee$, is the set of elements of $\mathcal{V}_n$ whose inner product with all elements of $\mathcal{C}$ is nonnegative,

$$\mathcal{C}^\vee := \{s \in \mathcal{V}_n \mid \langle s, c \rangle \geq 0 \text{ for all } c \in \mathcal{C}\}.$$

Given any subset S of $\mathcal{V}_n$ we can find a cone containing all of S by taking the set of all nonempty positive linear combinations of elements of S. This is is the smallest cone containing S and we denote it by $\langle \mathbf{R}_{>0} S \rangle$.

Note that $\mathcal{P}_n$, $\mathcal{P}_n^*$ and $\overline{\mathcal{P}}_n$ are cones.

Definition 1.2.2 (Semihull and Dual Semihull) A subset $\mathcal{M}$ of $\mathcal{V}_n$ is a *semihull* if

- $\mathcal{M}$ is convex, and
- $\mathcal{M}$ is closed under scaling by elements of $\mathbf{R}_{\geq 1}$.

The dual semihull of $\mathcal{M}$ is the set

$$\mathcal{M}^\sqcup := \{s \in \mathcal{V}_n \mid \langle s, m \rangle \geq 1 \text{ for all } m \in \mathcal{M}\}.$$

As with cones, we can find the smallest semihull containing a given subset S of $\mathcal{V}_n$, by taking all nonempty linear combinations of elements of S where the sum of all the coefficients is at least 1. We call this set the semihull generated by S and denote it $\langle \mathbf{R}_{\geq 1} S \rangle$.

1.3 Legendre Reduction

We cite the following result from the reduction theory of 2×2 symmetric matrices.

Proposition 1.3.1 *Let* $v \in \mathcal{P}_2^*$*. Then there exists a unique* $s \in \mathcal{P}_2^*$ *such that* $v \in s[G_Z]$ *and* s *is Legendre-reduced, i.e.* $0 \leq 2s_{12} \leq s_{11} \leq s_{22}$.

Proof See [PSY]. ∎

The $\mathcal{P}_2^*$-subset of Legendre-reduced matrices, $\mathcal{R}_2$, is a cone.

Given an $s \in \mathcal{P}_s^*$ we can find the Legendre-reduced representative of s by the following algorithm.

1. If $s_{11} > s_{22}$ then replace s with
$$s\left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right] = \begin{pmatrix} s_{22} & -s_{12} \\ -s_{12} & s_{11} \end{pmatrix},$$
i.e. swap s_{11} and s_{22} while negating s_{12}, so that $s_{11} \leq s_{22}$.

2. If $2|s_{12}| > s_{11}$ then set
$$\lambda = \left\lfloor -\frac{s_{12}}{s_{11}} + \frac{1}{2} \right\rfloor,$$
so that $-s_{12}/s_{11} - 1/2 < \lambda \leq -s_{12}/s_{11} + 1/2$ and hence
$$-s_{11}/2 < s_{12} + \lambda s_{11} \leq s_{11}/2.$$
Replace s by
$$s\left[\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}\right] = \begin{pmatrix} s_{11} & s_{12} + \lambda s_{11} \\ s_{12} + \lambda s_{11} & s_{22} + 2\lambda s_{12} + \lambda^2 s_{11} \end{pmatrix},$$
that is replace s_{12} by $s_{12} + \lambda s_{11}$ and s_{22} by $s_{22} + 2\lambda s_{12} + \lambda^2 s_{11}$, so now $2|s_{12}| \leq s_{11}$. Since this step preserves s_{11} and preserves the determinant $s_{11}s_{22} - s_{12}^2$ while it reduces $|s_{12}|$, it reduces s_{22} as well. If $s_{11} \leq s_{22}$, then go to Step 3. Otherwise, return to Step 1.

3. If $s_{12} < 0$ then replace s by
$$s\left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] = \begin{pmatrix} s_{11} & -s_{12} \\ -s_{12} & s_{22} \end{pmatrix},$$
i.e. negate s_{12} so that now $s_{12} \geq 0$.

4. s is now Legendre-reduced.

For a more thorough treatment of Legendre reduction, including a proof that the preceding algorithm converges to the Legendre-reduced representative of s, see [PSY].

1.4 Height Functions

Definition 1.4.1 (Height Function) Let $\mathcal{C}$ be a cone in $\overline{\mathcal{P}}_n$ containing $\mathcal{P}_n$. We say that $\varphi : \mathcal{C} \to \mathbf{R}_{\geq 0}$ is a height function if

- (Positivity on $\mathcal{P}_n$) $\varphi(s) > 0$ for all $s \in \mathcal{P}_n$,
- (Homogeneity) $\varphi(\lambda s) = \lambda\varphi(s)$ for all $s \in \mathcal{C}$ and $\lambda > 0$, and
- (Superadditivity) $\varphi(s + t) \geq \varphi(s) + \varphi(t)$ for all $s, t \in \mathcal{C}$.

Let $n = 1$. Note that then $\varphi(\lambda) = \lambda\varphi(1)$ and therefore the only height function is the identity up to scaling—we need to go up to $n \geq 2$ to get nontrivial height functions.

All height functions are continuous on $\mathcal{P}_n$. We present the following proof due to David S. Yuen ([PY00, Proposition 2.2]).

Proposition 1.4.2 (Continuity and Order Preservation) *Let $\varphi : \mathcal{C} \to \mathbf{R}_{\geq 0}$ be a height function. Then φ preserves order and φ restricted to $\mathcal{P}_n$ is continuous.*

Proof In order to show that φ is order preserving, let $s < v$. Then there exists a u such that $v = s + u$ and by positivity and superadditivity

$$\varphi(s) < \varphi(s) + \varphi(u) \leq \varphi(s + u) = \varphi(v).$$

To show that φ is continuous on $\mathcal{P}_n$, suppose we are given $\varepsilon > 0$. For any $s \in \mathcal{P}_n$ there exists an $\alpha \in (0, 1)$, such that $\alpha\varphi(s) < \varepsilon$. We use α to define the open neighborhood of s

$$N_s = \{t \in \mathcal{P}_n \mid (1 - \alpha)s < t < (1 + \alpha)s\}.$$

For all $t \in N_s$

$$\begin{aligned}
\varphi(s) - \varepsilon &< (1 - \alpha)\varphi(s) && \text{from the definition of } \alpha\\
&< \varphi((1 - \alpha)s) + \varphi(t - (1 - \alpha)s) && \text{by positivity and homogeneity}\\
&\leq \varphi(t) && \text{by superadditivity}\\
&< \varphi(t) + \varphi((1 + \alpha)s - t) && \text{by positivity}\\
&\leq (1 + \alpha)\varphi(s) && \text{by positivity and superadditivity}\\
&< \varphi(s) + \varepsilon. && \text{from the definition of } \alpha.
\end{aligned}$$

Therefore, $|\varphi(s) - \varphi(t)| < \varepsilon$ for all $t \in N_s$. ∎

Proposition 1.4.3 (Extrema of a Height Function) *Let φ be a height function. The minimum of φ on a closed line segment contained in its domain is on the endpoints. The minimum of φ on a ray contained in its domain is on the endpoint of the ray.*

PROOF Suppose that the line segment from s to t lies in the domain of φ. By superadditivity and homogeneity,

$$\varphi(\lambda s + (1-\lambda)t) \geq \varphi(\lambda s) + \varphi((1-\lambda)t) = \lambda\varphi(s) + (1-\lambda)\varphi(t),$$

and it follows that $\varphi(\lambda s + (1-\lambda)t) \geq \min\{\varphi(s), \varphi(t)\}$ for $\lambda \in [0, 1]$.

Suppose that the ray with endpoint s and direction t is in domain of φ (i.e. the set $\{s + \lambda t \mid \lambda \in \mathbf{R}_{>0}\}$). For all $\alpha > \lambda$ we then have

$$\begin{aligned}\varphi(s+\lambda t) &= \varphi\left(\left(1-\frac{\lambda}{\alpha}\right)s + \frac{\lambda}{\alpha}(s+\alpha t)\right)\\ &\geq \left(1-\frac{\lambda}{\alpha}\right)\varphi(s) + \frac{\lambda}{\alpha}\varphi(s+\alpha t)\\ &\geq \left(1-\frac{\lambda}{\alpha}\right)\varphi(s).\end{aligned}$$

Taking the limit as $\alpha \to \infty$ yields $\varphi(s+\lambda t) \geq \varphi(s)$. Therefore φ attains its minimum on the endpoint. ∎

The Dual Height Function

Definition 1.4.4 (Dual Height Function) Let $\varphi : \mathcal{C} \to \mathbf{R}_{\geq 0}$ be a height function. The function

$$\widehat{\varphi}(s) = \inf_{t \in \mathcal{P}_n} \frac{\langle s, t\rangle}{\varphi(t)}$$

defined on all of $\overline{\mathcal{P}}_n$ is the *dual function* of φ.

The dual function of a height function is also a height function (for a proof this fact see [PSY]).

Examples of Height Functions

We give the following examples of height functions, omitting the proof that they satisfy the conditions for being a height function. Suppose $s \in \overline{\mathcal{P}}_n$. Then

- the trace of s, $\mathrm{tr}(s)$;
- the least eigenvalue of s, $\lambda_1(s)$;
- the *minimum function* of s, the smallest value of s as a quadratic form over the nonzero integers, i.e.

$$m(s) := \inf_{s \in \mathbf{Z}^n \setminus \{0\}} s[v];$$

- the *dyadic trace* of s, the dual function of m,

$$w(s) := \inf_{t \in \mathcal{P}_n} \frac{\langle s, t \rangle}{m(t)};$$

- the *reduced determinant* of s, $\delta(s) := \sqrt[n]{\det(s)}$; and

- the *reduced trace* of s, the trace of the element with smallest trace in the G_Z-equivalence class of s,

$$\widetilde{\operatorname{tr}}(s) := \inf \operatorname{tr}(s[G_Z])$$

are height functions. Of these, λ_1 and tr, δ and $n\delta$, and m and w are dual pairs. Furthermore, m, w, $\widetilde{\operatorname{tr}}$ and δ are class functions on G_Z-equivalence classes.

The trace has the following useful property.

Proposition 1.4.5 (Dilational Dominance of the Trace) *Let $\varphi : \mathcal{C} \to \mathbf{R}_{\geq 0}$ be a height function. Then there exists $\alpha \in \mathbf{R}_{\geq 1}$ such that*

$$\varphi(s) \leq \alpha \operatorname{tr}(s) \textit{ for all } s \in \mathcal{C}.$$

Proof See [PSY]. ∎

As its name suggests, the dyadic trace of a matrix is related to its dyadic representations. In particular, given an $s \in \mathcal{P}_n^*$,

$$w(s) = \sup \left\{ \sum_{z \in \mathbf{Z}^n \setminus \{0\}} \alpha_z \right\}$$

taking the supremum over all dyadic representations $s = \sum_{z \in \mathbf{Z}^n \setminus \{0\}} \alpha_z z z^T$. Having the dyadic trace defined as both an infimum and a supremum proves to be computationally useful.

In the case of $n = 2$, Legendre-reduction simplifies the problem of computing height functions for $s \in \mathcal{P}_2^*$.

Proposition 1.4.6 *Let $s \in \mathcal{P}_2^*$ be Legendre-reduced. We then have*

- $m(s) = s_{11}$,

- $\widetilde{\operatorname{tr}}(s) = s_{11} + s_{22}$, *and*

- $w(s) = s_{11} + s_{22} - s_{12}$.

Proof See [PSY]. ∎

We can use this proposition to give the following alternative definitions for m, $\widetilde{\mathrm{tr}}$ and w.

Corollary 1.4.7 *Let* $s \in \mathcal{P}_2^*$. *Then*

$$\begin{aligned} m(s) &= \inf\langle e_1 e_1^T, s[G_Z]\rangle, \\ \widetilde{\mathrm{tr}}(s) &= \inf\langle I_n, s[G_Z]\rangle, \textit{ and} \\ w(s) &= \inf\langle A_2, s[G_Z]\rangle, \end{aligned}$$

where $A_2 = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$.

1.5 Kernels

Definition 1.5.1 (Kernel) A semihull $\mathcal{K}$ of $\mathcal{V}_n$ is a *kernel* if its closure does not contain 0 and if its cone contains $\mathcal{P}_n$ and is contained within $\overline{\mathcal{P}}_n$. That is

- $\mathcal{K}$ is convex,
- $\mathcal{K}$ is closed under $\mathbf{R}_{\geq 1}$-dilation,
- $\mathcal{K}$ is *far*, meaning $0 \notin \overline{\mathcal{K}}$, and
- $\mathcal{K}$ is *wide*, meaning $\mathcal{P}_n \subseteq \mathbf{R}_{>0}\mathcal{K} \subseteq \overline{\mathcal{P}}_n$.

Note that kernels are semihulls and thus all of the preceding results and properties of semihulls apply to kernels. In particular, we note that given a kernel $\mathcal{K}$, the dual semihull $\mathcal{K}^\sqcup$ is also a kernel.

We define the following relation on kernels.

Definition 1.5.2 The kernel $\mathcal{K}$ is *dilationally smaller* than the kernel $\mathcal{L}$ if $\alpha\mathcal{K} \subseteq \mathcal{L}$ for some $\alpha \in \mathbf{R}_{>0}$, and $\mathcal{K}$ and $\mathcal{L}$ are *dilationally comparable* if each is dilationally smaller than the other, i.e. if

$$\alpha\mathcal{K} \subseteq \mathcal{L} \subseteq \beta\mathcal{K} \quad \text{for some } \alpha, \beta \in \mathbf{R}_{>0}.$$

The Kernel-Height Function Correspondence

We want to show that given a height function φ, the set $\mathcal{K}_\varphi := \varphi^{-1}(\mathbf{R}_{\geq 1})$ is a kernel. Given an s in the domain of φ such that $\varphi(s) > 1$ it is clear that $\varphi(\alpha s) = \alpha\varphi(s) \geq \alpha \geq 1$ for $\alpha \in \mathbf{R}_{\geq 1}$ (thereby showing that $\mathcal{K}_\varphi$ is closed under $\mathbf{R}_{\geq 1}$-dilation). Given s, t in $\varphi^{-1}(\mathbf{R}_{\geq 1})$

$$\varphi(\lambda s + (1-\lambda)t) \geq \lambda\varphi(s) + (1-\lambda)\varphi(t) \geq \lambda + (1-\lambda) = 1$$

showing it is convex. To show that $\mathcal{K}_\varphi$ is far, consider any sequence $\{s_i\}$ in $\overline{\mathcal{P}}_n$ that approaches 0 and note that then $\{\mathrm{tr}(s_i)\}$ approaches 0 in $\mathbf{R}$. By the dilational dominance of the trace, it follows that $\{\varphi(s_i)\}$ also approaches 0 and so only finitely many terms of the sequence lie in $\varphi^{-1}(\mathbf{R}_{\geq 1})$. Therefore $0 \notin \overline{\mathcal{K}}_\varphi$.

Finally, to show that $\mathcal{K}_\varphi$ is wide, note that φ is defined on all of $\mathcal{P}_n$ and that therefore for any $s \in \mathcal{P}_n$, $\frac{1}{\varphi(s)}s \in \varphi^{-1}(\mathbf{R}_{\geq 1})$.

Similarly, given a kernel $\mathcal{K}$ we can obtain a height function such that its value at s is given by the amount $\lambda \in \mathbf{R}_{>0}$ by which we need to scale the kernel so that s is a boundary point of $\lambda\mathcal{K}$, i.e.

$$\varphi_{\mathcal{K}}(s) := \sup\{\lambda \in \mathbf{R}_{>0} \mid s \in \lambda\mathcal{K}\}.$$

The function $\varphi_{\mathcal{K}}(s)$ is defined on the cone of $\mathcal{K}$, and therefore by the definition of kernel, on a superset of $\mathcal{P}_n$ contained in $\overline{\mathcal{P}}_n$. Ideally, we would want to argue that $\varphi_{\mathcal{K}_\psi} = \psi$, but this need not be true on the boundary. To sidestep this issue we define the following equivalence relations on kernels and height functions:

- two kernels are equivalent if they have the same closure, and
- two height functions are equivalent if they agree on $\mathcal{P}_n$.

This gives us the following proposition.

Proposition 1.5.3 (Height Function-Kernel Correspondence) *The map from height functions to kernels and the map from kernels to height functions invert each other at the level of kernel closure classes and $\mathcal{P}_n$-restriction classes.*

Proof See [PSY]. ∎

In addition, the duality of kernels and the duality of height functions are compatible with the height function-kernel correspondence.

Proposition 1.5.4 *Let $\mathcal{K}_\varphi$ be the kernel of a height function φ. Then the dual of the kernel of the height function is the kernel of the dual of the height function, i.e.*

$$\mathcal{K}_\varphi^{\sqcup} = \mathcal{K}_{\widehat{\varphi}}.$$

Proof See [PSY]. ∎

The kernel-height function correspondence will allow us to prove properties about height functions by instead studying their kernels (or vice versa). In particular, we have the following results.

Proposition 1.5.5 *Let $\mathcal{K}$ be a kernel. Then $s \in \overline{\mathcal{P}}_n$ is in $\mathcal{K}$ if and only if $\varphi_{\mathcal{K}}(s) \geq 1$.*

Proof This is immediate from the definition of the height function of a kernel. ∎

Proposition 1.5.6 *A height function is a class function if and only if its associated kernel is* $G_{\mathbf{Z}}$*-invariant.*

Proof Let $\mathcal{K}$ be a $G_{\mathbf{Z}}$-invariant kernel and note that then for every $s \in \mathbf{R}_{>0}\mathcal{K}$ and $v \in G_{\mathbf{Z}}$ we have

$$\begin{aligned}\varphi_{\mathcal{K}}(s[v]) &= \sup\{\lambda \in \mathbf{R}_{>0} \mid s[v] \in \lambda\mathcal{K}\} \\ &= \sup\left\{\lambda \in \mathbf{R}_{>0} \mid s \in \lambda\mathcal{K}[v^{-1}]\right\} \\ &= \sup\{\lambda \in \mathbf{R}_{>0} \mid s \in \lambda\mathcal{K}\} \\ &= \varphi_{\mathcal{K}}(s).\end{aligned}$$

Conversely, suppose that $\varphi_{\mathcal{K}}$ is a class function and let $s \in \mathcal{K}$ and $v \in G_{\mathbf{Z}}$. Because $s \in \mathcal{K}$, $\varphi_{\mathcal{K}}(s) \geq 1$, but since $\varphi_{\mathcal{K}}$ is a class function, $\varphi_{\mathcal{K}}(s[v]) = \varphi_{\mathcal{K}}(s) \geq 1$ as well and therefore $s[v] \in \mathcal{K}$. ∎

We are particularly interested in the kernels of the minimum function and the dyadic trace. To that end, we have the following definition.

Definition 1.5.7 The *perfect core* is the kernel of the dyadic trace, $\mathcal{K}_w$, and the *perfect cocore* is the kernel of the minimum function $\mathcal{K}_m$.

The vertices of $\mathcal{K}_m$ consist of a finite number of $G_{\mathbf{Z}}$-equivalence classes called the *perfect forms*.

Siegel Modular Forms

2

2.1 Introduction

Definition 2.1.1 The *Siegel upper-half space* of degree n, denoted $\mathcal{H}_n$, is the set of all symmetric $n \times n$ matrices over the complex numbers with positive definite imaginary part, i.e.

$$\mathcal{H}_n := \{\Omega = x + iy \mid x \in \mathcal{V}_n, y \in \mathcal{P}_n\}.$$

Definition 2.1.2 Let J be the $2n \times 2n$ block matrix $\left(\begin{smallmatrix} 0 & -I_n \\ I_n & 0 \end{smallmatrix}\right)$. We define the *symplectic group* of degree n to be the J-stabilizer subgroup of $GL_{2n}(\mathbf{R})$ with respect to the $[\ ]$ action, i.e.

$$\mathrm{Sp}_n(\mathbf{R}) := \{M \in GL_{2n}(\mathbf{R}) \mid J[M] = J\}.$$

The *Siegel modular group* of degree n, denoted Γ_n, is the subgroup of $\mathrm{Sp}_n(\mathbf{R})$ consisting of integer matrices[1],i.e.

$$\Gamma_n := \mathrm{Sp}_n(\mathbf{Z}).$$

The group Γ_n acts on $\mathcal{H}_n$ from the left by

$$\begin{aligned} \Gamma_n \times \mathcal{H}_n &\to \mathcal{H}_n \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \Omega &\mapsto (a\Omega + b)(c\Omega + d)^{-1} \end{aligned} \qquad \text{where } a, b, c \text{ and } d \text{ are } n \times n \text{ matrices.}$$

We omit the proof, but note that the preceding action is well-defined (i.e. $c\Omega + d$ is always invertible).

Observe that $\Gamma_1 = SL_2(\mathbf{Z})$.

Definition 2.1.3 (Siegel Modular Forms) Let f be a function from $\mathcal{H}_n$ to $\mathbf{C}$. We say f is a *Siegel modular form* of degree n and weight k if:

- $f(\Omega)$ is holomorphic in the entries of Ω,
- for all $g \in \Gamma_n$ with block matrix form $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $\Omega \in \mathcal{H}_n$,

$$f(g \cdot \Omega) = \det(c\Omega + d)^k f(\Omega),$$

- and for any $s \in \mathcal{P}_n$, f is bounded on the set $\{\Omega \in \mathcal{H}_n \mid \mathrm{Im}(\Omega) > s\}$.

[1] Unlike in the case of $G_{\mathbf{Z}}$, it suffices for a matrix in $\mathrm{Sp}_n(\mathbf{R})$ to have integer entries for its inverse to also be an integer matrix, since all symplectic matrices have determinant 1.

We denote the vector space of degree n weight k modular forms over the complex numbers by $\mathcal{M}_n^k$. By convention $\mathcal{M}_n^0 = \mathbf{C}$. The direct sum of these vector spaces over k for a given n forms the graded ring of degree n Siegel modular forms,

$$\mathcal{M}_n = \bigoplus_{k=0}^{\infty} \mathcal{M}_n^k.$$

The reason we are summing only over the vector spaces of Siegel modular forms with nonnegative weight is because if $k < 0$, then $\mathcal{M}_n^k = \{0\}$. All nonzero Siegel modular forms have nonnegative weight.

Since $\Gamma_1 = SL_2(\mathbf{Z})$, $\mathcal{M}_1$ consists precisely of the classical modular forms. This shows that the Siegel modular forms truly are a generalization of the classical modular forms.

We give the following generalization of the notion of a cusp form.

Definition 2.1.4 (Siegel's Φ-map) Siegel's Φ-map,

$$\Phi : \mathcal{M}_n^k \to \mathcal{M}_{n-1}^k,$$

is given by

$$\Phi(f)(\Omega) := \lim_{y\to\infty} f(\Omega \oplus iy).$$

Definition 2.1.5 (Siegel Cusp Forms) Let f be a Siegel modular form of degree n and weight k. We say f is a *Siegel cusp form* if f is in the kernel of Φ, i.e. if

$$\lim_{y\to\infty} f(\Omega \oplus iy) = 0 \text{ for all } \Omega \in \mathcal{H}_{n-1}.$$

2.2 Fourier Expansion of Siegel Modular Forms

In this section we will argue that any Siegel modular form $f \in \mathcal{M}_n$ has a unique Fourier series.

Recall that $\mathcal{V}_n = M_n^{\mathrm{sym}}(\mathbf{R})$. We introduce the following sets

$$\begin{aligned}
\mathcal{V}_n(\mathbf{Z}) &:= \mathcal{V}_n \cap M_n(\mathbf{Z}), \\
\mathcal{V}_n(\mathbf{Z})^* &:= \{v \in \mathcal{V}_n \mid v_{ii} \in \mathbf{Z} \text{ and } v_{ij} \in \tfrac{1}{2}\mathbf{Z} \text{ for all } i < j\}, \\
\mathcal{X}_n^{\mathrm{semi}} &:= \mathcal{V}_n(\mathbf{Z})^* \cap \overline{\mathcal{P}}_n, \text{ and} \\
\mathcal{X}_n &:= \mathcal{V}_n(\mathbf{Z})^* \cap \mathcal{P}_n.
\end{aligned}$$

We claim that f has a Fourier series of the form

$$f(\Omega) = \sum_{t \in \mathcal{X}_n^{\mathrm{semi}}} a(t; f)\mathbf{e}(\langle t, \Omega \rangle),$$

where the coefficients $a(t; f)$ are complex numbers indexed by the set $\mathcal{X}_n^{\text{semi}}$ and where

$$\mathbf{e}(\omega) := e^{2\pi i\omega} \text{ for all } \omega \in \mathbf{C}.$$

We begin by showing that Siegel modular forms are invariant under translation by integers in the entries of the input matrix. Consider the matrix t_ν with block form

$$t_\nu = \begin{pmatrix} I_n & \nu \\ 0 & I_n \end{pmatrix}$$

where $\nu \in \mathcal{V}_n(\mathbf{Z})$. Since t_ν has integer entries and since

$$J[t_\nu] = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix},$$

the matrix t_ν is in Γ_n. For any $\Omega \in \mathcal{H}_n$ we have

$$t_\nu \cdot \Omega = \Omega + \nu.$$

The factor of automorphy for t_ν is then $\det(I_n) = 1$. Hence, for any $f \in \mathcal{M}_n^k$ and $\Omega \in \mathcal{H}_n$,

$$f(\Omega + \nu) = f(t_\nu \cdot \Omega) = f(\Omega).$$

It follows that Siegel modular forms are invariant under translation by integers.

In particular, the holomorphic function f of the entries $\{\Omega_{jk} = x_{jk} + iy_{jk} \mid 1 \le j, k \le n\}$ is $\mathbf{Z}$-periodic and smooth and therefore has the unique Fourier expansion

$$f(\Omega) = \sum_{t_{jk} \in \mathbf{Z}^{n(n+1)/2}} a(\{t_{jk}, y_{jk}\}; f)\mathbf{e}\left(\sum_{j,k} t_{jk}x_{jk}\right).$$

By the Cauchy-Riemann equations for each index (j, k), we have that $a(\{t_{jk}, y_{jk}\}; f) = a(\{t_{jk}\}; f)\mathbf{e}(\sum_{j,k} t_{jk} iy_{jk})$. Thus

$$f(\Omega) = \sum_{t_{jk} \in \mathbf{Z}^{n(n+1)/2}} a(t_{jk}; f)\mathbf{e}\left(\sum_{j,k} t_{jk}\Omega_{jk}\right).$$

To each $n(n+1)/2$-tuple $\{t_{jk}\}$ we associate the symmetric matrix t having diagonal entries t_{jj} and superdiagonal entries $\frac{1}{2}t_{jk}$; thus $\sum_{j,k} t_{jk}\Omega_{jk} = \langle t, \Omega\rangle$. The matrices t vary over $\mathcal{V}_n(\mathbf{Z})^*$, and so the Fourier series of f is a sum over this set,

$$f(\Omega) = \sum_{t \in \mathcal{V}_n(\mathbf{Z})^*} a(t; f)\mathbf{e}(\langle t, \Omega\rangle).$$

The sum is not yet in the desired form since it is taken over $t \in \mathcal{V}_n(\mathbf{Z})^*$ instead of $t \in \mathcal{X}_n^{\text{semi}}$.

Let $v \in G_{\mathbf{Z}}$ be an $n \times n$ matrix and consider the matrix

$$u_v := \begin{pmatrix} v & 0 \\ 0 & v^* \end{pmatrix}$$

in block matrix form. Since $\det(u_v) = \det(v)\det(v)^{-1} = 1$, $u_v \in \Gamma_n$, and we have

$$\begin{aligned} f(\Omega[v^T]) &= f(u_v \cdot \Omega) \\ &= (\det v)^k f(\Omega). \end{aligned}$$

We use this identity to establish that the Fourier coefficients of a Siegel modular form f satisfy

$$a(t[v]; f) = (\det v)^k a(t; f).$$

Since $\det v = \pm 1$, $a(t; f)$ is class function for even k and in general, the zeroness of $a(t; f)$ is a class function for all k and $t \in \mathcal{X}_n^{\mathrm{semi}}$.

To see this, note that

$$\begin{aligned} f(\Omega[v^*]) &= \sum_{t \in \mathcal{V}_n(\mathbf{Z})^*} a(t; f)\mathbf{e}(\langle t, \Omega[v^*]\rangle) \\ &= \sum_{t \in \mathcal{V}_n(\mathbf{Z})^*} a(t; f)\mathbf{e}(\langle t[v^{-1}], \Omega\rangle) \\ &= \sum_{t \in \mathcal{V}_n(\mathbf{Z})^*} a(t[v]; f)\mathbf{e}(\langle t, \Omega\rangle), \end{aligned}$$

and hence

$$(\det v)^k f(\Omega) = \sum_{t \in \mathcal{V}_n(\mathbf{Z})^*} (\det v)^k a(t; f)\mathbf{e}(\langle t, \Omega\rangle),$$

with the coefficients of the two last series matching by the uniqueness of the Fourier series of f.

We will use the third condition of the definition of Siegel modular forms (that f is bounded on the set $\{\Omega \in \mathcal{H}_n \mid \mathrm{Im}(\Omega) > y_0\}$ for any $y_0 \in \mathcal{P}_n$) to show that the summation can be taken over just the set $\mathcal{X}_n^{\mathrm{semi}}$.

For any $v \in \mathbf{R}^n$, consider $f(iI_n + vv^T\zeta)$ as a holomorphic function of the single complex variable $\zeta \in \mathcal{H}_1$. The resulting Fourier series is:

$$f(iI_n + vv^T\zeta) = \sum_{t \in \mathcal{V}_n(\mathbf{Z})^*} a(t; f)e^{-2\pi \operatorname{tr} t} q^{t[v]} \text{ where } q = e^{2\pi i \zeta}.$$

Since $f(iI_n + vv^T\zeta)$ is bounded as $\mathrm{Im}(\zeta) \to \infty$, it follows that for any t such that $t[v] < 0$ the coefficient $a(t; f)e^{-2\pi \operatorname{tr} t}$ of $q^{t[v]}$ is zero. But since $e^{-2\pi \operatorname{tr} t} \neq 0$ and v is arbitrary, $a(t; f) \neq 0$ only if $t \in \mathcal{X}_n^{\text{semi}}$, i.e. if $t[v] \geq 0$. Hence, the Fourier expansion of f is as claimed,

$$f(s) = \sum_{t \in \mathcal{X}_n^{\text{semi}}} a(t; f)\mathbf{e}(\langle t, s \rangle).$$

We can see from the preceding argument that a Siegel modular form $f \in \mathcal{M}_n^k$ is a cusp form if and only if $a(t; f) = 0$ for all $t \in \mathcal{X}_n^{\text{semi}}$ for which $\det(t) = 0$.

We note that by the *Köcher Principle*, the third condition of the definition of Siegel modular forms is redundant when $n \geq 2$. A proof of this result is available in [Fre83] and [PSY].

2.3 The Semihull Theorem

Definition 2.3.1 (Support and Semihull) Let $f \in \mathcal{M}_n^k$. The *support* of f is the set of all indices in $\mathcal{X}_n^{\text{semi}}$ such that the associated Fourier coefficient of f is not equal to 0, i.e.

$$\operatorname{supp}(f) := \{t \in \mathcal{X}_n^{\text{semi}} \mid a(t; f) \neq 0\}.$$

The *semihull* of f, $\nu(f)$, is the $\mathcal{V}_n$-closure of the semihull of the support of f, that is

$$\nu(f) := \overline{\langle \mathbf{R}_{\geq 1} \operatorname{supp}(f) \rangle}.$$

We will use the semihull of a Siegel modular form as a way to characterise the vanishing of its Fourier coefficients. We give the following results about semihulls.

Proposition 2.3.2 *Let* $f \in \mathcal{M}_n^k$. *Then* $\nu(f)$ *is a kernel.*

Proof See [PSY]. ∎

Proposition 2.3.3 (Valuation Property of ν) *Let* $f, g \in \mathcal{M}_n^k$. *Then*

$$\nu(fg) = \nu(f) + \nu(g) \text{ and } \nu(f + g) \subseteq \nu(f) \cup \nu(g).$$

Proof See [PSY]. ∎

We can estimate the order of vanishing required for a cusp form to be identically zero using height functions. To that end, we cite the following results from [PY00].

Theorem 2.3.4 (Semihull Theorem) *Let* $f \in \mathcal{S}_n^k$ *be a nonzero Siegel cusp form. Let* $\Omega_0 = x_0 + iy_0$ *maximize* $\det(y)^{k/2}|f(\Omega)|$ *over* $\Omega = x + iy$ *in* $\mathcal{H}_n$. *Then*

$$\frac{k}{4\pi} y_0^{-1} \in \nu(f).$$

Proof See [PY00, Theorem 1.2] or [PSY]. ∎

Theorem 2.3.5 (Vanishing Theorem) *Let* $f \in \mathcal{S}_n^k$ *be a Siegel cusp form and let* S *be a containment for the semihull of* f*, i.e.*

$$S \supset \nu(f).$$

If the set

$$\left\{ \Omega = x + iy \in \mathcal{H}_n \mid \frac{k}{4\pi} y^{-1} \notin S \right\}$$

contains a fundamental domain for $\Gamma_n \backslash \mathcal{H}_n$*, then* $f = 0$.

Proof See [PY00, Theorem 1.6] or [PSY]. ∎

Theorem 2.3.6 (Dyadic Trace Extraction Theorem) *For any Siegel cusp form* $f \in \mathcal{S}_n^k$ *the following conditions are equivalent:*

1. $f = 0$,
2. $a(t; f) = 0$ *for all* $t \in \mathcal{X}_n$ *such that* $w(t) < \frac{k}{2\sqrt{3}\pi} n$,
3. $a(t; f) = 0$ *for all* $t \in \mathcal{X}_n$ *such that* $w(t) < k c_n(w)$,

where

$$c_n(w) := \frac{1}{4\pi} \sup_{\Omega \in \mathcal{H}_n} \inf_{g \in \Gamma_n} w(\mathrm{Im}(g \cdot \Omega)^{-1}).$$

Proof See [PY00, Theorem 2.9] or [PSY]. ∎

A variant of the extraction theorem holds for all height functions, but the tightest bounds seem to be obtained using the dyadic trace.

In general, computing $c_n(w)$ is difficult and, in fact, computing the optimal value is an open problem for all n. It is, however, known that for $n = 2$ the third condition from the Extraction Theorem holds if we let $c_2(w) = \frac{1}{6}$.

2.4 Generators for the Siegel Modular Forms of Degree Two

For a general n determining the structure of the graded ring of Siegel modular forms $\mathcal{M}_n$ is an open problem. One exception is when $n = 2$, in which case the generators of $\mathcal{M}_n$ as a module over $\mathbf{C}$ are known. We cite the following results due to Jun-ichi Igusa.

Theorem 2.4.1 (Igusa 1962) *The graded ring of even weight Siegel modular forms in degree* 2, $\mathcal{M}_2^{\text{even}}$, *is generated over* $\mathbf{C}$ *by the four modular forms* E_4, E_6, χ_{10} *and* χ_{12} *of weight* 4, 6, 10 *and* 12 *respectively. The modular forms* χ_{10} *and* χ_{12} *are cusp forms, while* E_4 *and* E_6 *are not. Furthermore, all four modular forms are algebraically independent over* $\mathbf{C}$.

PROOF See [Igu62, Theorem 1]. ∎

Corollary 2.4.2 *Let n be a positive even integer. The dimension of the vector space of weight k Siegel modular forms in degree 2 is the same as the number of nonnegative integer solutions to the equation*

$$4i + 6j + 10k + 12l = k.$$

We also have a structure theorem for the degree 2 Siegel modular forms with integer Fourier coefficients. These modular forms are more convenient for computations, and as we will later see, they appear in the estimate for the extreme core.

Theorem 2.4.3 (Igusa 1979) *Let E_4, E_6, χ_{10}, χ_{12} be the Siegel modular forms from the previous theorem normalized so that they have Fourier series with integer coefficients and content 1. The graded ring of even weight Siegel modular forms with integer coefficients is generated over $\mathbf{Z}$ by the fourteen modular forms*

$$\begin{aligned}
X_4 &= E_4 & X_6 &= E_6 \\
X_{10} &= \chi_{10} & X_{12} &= \chi_{12} \\
Y_{12} &= 2^{-6}3^{-3}(X_4^3 - X_6^2) + 2^4 3^2 X_{12} & X_{16} &= 2^{-2}3^{-1}(X_4X_{12} - X_6X_{10}) \\
X_{18} &= 2^{-2}3^{-1}(X_6X_{12} - X_4^2X_{10}) & X_{24} &= 2^{-3}3^{-1}(X_{12}^2 - X_4X_{10}^2) \\
X_{28} &= 2^{-1}3^{-1}(X_4X_{24} - X_{10}X_{18}) & X_{30} &= 2^{-1}3^{-1}(X_6X_{24} - X_4X_{10}X_{16}) \\
X_{36} &= 2^{-1}3^{-2}(X_{12}X_{24} - X_{10}^2X_{16}) & X_{40} &= 2^{-2}(X_4X_{36} - X_{10}X_{30}) \\
X_{42} &= 2^{-2}3^{-1}(X_{12}X_{30} - X_4X_{10}X_{28}) & X_{48} &= 2^{-2}(X_{12}X_{36} - X_{24}^2).
\end{aligned}$$

PROOF See [Igu79, Theorem 1]. ∎

The Extreme Core

3

3.1 Cores and Cocores

Definition 3.1.1 A *core* is a kernel dilationally comparable to $\mathcal{K}_m$. A *cocore* is a kernel dilationally comparable to $\mathcal{K}_w$.

Proposition 3.1.2 *A $G_{\mathbf{Z}}$-invariant kernel that is contained in a core is a core.*

Proof Let $\mathcal{K}$ be a $G_{\mathbf{Z}}$-invariant kernel and suppose that $\mathcal{K} \subseteq \mathcal{C}$ where $\mathcal{C}$ is a core. We want to show that there exist $\alpha, \beta \in \mathbf{R}_{>0}$ such that $\alpha\mathcal{K} \subseteq \mathcal{K}_m$ and $\mathcal{K} \subseteq \beta\mathcal{K}_m$. Since $\mathcal{C}$ is a core, there exists an $\alpha \in \mathbf{R}_{>0}$ such that $\alpha\mathcal{C} \subseteq \mathcal{K}_m$, but since $\mathcal{K} \subseteq \mathcal{C}$, we also have $\alpha\mathcal{K} \subseteq \mathcal{K}_m$.

Let S_p be the set consisting of a single representative from each equivalence class of the set of perfect forms, and note that this set is finite by the finiteness of perfect classes. Since $\mathbf{R}_{>0}\mathcal{K} \supseteq \mathcal{P}_n$, for each $s_i \in S$ there exists a $\beta_i \in \mathbf{R}_{>0}$ such that $\beta_i s_i \in \mathcal{K}$. Letting $\beta := \min\{\beta_i\}$ it follows that $\mathcal{K}$ contains βS and therefore, by $G_{\mathbf{Z}}$-invariance, convex closure, and closure under $\mathbf{R}_{\geq 1}$-dilations, all of βK_m. ∎

3.2 The Extreme Core

The Valence Inequality is a well-known result from the theory of elliptic modular forms.

Theorem 3.2.1 (Valence Inequality) *Let $f \neq 0$ be a classical cusp form of weight k. We have*

$$\nu_\infty(f) \leq \frac{k}{12},$$

where $\nu_\infty(f)$ is the order of vanishing of f at infinity.

Proof See [Ser73, Chapter VII, Section 3, Theorem 3]. ∎

Due to the connection between the order of vanishing at infinity and the Fourier series of a function, we get the following corollary.

Corollary 3.2.2 *Let $f \neq 0$ be a classical cusp form of weight k. Then in the Fourier series expansion of f,*

$$f(\tau) = \sum_{j=1}^{\infty} a(j; f) q^j,$$

at least one of the $a(j; f)$ for $j \leq \lfloor \frac{k}{12} \rfloor$ is nonzero.

Our goal is to obtain an analogous result for Siegel modular forms. Note that by Theorem 2.3.6, the Dyadic Trace Extraction Theorem, we know that for a cusp form $f \in \mathcal{S}_n^k$,

$$f = 0 \text{ if and only if } a(t; f) = 0 \text{ for all } t \in \mathcal{X}_n \text{ such that } w(t) < \frac{k}{2\sqrt{3}\pi} n.$$

However, this result is dependent on our choice of height function (in this case, the dyadic trace) and does not intrinsically characterise the nonvanishing part of the support of Siegel cusp forms. To remedy this, we introduce the extreme core.

Definition 3.2.3 The *extreme core* in degree n is the weighted intersection of the semihulls of the nonzero degree n Siegel cusp forms, i.e.

$$\mathcal{C}_{\text{ext}} := \bigcap_{f \in \mathcal{S}_n \setminus \{0\}} \frac{1}{k} \nu(f) \quad \text{where } k \text{ is the weight of } f.$$

It follows that for any $f \in \mathcal{S}_n^k$, $\nu(f) \supseteq k\mathcal{C}_{\text{ext}}$. This is precisely the Valence Inequality-like result we were looking for.

As its name suggests, the extreme core is a core (i.e. it is a kernel dilationally comparable to $\mathcal{K}_m$). For a proof of this fact see [PY05, Corollary 3.14]. Combining the Valence Inequality with the existence of the weight 12 classical cusp form Δ, the modular discriminant, it is easy to see that in the language of this chapter, the extreme core for degree $n = 1$ is precisely the set $\left[\frac{1}{12}, \infty\right)$.

The geometry becomes more interesting for $n \geq 2$; the extreme core is then a kernel, not just a ray, and it is no longer the case that the extreme core is cut out by a single function.

Next, we introduce the kernel generated by a form, which we will later use to estimate the extreme core. While not all kernels are principally (or even finitely) generated, computations involving principally generated kernels are simpler, which makes these kernels particularly useful.

Definition 3.2.4 For $s \in \overline{\mathcal{P}}_n$ we define

$$\mathcal{K}(s) := \overline{\langle \mathbf{R}_{\geq 1} s[G_{\mathbf{Z}}] \rangle}$$

to be the $G_{\mathbf{Z}}$-invariant semihull generated by s.

Proposition 3.2.5 *Let* $s \in \mathcal{P}_n^*$. *Then* $\mathcal{K}(s)$ *is a kernel.*

Proof It is clear from the definition of semihull that $\mathcal{K}(s)$ is convex and closed under $\mathbf{R}_{\geq 1}$-dilation.

Since w is a height function, it attains its infimum on $\mathcal{K}(s)$ on the vertices. Combined with the fact that w is a class function, we have that

$$\mathcal{K}(s) \subseteq w(s)\mathcal{K}_w.$$

It follows that since $0 \notin \overline{\mathcal{K}}_w$, $0 \notin \overline{\mathcal{K}(s)}$.

It remains only to show that $\mathcal{K}(s)$ is wide, i.e. that $\mathcal{P}_n \subseteq \mathbf{R}_{>0}\mathcal{K}(s)$. This is clear for $n = 1$, so we can assume $n \geq 2$. Since $s \in \mathcal{P}_n^*$, there exists a $z \in \mathbf{Z}^n$ such that $s[z] > 0$, and we can replace s with $s[v]$, where v is a $G_{\mathbf{Z}}$ matrix with z as its second column. This ensures that $s_{22} > 0$.

First, we want to show that $s + t \in \mathcal{K}(s)$ for all $t \in \mathcal{P}_n$. Since all matrices of the form $s + t$ can be written as a convex combination

$$\frac{1}{k} \sum_{z \in \mathbf{Z}^n \setminus \{0\}} \left(s + \alpha_z z z^T\right)$$

where only k of the α_z's are nonzero, it suffices to show that $s + \alpha_z z z^T \in \mathcal{K}(s)$ for $\alpha_z \in \mathbf{R}_{>0}$ and $z \in \mathbf{Z}^n \setminus \{0\}$. For any $m \in \mathbf{Z}_{\geq 1}$ let $g_m = m e_{21} + I_n \in G_{\mathbf{Z}}$. Then $S[g_m] \in \mathcal{K}(s)$. Note that $S[g_m] = m^2 s_{22} e_{11} + O(m)$. There exists a $g \in G_{\mathbf{Z}}$ such that $g e_1 = z$, so that $e_{11}[g^T] = g e_1 e_1^T g^T = z z^T$. Thus $s[g_m g^T]$ lies in $\mathcal{K}(s)$ and

$$\frac{\alpha_z}{m^2 s_{22}} s[g_m g] = \alpha_z z z^T + O\left(\frac{1}{m}\right).$$

Since $\mathcal{K}(s)$ is closed under the metric on $\mathcal{V}_n$ and closed under superconvex combinations,

$$\lim_{m \to \infty} \left(s + \frac{\alpha_z}{m^2 s_{22}} s[g_m g]\right) = s + \alpha_z z z^T$$

is in $\mathcal{K}(s)$. Therefore $s + t \in \mathcal{K}(s)$. This completes the proof that $\mathcal{P}_n \subseteq \mathbf{R}_{>0}\mathcal{K}(s)$ since for any $t \in \mathcal{P}_n$, there exists a $\lambda \in \mathbf{R}_{>0}$ such that $\lambda t - s \in \mathcal{P}_n$. Therefore, $(\lambda t - s) + s = \lambda t \in \mathcal{K}(s)$. ∎

Corollary 3.2.6 *$\mathcal{K}(s)$ is a core if and only if $s \in \mathcal{P}_n$.*

Proof If $\mathcal{K}(s)$ is a core, then $\alpha\mathcal{K}(s) \subseteq \mathcal{K}_m$ for some $\alpha \in \mathbf{R}_{>0}$ and therefore $\mathcal{K}(s) \subseteq \mathcal{P}_n$ as well.

Conversely, if s is positive definite, $m(s) > 0$ and therefore $\mathcal{K}(s) \subseteq m(s)\mathcal{K}_s$ since $s \in m(s)\mathcal{K}(s)$. However, by Proposition 3.1.2, a kernel contained in a core is a core and thus we are done. ∎

Next, we prove some technical results which we will need in order to get the first estimate for the extreme core.

Proposition 3.2.7 *For all $s, t \in \mathcal{P}_n$ we have*

$$\varphi_{\mathcal{K}(s)}(t) = \inf_{u \in \mathcal{P}_n^* \setminus \{0\}} \frac{\widehat{\varphi}_{\mathcal{K}(u)}(t)}{\widehat{\varphi}_{\mathcal{K}(u)}(s)} = \inf_{u \in \mathcal{P}_n} \frac{\widehat{\varphi}_{\mathcal{K}(u)}(t)}{\widehat{\varphi}_{\mathcal{K}(u)}(s)}.$$

PROOF See [PY05, Proposition 3.9]. ∎

Lemma 3.2.8 *For* $s, t \in \mathcal{P}_n^*$ *we have* $\widehat{\varphi}_{\mathcal{K}(s)}(t) = \widehat{\varphi}_{\mathcal{K}(t)}(s) = \inf\langle s[G_Z], t[G_Z]\rangle$.

PROOF We have

$$\begin{aligned}
\widehat{\varphi}_{\mathcal{K}(s)}(t) &= \inf_{r\in\mathcal{P}_n} \langle r, t\rangle / \varphi_{\mathcal{K}(s)}(r) && \text{definition of dual function}\\
&= \inf_{r\in\mathcal{P}_n} \left\langle r/\varphi_{\mathcal{K}(s)}(r), t\right\rangle && \text{by the linearity of the trace}\\
&= \inf_{r\in\mathcal{K}(s)} \langle r, t\rangle && \text{since } \mathcal{K}(s) \text{ is a kernel}\\
&= \inf\langle \mathcal{K}(s), t\rangle \\
&= \inf\langle \mathcal{K}(s), \mathcal{K}(t)\rangle && \text{by the cyclity of the trace.}
\end{aligned}$$

∎

Definition 3.2.9 For $s, t \in \mathcal{P}_n$ we define

$$\eta(s,t) := \inf_{u\in\mathcal{P}_n} \varphi_{\mathcal{K}(u)}(s)\widehat{\varphi}_{\mathcal{K}(u)}(t).$$

Lemma 3.2.10 *Let* $s, t \in \mathcal{P}_n$. *If* $R \subseteq \mathcal{P}_n^*$ *is such that*

$$\varphi_{\mathcal{K}(s)}(t) = \inf_{r\in R} \frac{\widehat{\varphi}_{\mathcal{K}(r)}(t)}{\widehat{\varphi}_{\mathcal{K}(r)}(s)},$$

then

$$\eta(s,t) = \inf_{r\in R} \varphi_{\mathcal{K}(r)}(t)\widehat{\varphi}_{\mathcal{K}(r)}(s).$$

PROOF We have

$$\begin{aligned}
\eta(s,t) &= \inf_{u\in\mathcal{P}_n} \varphi_{\mathcal{K}(u)}(t)\widehat{\varphi}_{\mathcal{K}(u)}(s)\\
&= \inf_{u\in\mathcal{P}_n, r\in R} \frac{\widehat{\varphi}_{\mathcal{K}(r)}(t)}{\widehat{\varphi}_{\mathcal{K}(r)}(u)}\widehat{\varphi}_{\mathcal{K}(u)}(s)\\
&= \inf_{r\in R} \widehat{\varphi}_{\mathcal{K}(r)}(t) \inf_{u\in\mathcal{P}_n} \frac{\widehat{\varphi}_{\mathcal{K}(u)}(s)}{\widehat{\varphi}_{\mathcal{K}(r)}(u)}\\
&= \inf_{r\in R} \widehat{\varphi}_{\mathcal{K}(r)}(t) \inf_{u\in\mathcal{P}_n} \frac{\widehat{\varphi}_{\mathcal{K}(u)}(s)}{\widehat{\varphi}_{\mathcal{K}(u)}(r)} && \text{by Lemma 3.2.8}\\
&= \inf_{r\in R} \widehat{\varphi}_{\mathcal{K}(r)}(t)\varphi_{\mathcal{K}(r)}(s).
\end{aligned}$$

∎

We are now ready to prove the first main result of this chapter.

Proposition 3.2.11 *Let* $r, s, t \in \mathcal{P}_n$. *Then*

$$\frac{\widehat{\varphi}_{\mathcal{K}(t)}(r)}{\eta(s,t)}\mathcal{K}(s) \subseteq \mathcal{K}(r).$$

Proof It suffices to show that

$$\frac{\widehat{\varphi}_{\mathcal{K}(t)}(r)}{\eta(s,t)} \geq \frac{1}{\varphi_{\mathcal{K}(r)}(s)}.$$

However, $\eta(s,t) \leq \varphi_{\mathcal{K}(r)}(s)\widehat{\varphi}_{\mathcal{K}(r)}(t)$ from the definition of η and Lemma 3.2.8. ∎

Theorem 3.2.12 *Let* $t \in \mathcal{P}_n$ *and let* $\gamma \in \mathbf{R}_{>0}$. *Let* f *be a nonzero weight* k *cusp form and suppose that* $\inf\langle t, \nu(f)\rangle \leq \gamma k$. *Then for all* $s \in \mathcal{P}_n$ *we have*

$$\frac{\gamma}{\eta(s,t)}\mathcal{K}(s) \subseteq \frac{1}{k}\nu(f).$$

Proof Let r be an element of $\operatorname{supp}(f)$ such that $\langle t, r\rangle = \inf\langle t, \nu(f)\rangle$. By our hypothesis and the previous proposition we get

$$\frac{\gamma k}{\eta(s,t)}\mathcal{K}(s) \subseteq \frac{\langle t, r\rangle}{\eta(s,t)}\mathcal{K}(s) \subseteq \frac{\widehat{\varphi}_{\mathcal{K}(t)}(r)}{\eta(s,t)}\mathcal{K}(s) \subseteq \mathcal{K}(r) \subseteq \nu(f).$$ ∎

Our next goal is to improve on Theorem 3.2.12 in two ways. First, we would like to estimate $\mathcal{C}_{\text{ext}}$ using a kernel that is not necessarily principally generated, and second, we would like to make the computation of η more manageable. To that end, we introduce the *noble forms*. We will be using the fact that the perfect forms up to degree $n = 5$ are noble in order get an estimate for the extreme core.

Let $t \in \overline{\mathcal{P}}_n$. We define $\operatorname{Aut}_{\mathbf{Z}}(t)$ to be the t-stabilizer subgroup of $G_{\mathbf{Z}}$, i.e.

$$\operatorname{Aut}_{\mathbf{Z}}(t) := \{g \in GL_n(\mathbf{Z}) \mid t[g] = t\}.$$

$\operatorname{Aut}_{\mathbf{Z}}(t)$ can be infinite, but for invertible forms it is always finite. In addition,

$$\operatorname{Aut}_{\mathbf{Z}}(t^{-1}) = \{g^* \mid g \in \operatorname{Aut}_{\mathbf{Z}}(t)\}.$$

The group $\operatorname{Aut}_{\mathbf{Z}}(s)$ depends only on the $\mathbf{R}$-dilation class of s, i.e. $\operatorname{Aut}_{\mathbf{Z}}(s) = \operatorname{Aut}_{\mathbf{Z}}(\lambda s)$ for all $\lambda \in \mathbf{R}$.

Definition 3.2.13 (Noble Forms) An element $s \in \mathcal{P}_n^* \setminus \{0\}$ is called *noble* if for all $t \in \overline{\mathcal{P}}_n$,

$$\operatorname{Aut}_{\mathbf{Z}}(s) \subseteq \operatorname{Aut}_{\mathbf{Z}}(t) \text{ implies } t = \lambda s \text{ for some } \lambda \in \mathbf{R}.$$

We can think of the noble forms as matrices with maximal stabilizer subgroups.

The noble forms have been classified for $n \leq 11$ by Wilhelm Plesken and Michael Pohst ([PP77], [PP80]), and Bernd Souvignier ([Sou94]). The perfect forms are noble for degree up to $n = 5$, but when $n = 6$, $A_{6,1}$ and $A_{6,2}$ are perfect forms that are not noble (see [Mar03] for a definition of these forms).

Let $S, T \subseteq \overline{\mathcal{P}}_n$. We extend the definitions of $\mathcal{K}$ and η in the obvious fashion, letting

$$\mathcal{K}(S) := \overline{\langle \mathbf{R}_{\geq 1} \cup_{s \in S} \mathcal{K}(s) \rangle},$$

and

$$\eta(S, T) := \inf_{s \in S, t \in T} \eta(s, t).$$

Combining these definitions with the properties of noble forms leads to the following results.

Theorem 3.2.14 *Let* f *be a nonzero cusp form of weight* k *and let* $S, T \subseteq \mathcal{P}_n$ *be such that* $\eta(S, T) > 0$. *Let* $\gamma \in \mathbf{R}_{>0}$ *satisfy* $\inf\langle S, \nu(f)\rangle \leq \gamma k$. *Then we have*

$$\frac{\gamma}{\eta(S, T)} \mathcal{K}(T) \subseteq \frac{1}{k} \nu(f).$$

Proof See [PY05, Theorem 4.9]. ∎

To use the preceeding theorem, we need to be able to compute $\eta(S, T)$.

Proposition 3.2.15 *Let* S, T *be sets of noble forms in* $\mathcal{P}_n^*$. *Then*

$$\eta(S, T) = \inf_{s \in S, t \in T} \frac{n^2}{\widehat{\varphi}_{\mathcal{K}(s^*)}(t^*)}.$$

Proof See [PY05, Proposition 4.10]. ∎

We use Theorem 3.2.14 to find two kernels that are guaranteed to be contained in the extreme core for $n = 2, 3, 4, 5$. For the first case, we will take $S = T = S_p$, the set of perfect classes, and in the second $S = S_p$ and $T = \{I_n\}$. By Theorem 2.3.6, we can take $\gamma = \frac{n}{2\sqrt{3}\pi}$, since $\inf\langle S_p, \nu(f)\rangle = \inf w(\nu(f))$.

n	$\mathcal{K}_m$ estimate	$\mathcal{K}(I_n)$ estimate
2	$\frac{2}{3\sqrt{3}\pi}\mathcal{K}\left(\frac{1}{2}A_2\right)$	$\frac{2}{3\sqrt{3}\pi}\mathcal{K}(I_2)$
3	$\frac{5}{6\sqrt{3}\pi}\mathcal{K}\left(\frac{1}{3}A_3\right)$	$\frac{3}{4\sqrt{3}\pi}\mathcal{K}(I_3)$
4	$\frac{11}{10\sqrt{3}\pi}\mathcal{K}\left(\frac{1}{2}D_4, \frac{1}{2}A_4\right)$	$\frac{1}{\sqrt{3}\pi}\mathcal{K}(I_4)$
5	$\frac{13}{10\sqrt{3}\pi}\mathcal{K}\left(\frac{1}{2}D_5, \frac{1}{4}A_5^3, \frac{1}{2}A_5\right)$	$\frac{21}{20\sqrt{3}\pi}\mathcal{K}(I_5)$

Table 3.1: Table of inner containments for $\mathcal{C}_{\text{ext}}$ from [PY05] using the form names from [Mar03].

3.3 The Degree Two Extreme Core

We can further strengthen our results when $n = 2$ by using Legendre-reduction and the small dimension of $\mathcal{P}_2$. In particular, the following two propositions are useful for calculations in degree $n = 2$.

Proposition 3.3.1 *For all* $t \in \mathcal{P}_2^* \setminus \{0\}$ *and* $s \in \mathcal{P}_2$ *we have*

$$\varphi_{\mathcal{K}(s)}(t) = \inf_{u \in \{I_2, A_2, e_1 e_1^T\}} \frac{\widehat{\varphi}_{\mathcal{K}(u)}(t)}{\widehat{\varphi}_{\mathcal{K}(u)}(s)} = \min\left(\frac{m(t)}{m(s)}, \frac{w(t)}{w(s)}, \frac{\widetilde{\operatorname{tr}}(t)}{\widetilde{\operatorname{tr}}(s)}\right)$$

and

$$\eta(s,t) = \inf_{u \in \{I_2, A_2, e_1 e_1^T\}} \varphi_{\mathcal{K}(u)}(s)\widehat{\varphi}_{\mathcal{K}(u)}(t) = \min\left(w(t)m(s), m(t)w(s)\right).$$

Proof We have

$$\varphi_{\mathcal{K}(s)}(t) = \inf_{u \in \mathcal{P}_2^* \setminus \{0\}} \frac{\widehat{\varphi}_{\mathcal{K}(u)}(t)}{\widehat{\varphi}_{\mathcal{K}(u)}(s)} = \inf_{u \in \mathcal{P}_2^* \setminus \{0\}} \frac{\inf\langle u[G_Z], t\rangle}{\inf\langle u[G_Z], s\rangle}.$$

Note that since we are taking infimums over equivalence classes of u in the numerator and denominator, it suffices to take the outer infimum over just representatives of each class, say from the cone of Legendre-reduced forms given by $\mathcal{R}_2 = \left\{ \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \in \mathcal{P}_2^* \,\middle|\, 0 \le 2\beta \le \alpha \le \gamma \right\}$. In addition, note that since the trace is a linear function and since we are taking the trace of multiples of u in both the numerator and denominator, we may freely rescale u and therefore can also restrict the infimum to just $u \in \partial\mathcal{K}_w$.

Thus we have

$$\varphi_{\mathcal{K}(s)}(t) = \inf_{u \in \mathcal{R}_2 \cap \partial\mathcal{K}_w} \frac{\inf\langle u[G_Z], t\rangle}{\inf\langle u[G_Z], s\rangle},$$

which we can rewrite as

$$\varphi_{\mathcal{K}(s)}(t) = \inf_{u \in \mathcal{R}_2 \cap \partial\mathcal{K}_w} \frac{\inf\langle u, t[G_Z]\rangle}{\inf\langle u, s[G_Z]\rangle}$$

since the trace is cyclic.

Let $\widetilde{\mathcal{R}}_2 = \left\{ \begin{pmatrix} \gamma & -\beta \\ -\beta & \alpha \end{pmatrix} \in \mathcal{P}_2^* \,\middle|\, 0 \le 2\beta \le \alpha \le \gamma \right\}$. We can verify that $\langle x, y\rangle \le \langle x, y[g]\rangle$ for all $x \in \mathcal{R}_2$, $y \in \widetilde{\mathcal{R}}_2$ and $g \in G_Z$ by checking the inequality for the generators (as cones) of $\mathcal{R}_2$ and $\widetilde{\mathcal{R}}_2$, $\{e_{22}, A_2, I_2\}$ and $\{e_{11}, A_2^*, I_2\}$ respectively. We therefore know that the inner infimums are attained since s and t have unique representatives in $\widetilde{\mathcal{R}}_2$ (we know they have unique Legendre-reduced forms and we have the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ whose action gives us a bijection between $\mathcal{R}_2$ and $\widetilde{\mathcal{R}}_2$). Thus there exist $s', t' \in \widetilde{\mathcal{R}}_2$ such that

$$\varphi_{\mathcal{K}s}(t) = \inf_{u \in \mathcal{R}_2 \cap \partial \mathcal{K}_w} \frac{\langle u, t' \rangle}{\langle u, s' \rangle}.$$

Since $\frac{\langle u,t' \rangle}{\langle u,s' \rangle}$ is a linear fractional transformation in the entries of u, it is quasilinear and therefore it attains its infimum on any line segment on the endpoints. This in turn implies that it attains its infimum on the vertices of the triangle $\mathcal{R}_2 \cup \partial \mathcal{K}_w$, $\{e_{22}, \frac{1}{3}A_2, \frac{1}{2}I_2\}$. Hence

$$\begin{aligned}
\varphi_{\mathcal{K}(s)}(t) &= \inf_{u \in \mathcal{R}_2 \cap \partial \mathcal{K}_w} \frac{\langle u, t' \rangle}{\langle u, s' \rangle} \\
&= \inf_{u \in \{e_{22}, \frac{1}{3}A_2, \frac{1}{2}I_2\}} \frac{\langle u, t' \rangle}{\langle u, s' \rangle} \\
&= \min\left(\frac{m(t')}{m(s')}, \frac{w(t')}{w(s')}, \frac{\widetilde{\operatorname{tr}}(t')}{\widetilde{\operatorname{tr}}(s')}\right) \\
&= \min\left(\frac{m(t)}{m(s)}, \frac{w(t)}{w(s)}, \frac{\widetilde{\operatorname{tr}}(t)}{\widetilde{\operatorname{tr}}(s)}\right)
\end{aligned}$$

with the last equality following from the fact that m, w and $\widetilde{\operatorname{tr}}$ are class functions.

However, by Lemma 3.2.10, this implies that we can calculate $\eta(s,t)$ by also restricting the infimum to the endpoints:

$$\eta(s,t) = \inf_{u \in \{e_{22}, \frac{1}{2}A_2, I_2\}} \varphi_{\mathcal{K}(u)}(s) \widehat{\varphi}_{\mathcal{K}(u)}(t) = \min\left(w(s)m(t), m(s)w(t), \widetilde{\operatorname{tr}}(t)\varphi_{\mathcal{K}(I_2)}(s)\right),$$

so it remains only to show that we can omit $\widetilde{\operatorname{tr}}(t)\varphi_{\mathcal{K}(I_2)}(s)$. Notice that

$$\varphi_{\mathcal{K}(I_2)}(s) = \min\left(\frac{m(s)}{m(I_2)}, \frac{w(s)}{w(I_2)}, \frac{\widetilde{\operatorname{tr}}(s)}{\widetilde{\operatorname{tr}}(I_2)}\right) = \min\left(m(s), \frac{1}{2}w(s), \frac{1}{2}\widetilde{\operatorname{tr}}(s)\right)$$

and that by the formulas for w, m and $\widetilde{\operatorname{tr}}$ of a Legendre-reduced form, it is easy to check that $\widetilde{\operatorname{tr}}(t)m(s) \geq m(s)w(t)$, $\widetilde{\operatorname{tr}}(t)\frac{1}{2}w(s) \geq w(s)m(t)$ and $\widetilde{\operatorname{tr}}(t)\frac{1}{2}\widetilde{\operatorname{tr}}(s) \geq w(s)m(t)$. ∎

Proposition 3.3.2 *For all* $s, t \in \overline{\mathcal{P}}_2 \backslash \{0\}$, *there exists a* $u \in \mathcal{P}_2^*$ *such that* $\mathcal{K}(s) \cap \mathcal{K}(t) = \mathcal{K}(u)$.

PROOF Let

$$\begin{aligned}
u_{11} &= \max(m(s), m(t)) \\
u_{22} &= \max(\widetilde{\operatorname{tr}}(s), \widetilde{\operatorname{tr}}(t)) - u_{11} \\
u_{12} &= u_{11} + u_{22} - \max(w(s), w(t)).
\end{aligned}$$

Note that u is Legendre-reduced. Given a $v \in \mathcal{P}_2^*$, $v \in \mathcal{K}(u)$ if and only if $\varphi_{\mathcal{K}(u)}(v) \geq 1$ which, by Proposition 3.3.1, is true if and only if $m(v) \geq m(u)$, $\widetilde{\operatorname{tr}}(v) \geq \widetilde{\operatorname{tr}}(u)$ and $w(v) \geq w(u)$. However, since $m(u) = \max(m(s), m(t))$, $\widetilde{\operatorname{tr}}(u) = \max(\widetilde{\operatorname{tr}}(s), \widetilde{\operatorname{tr}}(t))$ and $w(u) = \max(w(s), w(t))$, this is true if and only if $v \in \mathcal{K}(s)$ and $v \in \mathcal{K}(t)$. ∎

Finally, we cite the main result from [PY05] about the extreme core of degree 2 Siegel cusp forms.

Theorem 3.3.3 *Let* $n = 2$. *Then*

$$\mathcal{K}\left[\frac{1}{10}\begin{pmatrix} 1 & 1/3 \\ 1/3 & 1 \end{pmatrix}\right] \subseteq \mathcal{C}_{\text{ext}} \subseteq \mathcal{K}\left[\frac{1}{10}\begin{pmatrix} 1 & 17/36 \\ 17/36 & 1 \end{pmatrix}\right].$$

PROOF See [PY05, Section 5]. ∎

In the following chapter we will present a method for finding the optimal outer containment for $\mathcal{C}_{\text{ext}}$ given by weight k cusp forms and will show that the preceding outer containment is optimal for forms of even weight up to 44.

Computing the Extreme Core in Degree Two

4

4.1 Finding the Semihull of a Cusp Form

We present an algorithm to compute the semihull of a degree 2 cusp form f with a truncated Fourier expansion F by finding a minimal set S such that $\nu(F) = \mathcal{K}(S)$.

While our Fourier expansion is accurate only up to some dyadic trace, most forms of larger dyadic trace will be inside the kernels cut out by the forms of smaller height (q.v. Proposition 3.3.1) and so we would still expect to capture most of the forms in the semihull of f. Unfortunately, this need not be all of them. We will later show that there exists a class of forms whose presence in the semihull of a given cusp form can not be determined from just looking at a finite Fourier expansion. Therefore, it is important to note that this method is merely a heuristic—while we can guarantee that the kernel computed by our algorithm is contained in the semihull of f, we can not prove we have captured all forms of large dyadic trace in $\nu(f)$. The algorithm is as follows:

1. Initialize an empty array S. For every term in F, find its corresponding index matrix and Legendre-reduce it. Add the matrix to S, avoiding duplicates.
2. The goal is to remove redundancies from S, i.e., to remove all forms that are in the kernel generated by some other form in S. Sort S in ascending order with the dyadic trace as the first key, the reduced trace as the second key, and the minimum function as the third key. At this point, by Proposition 3.3.1 it follows that no form is contained in the kernel of a form in S with a larger index.
3. For all i, j such that $j > i$ check whether $S[j]$ is in the kernel generated by $S[i]$ by using Proposition 3.3.1. If $S[j] \in \mathcal{K}(S[i])$, then remove $S[j]$ from S. While the worst case running time for this procedure is quadratic in the size of S, in most cases all but a few of the entries of S will be removed after checking for $S[j]$ contained in $S[1]$.
4. $\mathcal{K}(S)$ now equals $\nu(F)$.

The function `get_support` in `support.sage` implements this algorithm. In `igusa.sage` we use this function to compute the support of eleven of Igusa's generators for the ring of degree two modular forms with integer coefficients. We are interested in these cusp forms because it is their semihull that is used in [PY05] to obtain an outer bound for the extreme core ($\mathcal{C}_{\text{ext}} \subseteq (1/10)\nu(X_{10}) \cap (1/36)\nu(X_{36})$) and so it is natural to ask whether these cusp forms give the optimal containment of $\mathcal{C}_{\text{ext}}$ for a given weight. We will later see they do not.

f	$\nu(f)$	f	$\nu(f)$
X_{10}	$\mathcal{K}\left[\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}\right]$	X_{12}	$\mathcal{K}\left[\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}\right]$
X_{16}	$\mathcal{K}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]$	X_{18}	$\mathcal{K}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]$
X_{24}	$\mathcal{K}\left[\begin{pmatrix} 2 & 1/2 \\ 1/2 & 2 \end{pmatrix}\right]$	X_{28}	$\mathcal{K}\left[\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\right]$
X_{30}	$\mathcal{K}\left[\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\right]$	X_{36}	$\mathcal{K}\left[\begin{pmatrix} 3 & 1/2 \\ 1/2 & 3 \end{pmatrix}\right]$
X_{40}	$\mathcal{K}\left[\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 3/2 \\ 3/2 & 4 \end{pmatrix}\right]$	X_{42}	$\mathcal{K}\left[\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 3/2 \\ 3/2 & 4 \end{pmatrix}\right]$
X_{48}	$\mathcal{K}\left[\begin{pmatrix} 4 & 1/2 \\ 1/2 & 4 \end{pmatrix}, \begin{pmatrix} 4 & 2 \\ 2 & 5 \end{pmatrix}\right]$		

Table 4.1: The semihulls of Igusa's even weight cusp form generators over $\mathbf{Z}$. The shaded entry is outside the accuracy range of our Fourier series and therefore might be incorrect.

Since we are using finite Fourier expansions, our results for the semihull of f are accurate only up to a certain height. Running the algorithm will capture most of the forms in $\nu(f)$, but we can not guarantee that $\mathcal{K}(S)$ contains all of $\nu(f)$. In particular, note that given a prospective semihull $\mathcal{K}(s)$, with s Legendre-reduced and $s_{11} > 1$,

$$t = s - e_{11} + ce_{22} \text{ for } c \in \mathbf{Z}_{>1}$$

is a form with arbitrarily large dyadic trace, but with $t \notin \mathcal{K}(s)$, and therefore we can not determine whether $t \in \nu(f)$ without having a Fourier expansion of f that goes up to the dyadic trace of t. In practice, however, these pathological forms with large dyadic trace and small minimums do not seem to appear in the support of cusp forms.

There exists a class of cusp forms whose semihull is easy to compute exactly by hand. Recall that $A_2 = \left(\begin{smallmatrix} 1 & 1/2 \\ 1/2 & 1 \end{smallmatrix}\right)$. We want to argue that $\mathcal{K}(\frac{1}{2}A_2)$ is maximal in the sense that $t \in \mathcal{K}(\frac{1}{2}A_2)$ for all positive definite semi-integral t. To see this, note that for any positive definite semi-integral t, Proposition 3.3.1 gives

$$\begin{aligned}\varphi_{\mathcal{K}(\frac{1}{2}A_2)}(t) &= \min\left(\frac{m(t)}{m(\frac{1}{2}A_2)}, \frac{w(t)}{w(\frac{1}{2}A_2)}, \frac{\widetilde{\mathrm{tr}}(t)}{\widetilde{\mathrm{tr}}(\frac{1}{2}A_2)}\right) \\ &= \min\left(m(t), 2w(t), \widetilde{\mathrm{tr}}(t)\right) \\ &\geq 1.\end{aligned}$$

Therefore, if $\frac{1}{2}A_2 \in \mathrm{supp}(f)$ for some cusp form f, then $\nu(f) = \mathcal{K}(\frac{1}{2}A_2)$, since all other positive definite semi-integral forms will already be contained in $\mathcal{K}(\frac{1}{2}A_2)$. Because $\frac{1}{2}A_2 \in \mathrm{supp}(\chi_{10})$ and $\frac{1}{2}A_2 \in \mathrm{supp}(\chi_{12})$,

$$\nu(\chi_{10}) = \mathcal{K}(\tfrac{1}{2}A_2) \quad \text{and} \quad \nu(\chi_{12}) = \mathcal{K}(\tfrac{1}{2}A_2).$$

Combining this result with the valuation property of ν gives us that for any $m, n \in \mathbf{Z}_{>0}$,

$$\nu(\chi_{10}^n \chi_{12}^m) = n\nu(\chi_{10}) + m\nu(\chi_{12}) = (n+m)\mathcal{K}(\tfrac{1}{2}A_2).$$

4.2 Computing the Weight k Extreme Core

Definition 4.2.1 The *weight* k *extreme core*, $\mathcal{C}_{\mathrm{ext}}^k$, is the intersection of the semihulls of all nonzero cusp forms of weight k, i.e.

$$\mathcal{C}_{\mathrm{ext}}^k := \bigcap_{f \in \mathcal{S}_2^k \setminus \{0\}} \nu(f).$$

Note that we can then write

$$\mathcal{C}_{\mathrm{ext}} = \bigcap_{\substack{k \in \mathbf{Z}_{>0} \\ \mathcal{C}_{\mathrm{ext}}^k \neq \varnothing}} \frac{1}{k}\mathcal{C}_{\mathrm{ext}}^k,$$

so that $\frac{1}{k}\mathcal{C}_{\mathrm{ext}}^k$ is the optimal outer bound given by weight k cusp forms for $\mathcal{C}_{\mathrm{ext}}$.

Definition 4.2.2 Let $s \in \mathcal{P}_2^*$. We say $t \in \mathcal{P}_2^*$ is a *generator* of s if $s \in \mathcal{K}(t)$. Whether t generates s is a class property and therefore it makes sense to define G_s, the set of all equivalence classes of positive definite semi-integral generators of s.

The reason we are particularly interested in the positive definite semi-integral generators of a form s is because these are precisely the ones that appear in the support of cusp forms. Clearly, if $\nu(f)$ is principally generated (i.e. if $\nu(f) = \mathcal{K}(s)$ for some $s \in \mathcal{X}_2$) $a(g; f) = 0$ for all $g \in G_s$ if and only if $s \notin \nu(f)$. When $\nu(f)$ is not principal, the situation becomes more complicated. Suppose $\nu(f) = \mathcal{K}(S)$ for some set S and let $t \in \mathcal{P}_2$. If there exists some $s \in G_t$ such that $a(s; f) \neq 0$, then it follows that $t \in \mathcal{K}(s)$ and therefore $t \in \mathcal{K}(S)$, but the converse is not true—an element can be in $\mathcal{K}(S)$ without being in the kernels of any of the $s \in S$ (i.e. by being in the convex closure of the individual kernels).

Taking the Legendre-reduced forms as representatives of the classes in G_s, and assuming that s is Legendre-reduced as well, from Proposition 3.3.1 we get that $t \in G_s$ if and only if:

$$\begin{aligned}
&0 \le 2t_{12} \le t_{11} \le t_{22} && (t \text{ is Legendre-reduced})\\
&t_{11} \le s_{11} && (m(s)/m(t) \ge 1)\\
&t_{11} + t_{22} \le s_{11} + s_{22} && (\widetilde{\mathrm{tr}}(s)/\widetilde{\mathrm{tr}}(t) \ge 1)\\
&t_{11} + t_{22} - t_{12} \le s_{11} + s_{22} - s_{12} && (w(s)/w(t) \ge 1).
\end{aligned}$$

There exists only finitely many solutions such that $t_{11}, t_{22} \in \mathbf{Z}$ and $t_{12} \in \frac{1}{2}\mathbf{Z}$, and therefore G_s is finite.

We give an overview of the algorithm used to compute $\mathcal{C}^k_{\text{ext}}$:

1. Generate a basis $\beta = \{f_1, f_2, \ldots, f_n\}$ for S^k_2 using Igusa's generators.
2. Compute $C = \bigcap_{f \in \beta} \nu(f)$ using the previous algorithm. We let C be our candidate for $\mathcal{C}^k_{\text{ext}}$. Note that we can write C as $\mathcal{K}(S)$ where S is a set of positive definite forms which we may take to be Legendre-reduced.
3. For every $s \in S$ we want to show that either $s \in \nu(f)$ for every $f \in S^k_2 \backslash \{0\}$ or otherwise replace it with a form that generates a smaller kernel. Compute $G_s = \{g_1, g_2, \ldots, g_m\}$, the Legendre-reduced set of positive definite semi-integral generators of s, and consider the $m \times n$ matrix:

$$M = \begin{pmatrix} a(g_1; f_1) & a(g_1; f_2) & \ldots & a(g_1; f_n) \\ a(g_2; f_1) & a(g_2; f_2) & \ldots & a(g_2; f_n) \\ \vdots & \vdots & \ddots & \vdots \\ a(g_m; f_1) & a(g_m; f_2) & \ldots & a(g_m; f_n) \end{pmatrix}.$$

   Note that if $\mathrm{rank}(M) \ge n$ there is no nontrivial linear combination of the basis elements such that all of the generators of s have coefficient 0 and therefore s has to be an element of $\mathcal{C}^k_{\text{ext}}$. Otherwise, let b be a nonzero element of the null space of M and note that then $s \notin \nu\left(\sum_{i=1}^n b_i f_i\right)$. Replace C with $C \cap \nu\left(\sum_{i=1}^n b_i f_i\right)$, verifying that this reduced[1] C, and repeat Step 3.
4. C is now $\mathcal{C}^k_{\text{ext}}$.

The final claim is conditional on the accuracy of our algorithm for computing the semihull of a cusp form. If we can not guarantee that we are capturing all of the forms in the semihull, we can only prove that C has to be contained in $\mathcal{C}^k_{\text{ext}}$.

[1]If C is still the same, we need a more sophisticated algorithm that can account for forms that are in the convex closure, but not in the individual kernels of the generators of $\nu\left(\sum_{i=1}^n b_i f_i\right)$.

For an implementation of this algorithm see `extreme_core.sage` and the `cut_kernel` function in `support.sage` in the appendix. In `extreme_core.sage` we use the algorithm to find $\mathcal{C}_{\text{ext}}^k$ for even k between 14 and 44:

k	$\mathcal{C}_{\text{ext}}^k$	k	$\mathcal{C}_{\text{ext}}^k$	k	$\mathcal{C}_{\text{ext}}^k$
10	$\mathcal{K}\left[\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}\right]$	12	$\mathcal{K}\left[\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}\right]$	14	$\mathcal{K}\left[\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}\right]$
16	$\mathcal{K}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]$	18	$\mathcal{K}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]$	20	$\mathcal{K}\left[\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\right]$
22	$\mathcal{K}\left[\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\right]$	24	$\mathcal{K}\left[\begin{pmatrix} 2 & 1/2 \\ 1/2 & 2 \end{pmatrix}\right]$	26	$\mathcal{K}\left[\begin{pmatrix} 2 & 1/2 \\ 1/2 & 2 \end{pmatrix}\right]$
28	$\mathcal{K}\left[\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\right]$	30	$\mathcal{K}\left[\begin{pmatrix} 3 & 3/2 \\ 3/2 & 3 \end{pmatrix}\right]$	32	$\mathcal{K}\left[\begin{pmatrix} 3 & 3/2 \\ 3/2 & 3 \end{pmatrix}\right]$
34	$\mathcal{K}\left[\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}\right]$	36	$\mathcal{K}\left[\begin{pmatrix} 3 & 1/2 \\ 1/2 & 3 \end{pmatrix}\right]$	38	$\mathcal{K}\left[\begin{pmatrix} 3 & 1/2 \\ 1/2 & 3 \end{pmatrix}\right]$
40	$\mathcal{K}\left[\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}\right]$	42	$\mathcal{K}\left[\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}\right]$	44	$\mathcal{K}\left[\begin{pmatrix} 4 & 3/2 \\ 3/2 & 4 \end{pmatrix}\right]$

Table 4.2: The weight k extreme core for even weight up to 44. The shaded entry is outside the accuracy range of our Fourier series and therefore might be incorrect.

Note that although we were able to find several new cusp forms with very small semihulls (in particular, semihulls smaller than the ones for Igusa's generators over $\mathbf{Z}$), we failed to improve on the $\frac{1}{10}\begin{pmatrix} 1 & 17/36 \\ 17/36 & 1 \end{pmatrix}$ outer bound for $\mathcal{C}_{\text{ext}}$ given in [PY05], showing that this bound is optimal for cusp forms of even weight between 10 and 44.

A

Code

A.1 support.sage

```
R.<x,y,z> = QQ[]  # The Fourier expansions in polys.sage are polynomials in
                  # the variables x,y,z.
from copy import copy


# XXX: Most of the functions make assumptions about the format of their
# input, but perform no validation. Check the documentation for the function
# before using.


def legendre_reduce(m):
    """ Legendre-reduces a given positive definite 2x2 form. See PSY
        for a discussion of this algorithm.
    """
    _step_one(m)

def _step_one(m):
    if (m[0,0] > m[1,1]):
        m[0,0], m[0,1], m[1,0], m[1,1] = m[1,1], -m[0,1], -m[1,0], m[0,0]
    _step_two(m)

def _step_two(m):
    if (2*abs(m[1,0]) > m[0,0]):
        l = floor(-m[1,0]/m[0,0] + 1/2)

        m[0,1] = m[0,1] + l*m[0,0]
        m[1,0] = m[1,0] + l*m[0,0]
        m[1,1] = m[1,1] + 2*l*m[1,0] + l**2*m[0,0]

        if (m[0,0] > m[1,1]):
            _step_one(m)

    _step_three(m)

def _step_three(m):
    if (m[0,1] < 0):
        m[0,0], m[0,1], m[1,0], m[1,1] = m[0,0], -m[0,1], -m[1,0], m[1,1]
```

```
def is_legendre(m):
    """ Given a 2x2 form, checks if it is in Legendre-reduced form. """
    if (0 <= 2*m[1,0] <= m[0,0] <= m[1,1]):
        return True
    return False


def reduced_trace(m):
    """ Returns the reduced trace of a 2x2 form
        in Legendre-reduced form.
    """
    return (m[0,0] + m[1,1])


def minimum_function(m):
    """ Returns the minimum function of a 2x2 form
        in Legendre-reduced form.
    """
    return m[0,0]


def dyadic_trace(m):
    """ Returns the dyadic trace of a 2x2 form
        in Legendre-reduced form.
    """
    return (m[0,0] + m[1,1] - abs(m[1,0]))


def phi_s(s,t):
    """ Given two 2x2 Legendre-reduced forms s and t, returns the height
        function at t associated with the kernel generated by s.
    """
    a1 = minimum_function(t) / minimum_function(s)
    a2 = dyadic_trace(t) / dyadic_trace(s)
    a3 = reduced_trace(t) / reduced_trace(s)
    return min(a1, a2, a3)


def is_in_kernel(s,t):
    """ Given two 2x2 Legendre-reduced forms s and t, checks whether t is
        inside the kernel generated by s.
    """
    if (phi_s(s,t) >= 1):
        return True
    return False
```

```
def monomial_to_matrix(f):
    """ Given a monomial from a Fourier expansion in the format used in
        polys.sage, it returns the associated form.
    """
    m11 = f.degree(y)/8
    m22 = f.degree(z)/8
    m21 = f.degree(x)/8 - m11 - m22
    m = matrix([[m11, m21], [m21, m22]])

    return m


def get_coefficient(m, f):
    """ Given a Fourier expansion f in the format used in polys.sage
        and a form m, it returns the coefficient of m in f.
    """
    i = 8*(m[0,0] + m[0,1] + m[1,1])
    j = 8*m[0,0]
    k = 8*m[1,1]
    mon = x**i * y**j * z**k
    return f.monomial_coefficient(mon)


def matrix_to_tuple(m):
    """ Given a 2x2 form m, it returns a triple containing the entries of m.
        Useful when we need immutables in order to use matrices as
        dictionary keys.
    """
    return (m[0,0], m[0,1], m[1,1])


def get_support(f):
    """ Given a Fourier expansion f, it returns all Legendre-reduced
        matrices (without duplicates) which have nonzero coefficients
        in the expansion.
    """
    classes = {}  # Deduping dict.
    support = []
    monomials = f.monomials()
    for mon in monomials:
        m = monomial_to_matrix(mon)
        legendre_reduce(m)
        t = matrix_to_tuple(m)
        if t not in classes:  # Make sure m is not a dupe.
            classes[t] = 0
            support.append(m)
```

```
    return support


def dink_em_out(s):
    """ Given a list of 2x2 Legendre-reduced matrices, it removes all
        matrices which are in the kernel of another matrix in the
        list, i.e. it reduces a list to the generators of the original
        list.
    """
    s.sort(key=lambda x: (dyadic_trace(x), reduced_trace(x),
                          minimum_function(x)))
    i = 0
    while (i < len(s)):
        j = i + 1
        while (j < len(s)):
            if (is_in_kernel(s[i], s[j])):
                del s[j]
            else:
                j += 1
        i += 1


def intersect_kernels(s, t):
    """ Given two 2x2 Legendre-reduced positive definite matrices, it
        returns the matrix (Legendre-reduced) whose kernel is the
        intersection of the kernels of the two original matrices.
    """
    m = matrix(QQ, [[0, 0], [0, 0]])
    m[0,0] = max(minimum_function(s), minimum_function(t))
    m[1,1] = max(reduced_trace(s), reduced_trace(t)) - m[0,0]
    m[0,1] = m[0,0] + m[1,1] - max(dyadic_trace(s), dyadic_trace(t))
    m[1,0] = m[0,1]
    return m


def intersect_kernel_list(s, t):
    """ Same as intersect_kernels, only doesn't assume that the kernels
        are principally-generated. Works with lists of matrices instead.
    """
    result = []
    for m in s:
        for n in t:
            result.append(intersect_kernels(m, n))

    dink_em_out(result)
    return result
```

```
def preds(t):
    """ Given a 2x2 Legendre-reduced matrix s, it returns a list of all
         positive definite Legendre-reduced semi-integral matrices k such
         that s is in the kernel of k.
    """
    result = []
    s11 = 1
    while (s11 <= t[0,0]):
        s22 = s11
        while (s22 <= t[0,0] + t[1,1] - s11) and (s22 >= s11):
            s12 = s11/2
            while (s12 >= s11 + s22 + t[0,1] - t[0,0] - t[1,1]) and (s12 >= 0):
                s = matrix([[s11, s12], [s12, s22]])
                result.append(s)
                s12 -= 1/2
            s22 += 1
        s11 += 1

    return result


def cut_kernel(basis, ms):
    old = copy(ms)
    cuts = cut_kernel_h(basis, ms)

    while (old != cuts):
        old = copy(cuts)
        cuts = cut_kernel_h(basis, cuts)

    print('Got␣containment:␣')
    print(str(cuts))


def cut_kernel_h(basis, ms):
    intersected_kernel = ms

    for m in ms:
        print("Trying␣to␣beat:\n" + str(m))
        prevs = preds(m)
        t = matrix(QQ, len(prevs), len(basis))

        for i, tuple in enumerate(basis.keys()):
            for j, prev in enumerate(prevs):
                t[j,i] = get_coefficient(prev, basis[tuple])

        if (t.right_nullity() != 0):
```

```
            print('Found␣a␣nontrivial␣solution.')
            solv = t.right_kernel().matrix()[0]
        else:
            print("Cannot␣improve␣estimate.")
            continue

        f = 0
        for i, a in enumerate(solv):
            p = basis[basis.keys()[i]]
            f += a*p

        s = get_support(f)
        dink_em_out(s)
        intersected_kernel = intersect_kernel_list(s, intersected_kernel)
        if (intersected_kernel == ms):
            print("We␣failed␣to␣improve␣on␣the␣estimate␣---␣looks␣like␣s␣"
                  "was␣in␣the␣convex␣closure,␣but␣not␣in␣the␣individual␣"
                  "kernels.␣Need␣a␣better␣cut_kernel␣function␣to␣proceed.")
            raise NotImplementedError  # This is VERY unlikely to happen.

    return intersected_kernel
```

A.2 igusa.sage

```
R.<x,y,z> = QQ[]

# Include functions for working with the semihulls of Siegel modular forms.
load support.sage
# Load Fourier series of Igusa's 4 generators over C up to dyadic trace 14
# courtesy of Cris Poor and David S. Yuen.
load polys.sage

# Relabel the expansions from polys.sage.
X4 = e4
X6 = e6
X10 = xt
X12 = jt

# Compute Igusa's generators over Z.
Y12 = 2**(-6) * 3**(-3) * (X4*X4*X4 - X6*X6) + 2**(4) * 3**(2) * X12
X16 = 2**(-2) * 3**(-1) * (X4*X12 - X6*X10)
X18 = 2**(-2) * 3**(-1) * (X6*X12 - X4*X4*X10)
X24 = 2**(-3) * 3**(-1) * (X12*X12 - X4*X10*X10)
X28 = 2**(-1) * 3**(-1) * (X4*X24 - X10*X18)
X30 = 2**(-1) * 3**(-1) * (X6*X24 - X4*X10*X16)
X36 = 2**(-1) * 3**(-2) * (X12*X24 - X10*X10*X16)
X40 = 2**(-2) * (X4*X36 - X10*X30)
X42 = 2**(-2) * 3**(-1) * (X12*X30 - X4*X10*X28)
X48 = 2**(-2) * (X12*X36 - X24*X24)

X = [X4, X6, X10, X12, Y12, X16, X18, X24, X28, X30, X36, X40, X42, X48]
X_string = ['X4', 'X6', 'X10', 'X12', 'Y12', 'X16', 'X18', 'X24', 'X28',
            'X30', 'X36', 'X40', 'X42', 'X48']

for i, f in enumerate(X):
  print(X_string[i])
  s = get_support(f)
  dink_em_out(s)  # Find the generators for the support of f.
  print(s, '\n')
```

A.3 extreme_core.sage

```
# Include functions for working with the semihulls of Siegel modular forms.
load support.sage
# Load Fourier series of Igusa's 4 generators over C up to dyadic trace 14
# courtesy of Cris Poor and David S. Yuen.
load polys.sage

def get_tuples(n):
    """ Returns a list containing all tuples (i,j,k,l) such that
        X12^i * X10^j * X8^k * X4^l is a weight n cusp form.
    """
    tuples_list = []
    i = 0
    while (12*i <= n):
        j = 0 if i > 0 else 1  # cusp forms are multiples of X10 or X12.
        k = l = 0
        while (12*i + 10*j <= n):
            k = l = 0
            while (12*i + 10*j + 6*k <= n):
                if ((n - (12*i + 10*j + 6*k)) % 4 == 0):
                    l = (n - (12*i + 10*j + 6*k)) // 4
                    tuples_list.append((i,j,k,l))
                k += 1
            j += 1
        i += 1

    return(tuples_list)

# Relabel the Fourier series for Igusa's 4 generators over C from polys.sage
X4 = e4
X6 = e6
X10 = xt
X12 = jt

# We use memoization to speed the computation of a basis of cusp forms up to
# some weight. XXX: This is RAM intensive. Doing the computation up to
# weight 50 will require at least 8GB RAM.
minus_two_basis = {(1,0,0,0) : X12}
minus_four_basis = {(0,1,0,0) : X10}
minus_six_basis = {}
minus_eight_basis = {}
minus_ten_basis = {}
minus_twelve_basis = {}
current_basis = {}
```

```
def swap_levels():
    """ Adjusts the memo as we are going from weight n to weight n+2. """
    global minus_two_basis, minus_four_basis, minus_six_basis
    global minus_eight_basis, minus_ten_basis, minus_twelve_basis
    global current_basis

    minus_twelve_basis = minus_ten_basis
    minus_ten_basis = minus_eight_basis
    minus_eight_basis = minus_six_basis
    minus_six_basis = minus_four_basis
    minus_four_basis = minus_two_basis
    minus_two_basis = current_basis
    current_basis = {}


for n in range(14, 45, 2):
    tuples = get_tuples(n)
    # The kernel of this matrix contains everything.
    intersection = [matrix([[1, 1/2], [1/2, 1]])]
    count = 0

    for (i,j,k,l) in tuples:
        # Find the memoized predecessor from which it is easiest
        # to compute X(i,j,k,l)
        if (l > 0):
            p = minus_four_basis[(i, j, k, l-1)]
            p *= X4  # XXX: we are reassigning p to p*X4, not multiplying
                     # p by X4. the basis is not modified.
        elif (k > 0):
            p = minus_six_basis[(i, j, k-1, l)]
            p *= X6
        elif (j > 0):
            p = minus_ten_basis[(i, j-1, k, l)]
            p *= X10
        else:
            p = minus_twelve_basis[(i-1, j, k, l)]
            p *= X12

        current_basis[(i,j,k,l)] = p
        s = get_support(p)
        dink_em_out(s)
        intersection = intersect_kernel_list(intersection, s)
        count += 1

    print('Computed a basis of {} weight {} cusp forms.'.format(count, n))
    print('The intersection of their semihulls is:\n' + str(intersection))
```

```
    # Try to improve on the intersection as an estimate for
    # the weigth k extreme core.
    cut_kernel(current_basis, intersection)
    print('\n')
    swap_levels()  # Prepare the memo for the next weight.
```

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