COVERS OF ELLIPTIC CURVES AND THE MODULI SPACE OF STABLE CURVES

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ABSTRACT. Consider genus g curves that admit degree d covers of an elliptic curve. Varying a branch point, we get a one-parameter family W of simply branched covers. Varying the target elliptic curve, we get another one-parameter family Y of covers that have a unique branch point. We investigate the geometry of W and Y by using admissible covers to study their slopes, genera and components. The results can be applied to study slopes of effective divisors on the moduli space of stable genus g curves.

CONTENTS

1. INTRODUCTION AND MAIN RESULTS

Our motivation is to study the moduli space of stable genus g curves $\overline{\mathcal{M}}_g$ by using one-parameter families of covers of elliptic curves.

Recall that $\mathrm{Pic}(\overline{\mathcal{M}}_g)\otimes\mathbb{Q}$ is generated by the Hodge class λ and the boundary classes δ_i, $i = 0, 1, \ldots, [\frac{g}{2}]$. Let δ denote the total boundary. It is natural to ask what linear combinations of λ and δ are effective. To do this, people have associated a number called *slope* to a divisor class as follows.

Definition 1.1. Let $D = a\lambda - \sum_{i=0}^{[g/2]} b_i\delta_i$ denote a divisor class on $\overline{\mathcal{M}}_g$, $a, b_i > 0$. The slope of D is given by $s(D) = \frac{a}{min\{b_i\}}$.

The study of slopes of effective divisors on $\overline{\mathcal{M}}_g$ is interesting in a number of reasons. One of the most important facts is that the existence of an effective divisor with slope smaller than $\frac{13}{2}$ would imply that $\overline{\mathcal{M}}_g$ is of general type. Harris, Mumford and Eisenbud constructed certain Brill-Noether divisors with small slopes. They showed that $\overline{\mathcal{M}}_g$ is of general type when g is greater than 23, cf. [HMu], [H] and [EH]. More recently, Farkas, Popa and Khosla found a series of effective divisors with slopes even lower than the Brill-Noether bound $6 + \frac{12}{g+1}$, cf. [FP], [Kh], [F1], [F2] and [F3]. In particular, Farkas showed that $\overline{\mathcal{M}}_g$ is of general type for $g = 22$ and 23. Surprisingly, all the known effective divisors on $\overline{\mathcal{M}}_g$ have slope bigger than 6, which makes one guess that there could be a positive lower bound, independent of g, for slopes of effective divisors. Let s_g denote the *infimum* of the slope for all effective divisors on $\overline{\mathcal{M}}_g$. The exact value of s_g has been studied for small g, cf. [HM1], [CR], [T] and [FP]. However, s_g is unknown for large g. In [HM1], Harris and Morrison obtained an asymptotic lower bound for s_g. Their method is to construct moving curves on $\overline{\mathcal{M}}_g$ and compute their slopes.

Definition 1.2. Let B denote an irreducible one-dimensional family of stable genus g curves. There is a natural morphism $f : B \to \overline{\mathcal{M}}_g$. The *slope* of B is defined as $s(B) = \frac{\deg f^*\delta}{\deg f^*\lambda}$. B is *moving* iff the deformation of its image under f covers an open dense subset of $\overline{\mathcal{M}}_g$.

Since an effective divisor D cannot contain all the deformation of B, the intersection number $D. f_*B$ is positive, which implies $s(D) > s(B)$. Harris and Morrison considered simply branched genus g covers of $\mathbb{P}^1$ with large degree. When the branch points vary along a one-dimensional base, those covering curves form a moving curve in $\overline{\mathcal{M}}_g$. The calculation of its slope boils down to a rather complicated enumerative problem. The bound they obtained in general is not sharp and approaches zero when g goes to infinity. One way to generalize their idea is to consider one-parameter families of curves in higher dimensional projective spaces. In [CHS], Coskun, Harris and Starr studied canonical curves and obtained sharp bound s_g for genus up to 6. In [F], Fedorchuk considered one-parameter families of plane curves in the Severi variety and got a recursive formula for their slopes. The formula provides some new lower bounds for s_g when $g \leq 21$. But the asymptotic estimate of these bounds is unclear. Recently, Pandharipande computed certain intersection numbers of cotangent line bundles with λ and δ on the moduli space of curves with marked points. A lower bound $s_g \geq \frac{60}{g+4}$ was established in [P]. Note that this bound tends to zero for large g.

The slope of a one-parameter family of stable genus g curves is also interesting for its own sake. It can imply special geometric properties of this family, for instance, cf. [CH] and [S] for the related results on families of hyperelliptic and trigonal curves.

In this paper, we study covers of elliptic curves. Let $\overline{\mathcal{H}}_{d,g}$ denote the Hurwitz space parameterizing degree d genus g admissible covers of elliptic curves. A general point of $\overline{\mathcal{H}}_{d,g}$ corresponds to a degree d connected cover of an elliptic curve simply branched at $2g-2$ points. When the branch points meet, those covers can degenerate to prossess various ramification points in the sense of admissible covers, cf. [HM2, 3G] for an introduction on admissible covers. Among all the curve classes of $\overline{\mathcal{H}}_{d,g}$, we focus on two of them, the most "general" one-parameter family $W_{d,g}$ and the most "degenerate" one-parameter family $Y_{d,g,\sigma}$.

$W_{d,g}$ is constructed as follows. Fix an elliptic curve E. Take the product $S = E \times X$ over a smooth one-dimensional base X along with $2g-2$ sections $\Gamma_1, \ldots, \Gamma_{2g-2}$. A general fiber of S over X is isomorphic to E with $2g-2$ marked points given by those sections. We require that no three sections meet at a common point. If two sections meet, they must intersect transversely. Moreover, a fiber can possess at most one of such intersection points. X admits a map to $\overline{\mathcal{M}}_{1,2g-2}$. We define the one-parameter family $W_{d,g}$ by the following fiber product:

$$\begin{array}{ccc} W_{d,g} & \longrightarrow & \overline{\mathcal{H}}_{d,g} \\ \downarrow & & \downarrow \\ X & \longrightarrow & \overline{\mathcal{M}}_{1,2g-2} \end{array}$$

For $Y_{d,g,\sigma}$, we consider covers $C \to E$ with a unique branch point p on E. Fix the ramification type of the covers by a conjugacy class $\sigma = (l_1)\cdots(l_m)$ in S_d, i.e. the pre-image of p equals $\sum_{i=1}^m l_i q_i$ on C, where q_i's are distinct. Let $\overline{\mathcal{H}}_{d,g,\sigma}$ denote the one-dimensional subscheme of $\overline{\mathcal{H}}_{d,g}$ parameterizing covers with a unique branch point and the ramification class σ. $\overline{\mathcal{H}}_{d,g,\sigma}$ admits a morphism to $\overline{\mathcal{M}}_{1,1}$ by mapping a cover to its target elliptic curve (E, p). Let $Z \cong \mathbb{P}^1$ denote a general pencil of plane cubics. Blowing up a base point, we obtain a section for this elliptic fibration over Z. Hence, there is a morphism from

Z to $\overline{\mathcal{M}}_{1,1}$. We define the one-parameter family $Y_{d,g,\sigma}$ by the following fiber product:

$$\begin{array}{ccc} Y_{d,g,\sigma} & \longrightarrow & \overline{\mathcal{H}}_{d,g,\sigma} \\ \downarrow & & \downarrow \\ Z & \longrightarrow & \overline{\mathcal{M}}_{1,1} \end{array}$$

Intuitively speaking, $W_{d,g}$ is formed by varying the branch points along E. $Y_{d,g,\sigma}$ is formed by varying the j-invariant of E. The union of covers in such families $W_{d,g}$ contains an open dense subset of $\overline{\mathcal{H}}_{d,g}$. On the contrary, a cover in $Y_{d,g,\sigma}$ arises only if all the branch points of a cover meet. In this sense, we regard $W_{d,g}$ and $Y_{d,g,\sigma}$, respectively, as the most "general" and the most "degenerate" one-parameter families of covers. Note that $\overline{\mathcal{H}}_{d,g}$ admits a natural morphism to $\overline{\mathcal{M}}_g$, which induces maps from $W_{d,g}$ and $Y_{d,g,\sigma}$ to $\overline{\mathcal{M}}_g$. We want to study the geometry of $W_{d,g}$ and $Y_{d,g,\sigma}$, including their slopes, genera and components. Our strategy is to associate to a cover its monodromy data by using the symmetric group S_d.

Let us first consider $W_{d,g}$. Let E_b denote an elliptic curve E with a base point b on it. Take $2g-2$ general points $p_1, \ldots, p_{2g-2}$ on E and a group of generators $\alpha, \beta, \gamma_1, \ldots, \gamma_{2g-2}$ for $\pi_1(E_b, p_1, \ldots, p_{2g-2})$. Here α, β are a standard basis for $\pi_1(E_b)$ and γ_i is a closed loop around p_i. After choosing suitable directions for those paths, we have the relation $\beta^{-1}\alpha^{-1}\beta\alpha \sim \gamma_1 \cdots \gamma_{2g-2}$. See Figure 1.

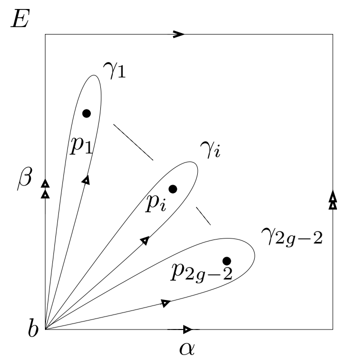


FIGURE 1. E with marked points and paths

A degree d cover of E simply branched at $p_1, \ldots, p_{2g-2}$ corresponds to a homomorphism

$$\pi_1(E_b, p_1, \ldots, p_{2g-2}) \to S_d,$$

such that the images of all γ_i's are simple transpositions. We still use $\alpha, \beta, \gamma_1, \ldots, \gamma_{2g-2}$ to denote their images in S_d.

Definition 1.3. Define the set $Cov_{d,g}$ as

$$Cov_{d,g} := \{(\alpha, \beta, \gamma_1, \ldots, \gamma_{2g-2}) \in S_d \times \cdots \times S_d \mid \beta^{-1}\alpha^{-1}\beta\alpha = \gamma_1 \cdots \gamma_{2g-2},$$

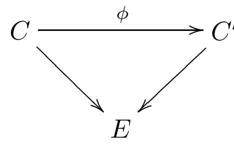
$$\gamma_i\text{'s are simple transpositions and } \langle \alpha, \beta, \gamma_1, \ldots, \gamma_{2g-2} \rangle \text{ is transitive}\}.$$

There is an *equivalence* relation $\sim$ among elements in $Cov_{d,g}$ defined as follows:

$$(\alpha, \beta, \gamma_1, \ldots, \gamma_{2g-2}) \sim (\alpha', \beta', \gamma_1', \ldots, \gamma_{2g-2}')$$

iff there exists $\tau \in S_d$ such that $\tau(\alpha, \beta, \gamma_1, \ldots, \gamma_{2g-2})\tau^{-1} = (\alpha', \beta', \gamma_1', \ldots, \gamma_{2g-2}')$. Moreover, let $N_{d,g} = |Cov_{d,g}/\sim|$ denote the cardinality of $Cov_{d,g}$ modulo the equivalence relation.

The above notation $\langle \cdot \rangle$ stands for the subgroup of S_d generated by the elements inside. The transitivity condition is to make sure that the covers are connected. The equivalence relation takes care of relabeling the d sheets of the cover. An isomorphism between two covers $C \to E$ and $C' \to E$ means that there exists a commutative diagram as follows:



where ϕ is an isomorphism between C and C'. Two elements in $Cov_{d,g}$ correspond to isomorphic covers iff they are equivalent. In this setting, $N_{d,g}$ equals the number of non-isomorphic degree d genus g connected covers of E simply branched at $p_1, \ldots, p_{2g-2}$. Hence, $W_{d,g} \to X$ is a finite map of degree $N_{d,g}$. It is necessary to understand which admissible covers appear when two branch points meet. Suppose locally two sections Γ_1 and Γ_2 meet. Since γ_1 and γ_2 are simple transpositions both permuting two letters of $\{1, \ldots, d\}$, their length-2 cycles can be the same, or contain only one common letter, or consist of four different letters. Denote the three cases by $\gamma_1 = \gamma_2$, $|\gamma_1 \cap \gamma_2| = 1$ and $\gamma_1 \cap \gamma_2 = \emptyset$, respectively. Define the following subsets of $Cov_{d,g}$ based on the relation between γ_1 and γ_2.

Definition 1.4. For $1 \leq h \leq [\frac{d}{2}]$, define the following subsets of $Cov_{d,g}$,

$$Cov_{d,g}(0) := \{(\alpha, \beta, \gamma_1, \ldots, \gamma_{2g-2}) \in Cov_{d,g} \mid \gamma_1 = \gamma_2,$$
$$\langle \alpha, \beta, \gamma_3, \ldots, \gamma_{2g-2} \rangle \text{ acts transitively on } \{1, \ldots, d\}\};$$
$$Cov_{d,g}(1,h) := \{(\alpha, \beta, \gamma_1, \ldots, \gamma_{2g-2}) \in Cov_{d,g} \mid \gamma_1 = \gamma_2,$$
$$\langle \alpha, \beta, \gamma_3, \ldots, \gamma_{2g-2} \rangle \text{ acts transitively on a partition } (h \mid d-h)\};$$
$$Cov_{d,g}(2) := \{(\alpha, \beta, \gamma_1, \ldots, \gamma_{2g-2}) \in Cov_{d,g} \mid \gamma_1 \cap \gamma_2 = \emptyset\};$$
$$Cov_{d,g}(3) := \{(\alpha, \beta, \gamma_1, \ldots, \gamma_{2g-2}) \in Cov_{d,g} \mid |\gamma_1 \cap \gamma_2| = 1\}.$$

Moreover, let

$$N_{d,g}(0) = |Cov_{d,g}(0)/\sim|, \ N_{d,g}(1,h) = |Cov_{d,g}(1,h)/\sim|,$$
$$N_{d,g}(2) = |Cov_{d,g}(2)/\sim|, \ N_{d,g}(3) = |Cov_{d,g}(3)/\sim|$$

denote the cardinalities of these subsets modulo the equivalence relation. Finally, let $N_{d,g}(1)$ denote the sum $\sum_{h=1}^{[d/2]} N_{d,g}(1,h)$.

The notation $(h \mid d-h)$ in the definition of $Cov_{d,g}(1,h)$ represents a partition of $\{1, \ldots, d\}$ into a subset of cardinality h and its complement of cardinality $d-h$. If $\langle \alpha, \beta, \gamma_1, \ldots, \gamma_{2g-2} \rangle$ acts transitively on $\{1, \ldots, d\}$ and if $\gamma_1 = \gamma_2$ are the same simple transposition, after taking away γ_1 and γ_2, the set $\{1, \ldots, d\}$ under the remaining permutations can at most break into two orbits corresponding to a partition of type $(h \mid d-h)$. Therefore, the subsets $Cov_{d,g}(0)$; $Cov_{d,g,}(1,h)$, $1 \leq h \leq [\frac{d}{2}]$ together fully cover the case $\gamma_1 = \gamma_2$. Note that the equivalence relation is well-defined for the above subsets. The subsets $Cov_{d,g}(0)$; $Cov_{d,g}(1,h)$, $1 \leq h \leq [\frac{d}{2}]$; $Cov_{d,g}(2)$; $Cov_{d,g}(3)$ yield a decomposition of $Cov_{d,g}$. Hence, we have the equality $N_{d,g} = N_{d,g}(0) + N_{d,g}(1) + N_{d,g}(2) + N_{d,g}(3)$. Now we can state one of our main results, a slope formula for $W_{d,g}$.

Theorem 1.5. *The slope of $W_{d,g}$ is given by*

$$s(W_{d,g}) = \frac{72\big(N_{d,g}(0) + N_{d,g}(1)\big)}{9\big(N_{d,g}(0) + N_{d,g}(1)\big) + N_{d,g}(3)}.$$

Moreover, the slope of an irreducible component of $W_{d,g}$ is equal to the slope of $W_{d,g}$.

Note that $s(W_{d,g})$ only depends on the ratio of $N_{d,g}(0) + N_{d,g}(1)$ and $N_{d,g}(3)$, which is independent of the base X and the sections.

Corollary 1.6. *When $g = 2$, the equality $5N_{d,2}(3) = 27N_{d,2}(1) - 9N_{d,2}(0)$ holds.*

Corollary 1.7. *When $g = 2$ and d is odd, the slope $s(W_{d,2})$ converges to 5 as d approaches infinity.*

Corollary 1.8. *When $d = 2$, the slope $s(W_{2,g})$ equals 8. Moreover, the stable base locus of an effective divisor on $\overline{\mathcal{M}}_g$ with slope smaller than 8 must contain the locus of genus g curves that admit a double cover of an elliptic curve.*

To figure out the genus and irreducible components for $W_{d,g}$, we need to study certain monodromy actions for the map $W_{d,g} \to X$. Below we analyze a special case when the base X is isomorphic to E, $\Gamma_1, \ldots, \Gamma_{2g-3}$ are distinct horizontal sections and Γ_{2g-2} is the diagonal of $E \times E$, cf. Figure 2.

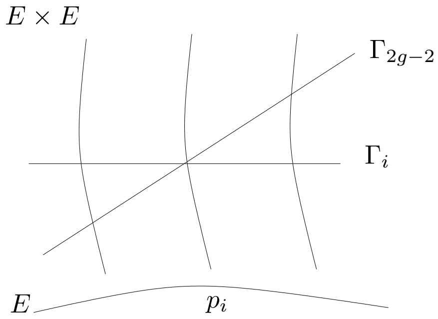


FIGURE 2. $E \times E$ with sections

This case makes sense in that a section of $E \times X$ induces a map $X \to E$, which pulls back the covering family over $E \times E$ to a family over $E \times X$. We denote this special one-parameter family over E by $W_{d,g}(E)$. Intuitively speaking, $W_{d,g}(E)$ is formed by moving the last branch point p_{2g-2} along E while fixing the other branch points. The morphism $W_{d,g}(E) \to E$ is of degree $N_{d,g}$. Fix a base point $b \in E$. Let η_i denote a closed path separating $p_i, \ldots, p_{2g-3}$ from the other marked points on the base curve E. Then $\eta_1, \ldots, \eta_{2g-3}, \alpha, \beta$ generate $\pi_1(E_b, p_1, \ldots, p_{2g-3})$, cf. Figure 3.

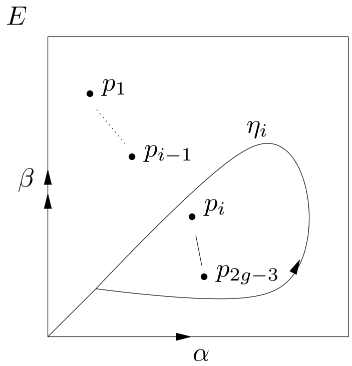


FIGURE 3. The path η_i on E

Definition 1.9. Going along a closed path $\gamma \in \pi_1(E_b, p_1, \ldots, p_{2g-3})$, there is an action permuting the $N_{d,g}$ sheets of $W_{d,g}(E)$ over the ending point $\gamma(0) = \gamma(1)$. It can be regarded as an action on $Cov_{d,g}/\sim$. In this sense, define g_i, g_α and g_β as the *monodromy actions* corresponding to the paths η_i, α and β, respectively.

Theorem 1.10. *For a cover in $W_{d,g}(E)$ over b that corresponds to $(\alpha, \beta, \gamma_1, \ldots, \gamma_{2g-2}) \in Cov/\sim$, the monodromy action g_i acts as follows:*

$$g_i(\alpha) = \alpha,\ g_i(\beta) = \beta,\ g_i(\gamma_j) = \gamma_j \text{ for } j < i,$$
$$g_i(\gamma_j) = \gamma_{2g-2}^{-1}\gamma_j\gamma_{2g-2} \text{ for } i \le j \le 2g-3,$$
$$g_i(\gamma_{2g-2}) = (\gamma_i \cdots \gamma_{2g-2})^{-1}\gamma_{2g-2}(\gamma_i \cdots \gamma_{2g-2}).$$

The monodromy actions g_α and g_β act as follows:

$$g_\alpha(\alpha) = \alpha,\ g_\alpha(\beta) = \beta\gamma_{2g-2},\ g_\alpha(\gamma_{2g-2}) = \alpha^{-1}\gamma_{2g-2}\alpha,$$
$$g_\alpha(\gamma_j) = \gamma_{2g-2}^{-1}\gamma_j\gamma_{2g-2}\ \textit{for}\ j \leq 2g-3;$$
$$g_\beta(\alpha) = \alpha\gamma_{2g-2}^{-1},\ g_\beta(\beta) = \beta,\ g_\beta(\gamma_j) = \gamma_j\ \textit{for}\ j \leq 2g-3,$$
$$g_\beta(\gamma_{2g-2}) = (\beta\gamma_1\cdots\gamma_{2g-3})^{-1}\gamma_{2g-2}(\beta\gamma_1\cdots\gamma_{2g-3}).$$

In particular, two covers are contained in the same component of $W_{d,g}(E)$ iff their corresponding data in $Cov_{d,g}/\sim$ belong to the same orbit under the monodromy actions generated by $g_i,\ 1 \leq i \leq 2g-3,\ g_\alpha$ and g_β.

Corollary 1.11. *The geometric genus of $W_{d,g}$ is given by*

$$g(W_{d,g}) = 1 + N_{d,g}\big(g(X) - 1\big) + kN_{d,g}(3),$$

where $g(X)$ is the genus of the smooth base curve X and $k = \sum_{i<j}\Gamma_i.\Gamma_j$ is the number of pairwise intersections between the $2g-2$ sections.

Corollary 1.12. *When $d = 2$, the genus of $W_{2,g}$ equals $4g(X) - 3$.*

Corollary 1.13. *When $g = 2$, $X \cong E$ and the family is formed by moving the last branch point, the genus of $W_{d,2}(E)$ equals $1 + \frac{8}{9}\big(\sigma_3(d) - 2d\sigma_1(d) + \sigma_1(d)\big)$, where $\sigma_i(d) = \sum\limits_{l|d,\ l>0} l^i$.*

Now let us consider $Y_{d,g,\sigma}$. Denote by α and β a standard basis of $\pi_1(E_b, p)$. Pick a loop $\gamma = \beta^{-1}\alpha^{-1}\beta\alpha$ around the branch point p, cf. Figure 4.

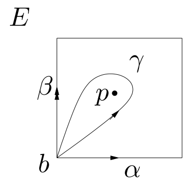


FIGURE 4. E with a branch point p

Recall that $\sigma = (l_1)\cdots(l_m)$ is a conjugacy class of S_d. A cover in $Y_{d,g,\sigma}$ has a unique branch point at p with the ramification class σ. This cover corresponds to a homomorphism $\pi_1(E_b, p) \to S_d$ such that the image of γ belongs to the conjugacy class σ. Below we will also use another notation $1^{a_1}\ldots d^{a_d}$ to denote a conjugacy class that has a_i cycles of length-i, where $\sum_{i=1}^d ia_i = d$.

Definition 1.14. Define the set $Cov_{d,g,\sigma}$ as

$$Cov_{d,g,\sigma} := \{(\alpha, \beta) \in S_d \times S_d \mid \beta^{-1}\alpha^{-1}\beta\alpha \in \sigma,$$
$$\langle\alpha, \beta\rangle \text{ is a transitive subgroup of } S_d\}.$$

Define its subset $Cov_{d,g,\sigma}^{1^{a_1}\ldots d^{a_d}}$ as

$$Cov_{d,g,\sigma}^{1^{a_1}\ldots d^{a_d}} := \{(\alpha, \beta) \in Cov_{d,g,\sigma} \mid \beta \in 1^{a_1}\ldots d^{a_d}\}$$

based on the conjugacy type of β. There is an *equivalence* relation $\sim$ among elements in $Cov_{d,g,\sigma}$:

$$(\alpha, \beta) \sim (\alpha', \beta')$$

iff there exists $\tau \in S_d$ such that $\tau(\alpha, \beta)\tau^{-1} = (\alpha', \beta')$. Moreover, let $N_{d,g,\sigma}$ and $N_{d,g,\sigma}^{1^{a_1}\ldots d^{a_d}}$ denote, respectively, the cardinalities of $Cov_{d,g,\sigma}/\sim$ and $Cov_{d,g,\sigma}^{1^{a_1}\ldots d^{a_d}}/\sim$. Define a total sum

$$N_{d,g,\sigma} = \sum_{1^{a_1}\ldots d^{a_d}} N_{d,g,\sigma}^{1^{a_1}\ldots d^{a_d}}$$

and a weighted sum

$$M_{d,g,\sigma} = \sum_{1^{a_1}\dots d^{a_d}} (\frac{a_1}{1} + \dots + \frac{a_d}{d}) N_{d,g,\sigma}^{1^{a_1}\dots d^{a_d}},$$

where the summations range over all conjugacy classes of S_d.

In this setting, $N_{d,g,\sigma}$ equals the number of non-isomorphic degree d genus g connected covers of elliptic curves with a unique branch point and the ramification class σ. The morphism from $Y_{d,g,\sigma}$ to the pencil Z of plane cubics is finite of degree $N_{d,g,\sigma}$. We have a slope formula for $Y_{d,g,\sigma}$ as follows.

Theorem 1.15. *Let $\sigma = (l_1)\cdots(l_m)$ denote the ramification class. The slope of $Y_{d,g,\sigma}$ is given by*

$$s(Y_{d,g,\sigma}) = \frac{12M_{d,g,\sigma}}{M_{d,g,\sigma} + \frac{1}{12}(d - \sum_{i=1}^{m}\frac{1}{l_i})N_{d,g,\sigma}}.$$

The slope $s(Y_{d,g,\sigma})$ only depends on the ratio of $N_{d,g,\sigma}$ and $M_{d,g,\sigma}$. We apply it to the case $g = 2$. By the Riemann-Hurwitz formula, a degree d genus two cover of an elliptic curve with a unique branch point has the ramification class σ equal to $1^{d-3}3^1$ or $1^{d-4}2^2$.

Corollary 1.16. *When $g = 2$ and σ is the conjugacy class $1^{d-3}3^1$ or $1^{d-4}2^2$, the slope of $Y_{d,2,\sigma}$ is equal to 10.*

Note that 10 is the sharp lower bound s_2 for the slope of effective divisors on $\overline{\mathcal{M}}_2$. For $g = 3$ and $\sigma = 1^{d-5}5^1$, a computer check implies that the slope of $Y_{d,3,1^{d-5}5^1}$ goes to 9 as d approaches infinity. For the slope of effective divisors on $\overline{\mathcal{M}}_3$, 9 is indeed the sharp lower bound s_3.

The study of the genus and components for $Y_{d,g,\sigma}$ also boils down to study certain monodromy actions.

Definition 1.17. Define two *monodromy actions* $h_\alpha : (\alpha,\beta) \to (\alpha, \alpha\beta)$ and $h_\beta : (\alpha,\beta) \to (\alpha\beta, \beta)$ on the set $Cov_{d,g,\sigma}/\sim$.

Note that both actions are well-defined with respect to the equivalence relation.

Theorem 1.18. *Two covers in $Y_{d,g,\sigma}$ are contained in the same component iff their corresponding data in $Cov_{d,g,\sigma}/\sim$ belong to the same orbit under the actions generated by h_α and h_β.*

Corollary 1.19. *When $g = 2$, d is prime and σ is the conjugacy class $1^{d-3}3^1$, the geometric genus of $Y_{d,2,1^{d-3}3^1}$ equals*

$$1 + \frac{1}{8}(d-1)(d-2)(15d+23) - 6\Big(\sum_{\substack{a_1l_1+a_2l_2=d \\ l_1>l_2}} (l_1,a_1)(l_2,a_2)\Big),$$

where (m,n) denotes the greatest common divisor of m and n. In particular, asymptotically $g(Y) \sim \frac{15}{8}d^3$.

Corollary 1.20. *When $g = 2$, d is prime and σ is the conjugacy class $1^{d-4}2^2$, the geometric genus of $Y_{d,2,1^{d-4}2^2}$ equals*

$$1 + \frac{1}{12}(d-1)(d-3)(10d^2 - 13d - 14) - 6\Big(\sum_{\substack{a_1l_1+a_2l_2+a_3l_3=d \\ l_1=l_2+l_3>l_2>l_3}} \prod_{i=1}^{3}(l_i,a_i)$$
$$- \sum_{\substack{a_1l_1+a_2l_2=d \\ l_1>l_2}} (l_1-2)(l_1,a_1)(l_2,a_2) - \sum_{2a_2+a_1=d} \frac{a_1-1}{(a_2,2)}\Big).$$

In particular, asymptotically $g(Y) \sim \frac{5}{6}d^4$.

Note that a general genus g curve cannot be a cover of an elliptic curve. Hence, the deformation of $W_{d,g}$ or $Y_{d,g,\sigma}$ may not cover an open set of $\overline{\mathcal{M}}_g$. However, for the purpose of bounding slopes, it suffices to verify that the union of $W_{d,g}$ or $Y_{d,g,\sigma}$ for all d maps to a Zariski dense subset of $\overline{\mathcal{M}}_g$. Then an effective divisor on $\overline{\mathcal{M}}_g$ cannot contain all $W_{d,g}$ or $Y_{d,g,\sigma}$. Indeed, we have such a density result as follows.

Theorem 1.21. *When g is fixed and d varies, the union* $\bigcup_d^{\infty} W_{d,g}$ *maps to a Zariski dense subset of* $\overline{\mathcal{M}}_g$. *When g and the ramification type are fixed but d varies, the union* $\bigcup_d^{\infty} Y_{d,g,\sigma}$ *maps to a Zariski dense subset of* $\overline{\mathcal{M}}_g$ *iff the number of ramification points is bigger than $g-2$, i.e. the number of cycles of length $l_i \geq 2$ in σ is bigger than $g-2$.*

The paper is organized as follows. In section 2 and 3, we study the geometry of $W_{d,g}$ and $Y_{d,g,\sigma}$ respectively, including their slopes, genera and components. In section 4, we verify the density result. In section 5, we discuss related results and open problems. Throughout the paper, we work over $\mathbb{C}$. A divisor on $\overline{\mathcal{M}}_g$ means a $\mathbb{Q}$-Cartier divisor. We always assume that $g \geq 2$.

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2. Geometry of $W_{d,g}$

2.1. **Slope.** In this part, we will prove Theorem 1.5. Recall that $W_{d,g}$ is the space of simply branched admissible covers of an elliptic curve over a one-dimensional base X. X admits $2g-2$ sections $\Gamma_1, \ldots, \Gamma_{2g-2}$ to $E \times X$. No three sections meet at a common point. If two sections meet, they meet transversely and a fiber can have at most one such intersection point. Suppose two sections Γ_1 and Γ_2 meet at p on a fiber E over a base point $b_l \in X$. Blow up $E \times X$ at p. The fiber becomes a nodal union of E with an exceptional curve $\mathbb{P}^1$. The proper transforms $\widetilde{\Gamma}_1$ and $\widetilde{\Gamma}_2$ of Γ_1 and Γ_2 meet the exceptional $\mathbb{P}^1$ at p_1 and p_2, respectively. Suppose that E meets the other sections at $p_3, \ldots, p_{2g-2}$. See Figure 5.

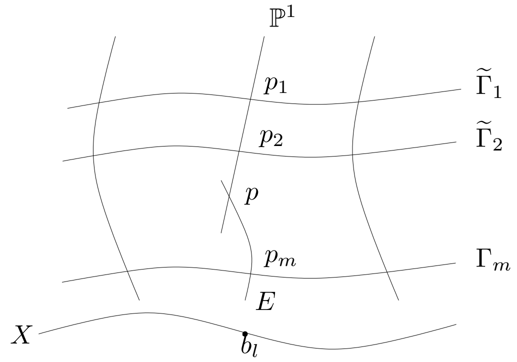


FIGURE 5. The blow-up of $E \times X$ at p

A smooth cover in $W_{d,g}$ over a general point $b \in X$ can degenerate to an admissible cover of $E \cup \mathbb{P}^1$ when b approaches b_l. Decompose $Cov_{d,g}$ using the subsets $Cov_{d,g}(0)$; $Cov_{d,g}(1,h)$, $1 \leq h \leq [\frac{d}{2}]$; $Cov_{d,g}(2)$; $Cov_{d,g}(3)$ in Definition 1.4. The type of the degenerate cover is determined by the subset that contains $(\alpha, \beta, \gamma_1, \ldots, \gamma_{2g-2})$. We will follow a similar description in [HM1] to draw the admissible covers and illustrate their stable limits.

The admissible covers in Figure 6 correspond to the degeneration of covers in $Cov_{d,g}(0)$. C_0 is a smooth genus $g-1$ curve admitting a degree d cover of E simply branched at $p_3, \ldots, p_{2g-2}$. There are $d-1$ rational curves attached to C_0. The first one meets C_0 at two distinct points in the pre-image of p and admits a double cover of the exceptional $\mathbb{P}^1$ simply branched at p_1, p_2. Each of the other $d-2$ rational curves meets C_0 at only one point in the pre-image of p and maps to the exceptional $\mathbb{P}^1$ isomorphically. The stable limit of this cover is an irreducible one-nodal curve of geometric genus $g-1$.

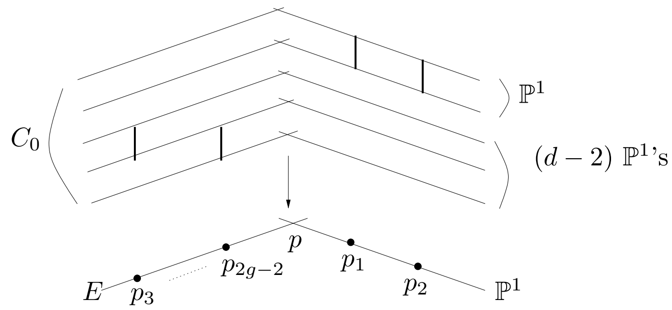


FIGURE 6. Degeneration of covers in $Cov_{d,g}(0)$

The admissible covers in Figure 7 correspond to the degeneration of covers in $Cov_{d,g}(1,h)$, $1 \leq h \leq [\frac{d}{2}]$. Without loss of the generality, assume that $\langle \alpha, \beta, \gamma_3, \ldots, \gamma_{2g-2} \rangle$ acts transitively on $\{1, \ldots, h\}$ and $\{h+1, \ldots, d\}$, respectively. Suppose there are m cycles out of $(\gamma_3, \ldots, \gamma_{2g-2})$ that act on $\{1, \ldots, h\}$ and $2g-4-m$ cycles that act on $\{h+1, \ldots, d\}$. Then the genera of C_1 and C_1' satisfy $2g(C_1) - 2 = m$, $2g(C_1') - 2 = 2g-4-m$, which implies that m must be even. C_1 admits a degree h map to E simply branched at the m intersection points of E and the corresponding m sections. Similarly, C_1' admits a degree $d-h$ map to E simply branched at the other $2g-4-m$ intersection points on E. There is a rational curve connecting C_1 and C_1' at two points in the pre-image of p. Moreover, it admits a double cover of the exceptional $\mathbb{P}^1$ simply branched at p_1, p_2. There are $h-1$ rational tails attached to C_1 and $d-h-1$ rational tails attached to C_1' at the other points in the pre-image of p. Each rational tail maps to the exceptional $\mathbb{P}^1$ isomorphically. The stable limit of this cover is a reducible one-nodal union of C_1 and C_1'. Note that $g(C_1) + g(C_1') = g$.

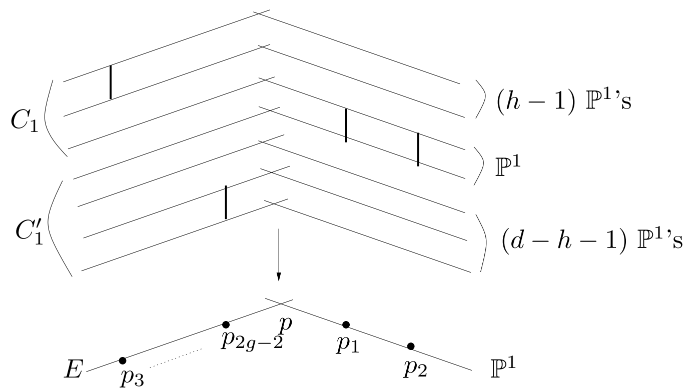


FIGURE 7. Degeneration of covers in $Cov_{d,g}(1,h)$

The admissible covers in Figure 8 correspond to the degeneration of covers in $Cov_{d,g}(2)$. C_2 is a smooth genus g curve that admits a degree d cover of E simply branched at $p_3, \ldots, p_{2g-2}$ and simply ramified at two points x, y over p. Attach to C_2 a rational curve that admits a double cover of the exceptional $\mathbb{P}^1$ by identifying one of the ramification points with x. In the same manner, attach another rational curve to C_2 at y. Locally around x, y and p, the covering map is given by $(u,v) \to (u^2, v^2)$. Finally, attach $d-4$ rational tails to C_2 at the other points in the pre-image of p. Each rational tail admits an isomorphic map to the exceptional $\mathbb{P}^1$. The stable limit of this cover is isomorphic to C_2.

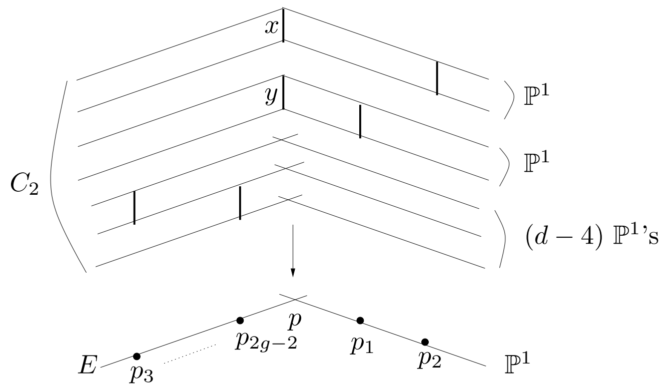


FIGURE 8. Degeneration of covers in $Cov_{d,g}(2)$

For covers in $Cov_{d,g}(3)$, their degeneration corresponds to the admissible covers in Figure 9. C_3 is a smooth genus g curve that admits a degree d cover of E simply branched at $p_3, \ldots, p_{2g-2}$ and triply ramified at a point x over p. Take a rational triple cover of the exceptional $\mathbb{P}^1$ that has a triple ramification point y over p and two other simple ramification points over p_1, p_2. Attach this rational curve to C_3 by gluing y to x. Locally around x, y and p, the covering map is given by $(u, v) \to (u^3, v^3)$. Finally, attach $d-3$ rational tails to C_3 at the other points in the pre-image of p. Each rational tail maps isomorphically to the exceptional $\mathbb{P}^1$. The stable limit of this cover is isomorphic to C_3.

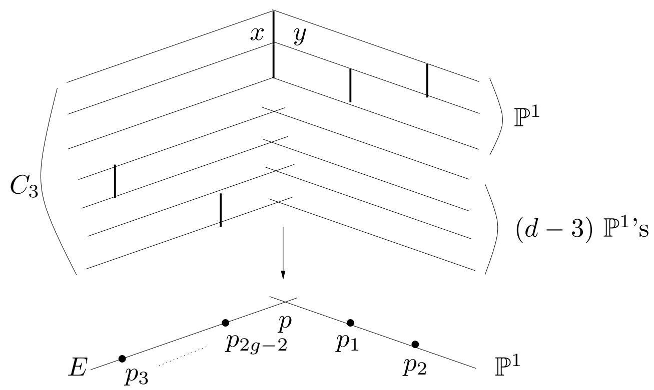


FIGURE 9. Degeneration of covers in $Cov_{d,g}(3)$

Let us first compute the degree of the boundary classes of $\overline{\mathcal{M}}_g$ on $W_{d,g}$.

Proposition 2.1. *In the above setting, let $k = \sum_{i<j} \Gamma_i . \Gamma_j$ denote the pairwise intersection numbers between the $2g-2$ sections. The degree of δ_0 on $W_{d,g}$ equals $2kN_{d,g}(0)$. The degree of $\delta_1 + \cdots + \delta_{[\frac{g}{2}]}$ on $W_{d,g}$ equals $2kN_{d,g}(1)$. In particular, the degree of the total boundary δ on $W_{d,g}$ equals $2k\big(N_{d,g}(0) + N_{d,g}(1)\big)$.*

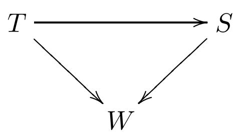
Proof. The degeneration of each cover in $Cov_{d,g}(0)$ contributes 2 to $W_{d,g}.\,\delta_0$, since the stable limit is only contained in δ_0 and the rational curve that admits a double cover of the exceptional $\mathbb{P}^1$ has self-intersetion -2. The degeneration of each cover in $Cov_{d,g}(1, h)$, $1 \leq h \leq [\frac{d}{2}]$ contributes 2 to $W_{d,g}.\,(\delta_1 + \cdots + \delta_{[\frac{g}{2}]})$, since the stable limit is a reducible one-nodal union of two smooth curves and the rational curve that admits a double cover has self-intersetion -2. For covers in $Cov_{d,g}(2)$ and $Cov_{d,g}(3)$, their degenerations are not contained in the boundary of $\overline{\mathcal{M}}_g$, since the stable limits are smooth genus g curves. Since the $2g-2$ sections meet pairwise at k points, the proposition follows immediately. □

Next, let us verify the degree of λ on $W_{d,g}$.

Proposition 2.2. *In the above setting, let* $k = \sum_{i<j} \Gamma_i . \Gamma_j$ *denote the pairwise intersection numbers between the* $2g-2$ *sections. The degree of* λ *on* $W_{d,g}$ *equals*

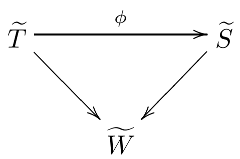
$$k\Big(\frac{N_{d,g}(0)+N_{d,g}(1)}{4}+\frac{N_{d,g}(3)}{36}\Big).$$

Proof. Let $\mathrm{Bl}(E\times X)$ denote the blow-up of $E\times X$ at the k intersection points of $\Gamma_1,\ldots,\Gamma_{2g-2}$. There are k special fibers of $\mathrm{Bl}(E\times X)$ over X. Each of them is a union of E and an exceptional $\mathbb{P}^1$. Let $\widetilde{\Gamma}_i$ denote the proper transform of Γ_i. Suppose that a section Γ_i meets the other sections k_i times. We have $\sum_{i=1}^{2g-2} k_i = 2k$. Pull back $\mathrm{Bl}(E\times X)$ via the map $W_{d,g}\to X$ to a surface S over $W_{d,g}$. We would like to form a universal admissible cover as follows:



such that over a singular fiber of S, a suitable degenerate admissible cover described before can appear as the corresponding fiber of T. This is not possible without a *base change*, noticed in [HM1], due to the fact that $\overline{\mathcal{H}}_{d,g}$ is in general a *coarse* moduli space rather than fine. In particular, the admissible cover corresponding to the degeneration of covers in $Cov_{d,g}(2)$, cf. Figure 8, has automorphisms coming from the involution of the two rational curves that admit double covers of the exceptional $\mathbb{P}^1$. We need a degree 2 base change such that around the node p of the singular fiber, the surface S is locally given by $zw = s^2$. Upstairs there are two nodes x and y simply ramified over p. Their local equations for T look like $uv = s$. The covering map is given by $z = u^2, w = v^2$. Note that a singularity of type A_1 arises after the base change. This also implies why we count 2 for its contribution to the degree of the boundary of $\overline{\mathcal{M}}_g$ restricted to $W_{d,g}$.

An *ad hoc* analysis for the local information around a degenerate cover was studied in [HM1, Theorem 2.15]. One can follow that method to derive the desired calculation. Nevertheless, here we take a different approach by passing to a base change of the Hurwitz space, such that a universal family of admissible covers exists after the base change. This kind of calculation is less cumbersome and can be generalized to more complicated ramification types, cf. the proof of Theorem 1.15. In particular, the slope as a quotient is invariant under the base change. Hence, let us take a degree n base change $\widetilde{W}\to W_{d,g}$ to realize the desired universal family of covers as follows:



It suffices to show that the degree of λ on $\widetilde{W}$ equals

$$nk\Big(\frac{N_{d,g}(0)+N_{d,g}(1)}{4}+\frac{N_{d,g}(3)}{36}\Big).$$

We use the relation $12\lambda = \delta_{\widetilde{T}} + \kappa$ for the family $\widetilde{T}\to\widetilde{W}$. $\delta_{\widetilde{T}}$ counts the total number of nodes in the singular fibers of $\widetilde{T}$. By enumerating the nodes of the degenerate admissible covers described before, we have

$$\begin{aligned}\delta_{\widetilde{T}} &= nk\Big(d(N_{d,g}(0)+N_{d,g}(1)) + (d-4+\frac{1}{2}+\frac{1}{2})N_{d,g}(2) + (d-3+\frac{1}{3})N_{d,g}(3)\Big)\\ &= nk\Big(dN - 3N_{d,g}(2) - \frac{8}{3}N_{d,g}(3)\Big).\end{aligned}$$

Note that if locally around a node the admissible cover is given by $(u,v)\to(u^m,v^m)$, we count the node with weight $\frac{1}{m}$ since m nearby sheets in $\widetilde{W}$ corresponding to smooth covers approach together.

Let ω denote the first Chern class of the relative dualizing sheaf. To calculate $\kappa = \omega^2_{\widetilde{T}/\widetilde{W}}$, we will use the branched cover $\widetilde{T} \to \widetilde{S}$. The calculation eventually boils down to a calculation on $\mathrm{Bl}(E \times X)$. By the Riemann-Hurwitz formula, $\omega_{\widetilde{T}/\widetilde{W}} = \phi^*\omega_{\widetilde{S}/\widetilde{W}} + \sum_{i=1}^{2g-2} D_i$, where D_i is the ramification section on $\widetilde{T}$ whose image on $\widetilde{S}$ corresponds to the pull-back of $\widetilde{\Gamma}_i$. Hence,

$$\omega^2_{\widetilde{T}/\widetilde{W}} = (\phi^*\omega_{\widetilde{S}/\widetilde{W}})^2 + 2\big(\sum_{i=1}^{2g-2} \phi^*\omega_{\widetilde{S}/\widetilde{W}}.\, D_i\big) + \big(\sum_{i=1}^{2g-2} D_i\big)^2.$$

We have

$$(\phi^*\omega_{\widetilde{S}/\widetilde{W}})^2 = d(\omega_{\widetilde{S}/\widetilde{W}})^2 = nd(\omega_{S/W})^2 = ndN_{d,g}(\omega_{Bl(E\times X)/X})^2 = -kndN_{d,g}.$$

Moreover,

$$\phi^*\omega_{\widetilde{S}/\widetilde{W}}.\, D_i = \omega_{\widetilde{S}/\widetilde{W}}.\, (\phi_* D_i) = -nN_{d,g}\widetilde{\Gamma}_i^2 = -nN_{d,g}(\Gamma_i^2 - k_i)$$

and

$$D_i^2 = \frac{1}{2}nN_{d,g}\widetilde{\Gamma}_i^2 = \frac{1}{2}nN_{d,g}(\Gamma_i^2 - k_i),\ D_i.\, D_j = 0,\ i \neq j.$$

Therefore,

$$\begin{aligned}\omega^2_{\widetilde{T}/\widetilde{W}} &= -kndN_{d,g} + 2nN_{d,g}\Big(\sum_{i=1}^{2g-2}(k_i - \Gamma_i^2)\Big) + \frac{1}{2}nN_{d,g}\Big(\sum_{i=1}^{2g-2}(\Gamma_i^2 - k_i)\Big) \\ &= nN_{d,g}\Big(-kd + 3k - \frac{3}{2}\sum_{i=1}^{2g-2}\Gamma_i^2\Big) = nN_{d,g}k(3-d),\end{aligned}$$

since $\Gamma_i^2 = 0$ on $E \times X$.

Plugging the above into the relation $\lambda = \frac{1}{12}(\delta_{\widetilde{T}} + \kappa)$, we get the desired equality. □

Proposition 2.3. *For an irreducible component W of $W_{d,g}$, the slope of W equals the slope of $W_{d,g}$.*

Proof. The union of the $2g-2$ branch points corresponds to a degree $2g-2$ divisor on E. Forget the order among the $2g-2$ points. Then $W_{d,g}$ and W both map to effective curves in $\mathrm{Sym}^{2g-2}E$, which is a projective bundle over $\mathrm{Pic}(E) \cong E$. In particular, $\mathrm{Sym}^{2g-2}E$ has Picard number equal to *two*. Take a curve class C_1 of $\mathrm{Sym}^{2g-2}E$ by varying a point of a degree $2g-2$ divisor along E. Take another curve class C_2 by moving a degree $2g-2$ divisor via the translation of E. Any curve class of $\mathrm{Sym}^{2g-2}E$ is numerically equivalent to $aC_1 + bC_2$. Let W_1 and W_2 denote the pre-images of C_1 and C_2, respectively, in the Hurwitz space $\overline{\mathcal{H}}_{d,g}$. W_2 maps to a point in $\overline{\mathcal{M}}_g$, since all the covers in W_2 are isomorphic. Therefore, the slopes of $W_{d,g}$ and W are both equal to the slope of W_1. □

Proposition 2.1, 2.2 and 2.3 complete the proof of Theorem 1.5.

Remark 2.4. In practice, the slope formula for $W_{d,g}$ can be applied to its components as well. An irreducible component of $W_{d,g}$ corresponds to a subset of $Cov_{d,g}/\sim$ that is an orbit under the monodromy actions in Theorem 1.10. We only need to decompose the subset as what we did for $Cov_{d,g}$ and re-enumerate the $N_{d,g}(i)$'s. By the proof of Proposition 2.1, the slope formula does not change its form.

2.2. **Monodromy.** In this part, we will prove Theorem 1.10. Recall that $W_{d,g}(E)$ is the space of degree d genus g admissible covers of E simply branched at $2g-2$ points $p_1, \ldots, p_{2g-2}$, where the first $2g-3$ points are fixed and the last point moves along E. For a general base point $b \in E$ and the closed paths $\alpha, \beta, \eta_1, \ldots, \eta_{2g-3}$ that generate $\pi_1(E_b, p_1, \ldots, p_{2g-3})$, cf. Figure 3, we want to study the monodromy actions $g_\alpha, g_\beta, g_1, \ldots, g_{2g-3}$ associated to those paths. Let $(\alpha, \beta, \gamma_1, \ldots, \gamma_{2g-2}) \in Cov_{d,g}/\sim$ correspond to a cover in $W_{d,g}(E)$ over b.

Proposition 2.5. *Going along the path η_i once, $1 \le i \le 2g-3$, the monodromy action g_i on $(\alpha, \beta, \gamma_1, \dots, \gamma_{2g-2}) \in$ Cov acts as follows:*

$$g_i(\alpha) = \alpha,\ g_i(\beta) = \beta,\ g_i(\gamma_j) = \gamma_j \text{ for } j < i,$$

$$g_i(\gamma_j) = \gamma_{2g-2}^{-1}\gamma_j\gamma_{2g-2} \text{ for } i \le j \le 2g-3,$$

$$g_i(\gamma_{2g-2}) = (\gamma_i \cdots \gamma_{2g-2})^{-1}\gamma_{2g-2}(\gamma_i \cdots \gamma_{2g-2}).$$

Proof. Since p_{2g-2} is the moving branch point which corresponds to the diagonal section of $E \times E$, the fiber over $p_i \in E$ has an intersection point between Γ_i and Γ_{2g-2}. Going along η_i once, it amounts to circling p_{2g-2} around $p_i, \dots, p_{2g-3}$ once while keeping all the other paths α, β, γ_j's in a *relatively fixed* position. Figure 10 illustrates what happens when p_{2g-2} moves along the dashed circle homotopic to η_i.

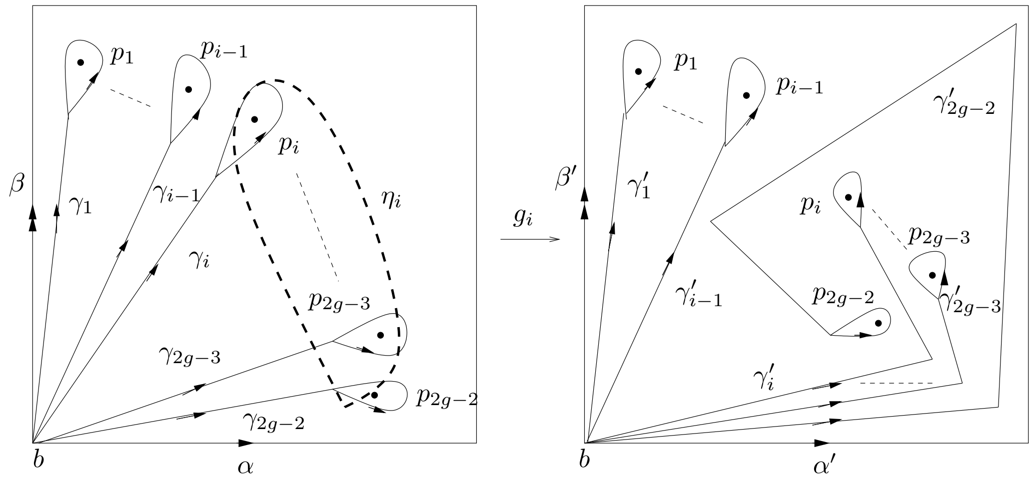


FIGURE 10. The action of g_i

Note that in $\pi_1(E_b, p_1, \dots, p_{2g-3})$, we can express the new paths by the original paths as follows:

$$\gamma'_j \sim \gamma_j,\ j < i,$$

$$\gamma'_j \sim \gamma_{2g-2}^{-1}\gamma_j\gamma_{2g-2},\ i \le j \le 2g-3,$$

$$\gamma'_{2g-2} \sim (\gamma_i \cdots \gamma_{2g-2})^{-1}\gamma_{2g-2}(\gamma_i \cdots \gamma_{2g-2}).$$

Moreover, $\alpha' \sim \alpha$ and $\beta' \sim \beta$ do not change under g_i. The relations are the same as those in the proposition. □

Proposition 2.6. *Going along the path α once, the monodromy action g_α on $(\alpha, \beta, \gamma_1, \dots, \gamma_{2g-2}) \in$ Cov acts as follows:*

$$g_\alpha(\alpha) = \alpha,\ g_\alpha(\beta) = \beta\gamma_{2g-2},\ g_\alpha(\gamma_{2g-2}) = \alpha^{-1}\gamma_{2g-2}\alpha,$$

$$g_\alpha(\gamma_j) = \gamma_{2g-2}^{-1}\gamma_j\gamma_{2g-2} \text{ for } j \le 2g-3.$$

Proof. Let us draw what happens to those paths as p_{2g-2} moves along the dashed path homotopic to α, cf. Figure 11. We have the following relations to express the new paths by the original paths:

$$\alpha' \sim \alpha,\ \beta' \sim \beta\gamma_{2g-2},\ \gamma'_{2g-2} \sim \alpha^{-1}\gamma_{2g-2}\alpha,$$

$$\gamma'_j \sim \gamma_{2g-2}^{-1}\gamma_j\gamma_{2g-2},\ j \le 2g-3.$$

These relations imply the desired proposition. □

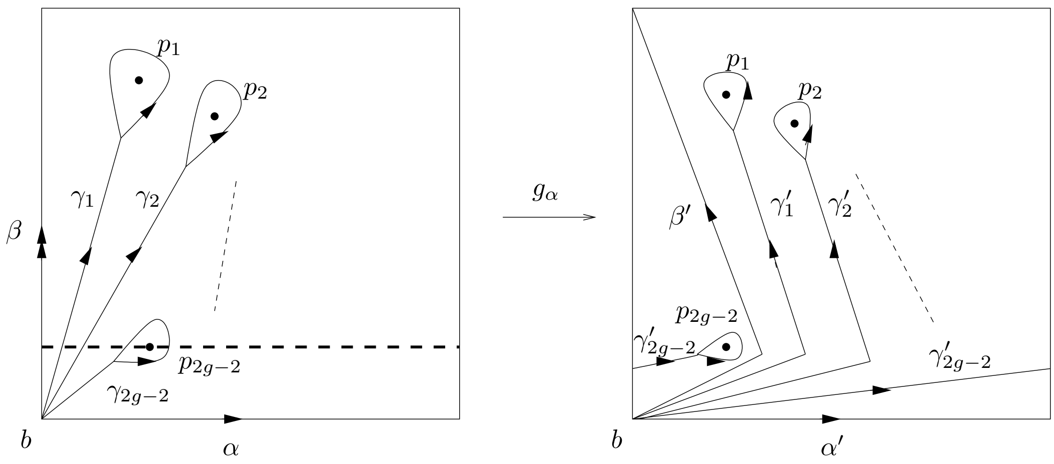


FIGURE 11. The action of g_α

Proposition 2.7. *Going along the path β once, the monodromy action g_β on $(\alpha, \beta, \gamma_1, \ldots, \gamma_{2g-2}) \in Cov$ acts as follows:*

$$g_\beta(\alpha) = \alpha\gamma_{2g-2}^{-1},\ g_\beta(\beta) = \beta,\ g_\beta(\gamma_j) = \gamma_j \text{ for } j \le 2g-3,$$
$$g_\beta(\gamma_{2g-2}) = (\beta\gamma_1 \cdots \gamma_{2g-3})^{-1}\gamma_{2g-2}(\beta\gamma_1 \cdots \gamma_{2g-3}).$$

Proof. The monodromy action changes the paths as in Figure 12, when p_{2g-2} moves along the dashed path homotopic to β. We have the following relations to express the new paths by the original paths:

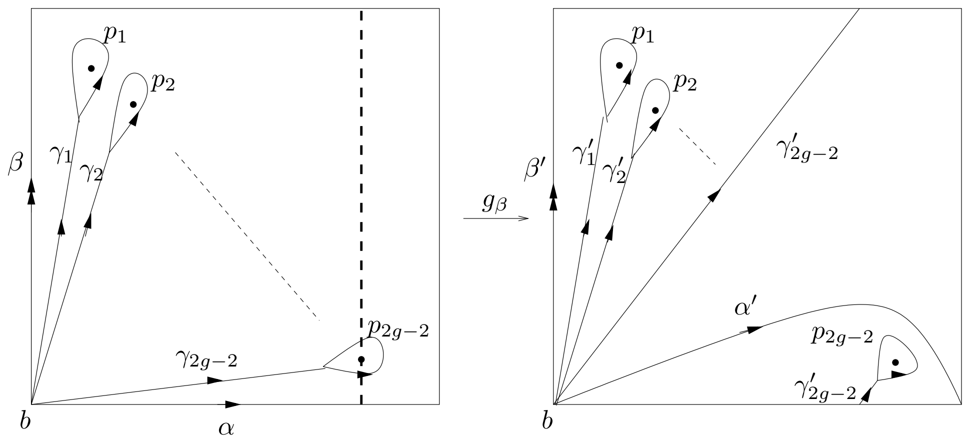


FIGURE 12. The action of g_β

$$\alpha' \sim \alpha\gamma_{2g-2}^{-1},\ \beta' \sim \beta,\ \gamma'_j \sim \gamma_j,\ j \le 2g-3,$$
$$\gamma'_{2g-2} \sim (\beta\gamma_1 \cdots \gamma_{2g-3})^{-1}\gamma_{2g-2}(\beta\gamma_1 \cdots \gamma_{2g-3}).$$

These relations imply the proposition. □

Proof of Theorem 1.10. The monodromy actions g_i for $1 \le i \le 2g-3$, g_α and g_β are verified in Proposition 2.5, 2.6 and 2.7, respectively. Moreover, the morphism $W_{d,g}(E) \to E$ is unramified over $E - \{p_1, \ldots, p_{2g-3}\}$. Since $\alpha, \beta, \eta_1, \ldots, \eta_{2g-3}$ generate $\pi_1(E_b, p_1, \ldots, p_{2g-3})$, two covers are contained in the same irreducible component of $W_{d,g}(E)$ iff their corresponding data in $Cov_{d,g}/\sim$ belong to the same orbit under the monodromy actions generated by $g_\alpha, g_\beta, g_1, \ldots, g_{2g-3}$. □

2.3. **Application.** In this part, we apply the general theory of $W_{d,g}$ obtained above to various examples.

2.3.1. *The slope of* $W_{d,2}$. When $g = 2$, let us compute those numbers $N_{d,2}(i)$'s in Definition 1.4. For simplicity, we only consider the case when d is *odd*. The set $Cov_{d,2}$ in Definition 1.4 is reduced to:

$$Cov_{d,2} = \{(\alpha, \beta, \gamma_1, \gamma_2) \in S_d \times S_d \times S_d \times S_d \mid \beta^{-1}\alpha^{-1}\beta\alpha = \gamma_1\gamma_2,$$
$$\gamma_1, \gamma_2 \text{ are simple transpositions and } \langle \alpha, \beta, \gamma_1, \gamma_2 \rangle \text{ acts transitively on } \{1, \ldots, d\}\}.$$

Its subsets in Definition 1.4 are reduced to:

$$Cov_{d,2}(0) = \{(\alpha, \beta, \gamma_1, \gamma_2) \in Cov_{d,2} \mid \gamma_1 = \gamma_2,\ \langle \alpha, \beta \rangle \text{ acts transitively on } \{1, \ldots, d\}\};$$
$$Cov_{d,2}(h, 1) = \{(\alpha, \beta, \gamma_1, \gamma_2) \in Cov_{d,2} \mid \gamma_1 = \gamma_2,\ \langle \alpha, \beta \rangle \text{ acts transitively on a partition } (h \mid d-h)\};$$
$$Cov_{d,2}(2) = \{(\alpha, \beta, \gamma_1, \gamma_2) \in Cov_{d,2} \mid \gamma_1 \cap \gamma_2 = \emptyset\};$$
$$Cov_{d,2}(3) = \{(\alpha, \beta, \gamma_1, \gamma_2) \in Cov_{d,2} \mid |\gamma_1 \cap \gamma_2| = 1\}.$$

Recall that $1^{a_1} \ldots d^{a_d}$ denotes a conjugacy class of S_d that has a_i cycles of length-i. We need the following lemma and definition.

Lemma 2.8. *For an element τ in S_d, if τ commutes with all the elements in a transitive subgroup G of S_d, then τ consists of m cycles of length-l, $lm = d$. Namely, τ belongs to the conjugacy class l^m.*

Proof. It suffices to prove that for any $t \in \mathbb{Z}$, if τ^t fixes an element in $\{1, 2, \ldots, d\}$, then $\tau^t = id$. Suppose that $\tau^t(i) = i$. For any $j \neq i$, there exists $\xi \in G$ such that $\xi(i) = j$. Since τ commutes with ξ, we have $\xi\tau^t\xi^{-1} = \tau^t$. Hence, $\tau^t(j) = j$ and $\tau^t = id$. ☐

Definition 2.9. For positive integers d and i, define the sum $\sigma_i(d) := \sum_{l|d} l^i$, where the summation ranges over all positive divisors l of d.

Let us first figure out the number $N_{d,2}(0) = |Cov_{d,2}(0)/\sim|$. Consider the conjugate action of S_d on the set $Cov_{d,2}(0)$. Each orbit corresponds to an element of $Cov_{d,2}(0)/\sim$. Introduce a new set

$$Cov_{d,2}(0)(\tau) = \{(\alpha, \beta, \gamma_1, \gamma_2) \in Cov_{d,2}(0) \mid \tau \text{ commutes with } \alpha, \beta, \gamma_1, \gamma_2\}$$

parameterizing data in $Cov_{d,2}(0)$ that have τ as a stabilizer under the conjugate action of S_d. By Burnside's Lemma, we have

$$N_{d,2}(0) = \frac{1}{d!} \sum_{\tau \in S_d} |Cov_{d,2}(0)(\tau)|.$$

Since γ_1 and γ_2 are both equal to the simple transposition $(t_1 t_2)$, τ commutes with γ_1 and γ_2 iff τ either fixes t_1, t_2 or permutes them. By Lemma 2.8 and the assumption that d is odd, τ cannot have a cycle of length-2. Therefore, τ must be the identity. Then we have

$$N_{d,2}(0) = \frac{1}{d!}|Cov_{d,2}(0)|.$$

For an element $(\alpha, \beta, \gamma_1, \gamma_2) \in Cov_{d,2}(0)$, $\gamma_1 = \gamma_2$ can have $\binom{d}{2}$ options. Fix a choice. We have $\alpha^{-1}\beta\alpha = \beta$. In this form, if β has two cycles of different lengths, the letters in these two cycles can never be exchanged by α, which contradicts to the transitivity of $\langle \alpha, \beta \rangle$. Hence, β belongs to a conjugacy class l^a, where $la = d$. β has $\frac{d!}{l^a a!}$ choices. For a fixed β, α has $(a-1)!l^a$ choices. In total, we get

$$|Cov_{d,2}(0)| = \binom{d}{2} \sum_{l|d} \frac{d!}{l} \text{ and } N_{d,2}(0) = \frac{d-1}{2}\sigma_1(d).$$

Similarly, one can verify that

$$N_{d,2}(1, h) = \sum_{\substack{a_1 l_1 = h, \\ a_2 l_2 = d-h}} l_1 l_2,$$

$$N_{d,2}(1) = \frac{1}{2} \sum_{h=1}^{d-1} \sigma_1(h)\sigma_1(d-h),$$

$$N_{d,2}(3) = \frac{3}{2}\Big(\sum_{h=1}^{d-1}\sigma_1(h)\sigma_1(d-h)\Big) - (\frac{3}{2}d-1)\sigma_1(d) + \frac{1}{2}\sigma_3(d).$$

Since $N_{d,2}(2)$ does not appear in the slope formula, we skip its enumeration. In order to simplify those summations, the following equality will be used frequently.

Lemma 2.10. *For a positive integer d, we have the following equality*

$$\sum_{k=1}^{d-1}\sigma_1(k)\sigma_1(d-k) = (\frac{1}{12}-\frac{d}{2})\sigma_1(d) + \frac{5}{12}\sigma_3(d).$$

Proof. Note that

$$\sum_{k=1}^{d-1}\sigma_1(k)\sigma_1(d-k) = \Bigg[\Big(\sum_{k=1}^{\infty}\sigma_1(k)q^k\Big)^2\Bigg]_d,$$

where $[\cdot]_d$ means the coefficient of the degree d term in the series expansion. Moreover,

$$\sum_{k=1}^{\infty}\sigma_1(k)q^k = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}kq^{kj} = \sum_{k=1}^{\infty}\frac{kq^k}{1-q^k}.$$

Similarly, we have

$$\sum_{k=1}^{\infty}\sigma_3(k)q^k = \sum_{k=1}^{\infty}\frac{k^3q^k}{1-q^k}.$$

Define three series:

$$P = 1 - 24\sum_{k=1}^{\infty}\frac{kq^k}{1-q^k},$$
$$Q = 1 + 240\sum_{k=1}^{\infty}\frac{k^3q^k}{1-q^k},$$
$$R = 1 - 504\sum_{k=1}^{\infty}\frac{k^5q^k}{1-q^k}.$$

There are relations among P, Q and R given by the Ramanujan differential equations, cf. [BY]:

$$q\frac{dP}{dq} = \frac{P^2-Q}{12},\ q\frac{dQ}{dq} = \frac{PQ-R}{3},\ q\frac{dR}{dq} = \frac{PR-Q^2}{2}.$$

Using the first relation, we get

$$\Bigg[\Big(\sum_{k=1}^{\infty}\sigma_1(k)q^k\Big)^2\Bigg]_d = \Bigg[(\frac{1-P}{24})^2\Bigg]_d = \Bigg[\frac{1}{24^2} - \frac{P}{24\cdot 12} + \frac{1}{24^2}(Q+12q\frac{dP}{dq})\Bigg]_d$$
$$= (\frac{1}{12}-\frac{d}{2})\sigma_1(d) + \frac{5}{12}\sigma_3(d).$$

□

Proof of Corollary 1.6. We provide two proofs here. The first one only works for odd d. Plugging in the expressions of those $N_{d,2}(i)$'s, the desired equality is equivalent to the following

$$\sum_{h=1}^{d-1}\sigma_1(h)\sigma_1(d-h) = (\frac{1}{12}-\frac{d}{2})\sigma_1(d) + \frac{5}{12}\sigma_3(d),$$

which has been proved in Lemma 2.10.

The other proof is indirect but more interesting. The moduli space $\overline{\mathcal{M}}_2$ is special in that the equality $\lambda = \frac{1}{10}\delta_0 + \frac{1}{5}\delta_1$ holds. Note that in Proposition 2.1 and 2.2, the restrictions of λ, δ_0 and δ_1 on $W_{d,2}$ have degree equal to $\frac{1}{4}(N_{d,2}(0)+N_{d,2}(1)) + \frac{1}{36}N_{d,2}(3)$, $2N_{d,2}(0)$ and $2N_{d,2}(1)$, respectively. Plugging them into the relation $\lambda = \frac{1}{10}\delta_0 + \frac{1}{5}\delta_1$, we get the desired equality. □

Proof of Corollary 1.7. Using Corollary 1.6, we have the slope

$$s(W_{d,2}) = \frac{10\big(N_{d,2}(0) + N_{d,2}(1)\big)}{N_{d,2}(0) + 2N_{d,2}(1)}.$$

This quotient approaches 5 for large d, since $N_{d,2}(0)/N_{d,2}(1)$ approaches zero by their expressions and Lemma 2.10. □

2.3.2. *The slope of $W_{2,g}$.* Let us consider double covers of an elliptic curve. This case is easy to analyze since we are working with S_2. For $(\alpha, \beta, \gamma_1, \ldots, \gamma_{2g-2}) \in Cov_{2,g}$, all the γ_i's are the same simple transposition (12) permuting the two sheets of the cover. α and β can take value on any element of S_2. There are in total four different combinations. All of them are not equivalent. Hence, we get $N_{2,g} = 4$. In particular, one can check that $N_{2,g}(0) = 3, N_{2,g}(1) = 1, N_{2,g}(2) = N_{2,g}(3) = 0$. By the slope formula, we get $s(W_{2,g}) = 8$. This was also noticed in [X, Example 3]. Moreover, the four equivalence classes in $Cov_{2,g}/\sim$ can be sent to each other by the monodromy actions in Theorem 1.10. Hence, $W_{2,g}(E)$ is an irreducible degree 4 cover of E. If an effective divisor D on $\overline{\mathcal{M}}_g$ has slope smaller than 8, D must contain the image of $W_{2,g}(E)$ in $\overline{\mathcal{M}}_g$. This completes the proof of Corollary 1.8.

When the Brill-Noether number equals -1, the Brill-Noether divisor parameterizing curves with a g^r_d has slope $6 + \frac{12}{g+1}$ on $\overline{\mathcal{M}}_g$, cf. [HMu]. For $g = 5$, $r = 1$ and $d = 3$, it has slope equal to 8. Since $s(W_{2,g}) = 8$, this divisor has zero intersection with $W_{2,5}$. This conclusion coincides with a classical fact that a genus 5 bielliptic curve cannot be trigonal. For $g > 5$, the Brill-Noether divisor has slope strictly smaller than 8. Hence, a bielliptic curve is contained in the Brill-Noether locus in that case. A similar discussion can also be drawn for the Gieseker-Petri divisor.

2.3.3. *The genus of $W_{d,g}$.* We can obtain the genus of $W_{d,g}$ by the following result.

Proposition 2.11. *The morphism $W_{d,g} \to X$ is triply ramified at a point corresponding to a local degeneration of the covers in $Cov_{d,g}(3)/\sim$. It is unramified everywhere else.*

Proof. Since the ramification type is local, the proof is the same as [HM1, Proposition 2.9.1]. □

Applying the Riemann-Hurwitz formula to $W_{d,g} \to X$, we get

$$g(W_{d,g}) = 1 + N_{d,g}(g(X) - 1) + kN_{d,g}(3).$$

This completes the proof of Corollary 1.11.

For the case $g = 2$, d is odd and $X \cong E$, using the explicit enumeration for $N_{d,2}(3)$ obtained previously, one can verify Corollary 1.13 immediately. For the case $d = 2$, using Proposition 1.11, one can also verify Corollary 1.12 directly.

Remark 2.12. Note that if $W_{d,g}$ is reducible, each component of $W_{d,g}$ admits a finite map to X. Once we know the subset of $Cov_{d,g}/\sim$ corresponding to a component of $W_{d,g}$, the above formulae can be easily modified to calculate the genus of that component.

2.3.4. *The components of $W_{d,g}$.* We present two examples where $W_{d,g}$ are irreducible and reducible, respectively. Use $(a_1 \ldots a_n)$ to denote a cycle of a permutation in S_d that sends the letter a_i to a_{i+1}.

Example 2.13. For genus two triple covers of an elliptic curve, one can check that $N_{3,2} = 16$. Moreover, $N_{3,2}(0) = 4, N_{3,2}(1) = 3, N_{3,2}(2) = 0$ and $N_{3,2}(3) = 9$. Note that these numbers satisfy the equality in Corollary 1.6. The slope of $W_{3,2}$ equals 7 by the slope formula. Finally, the monodromy actions in Theorem 1.10 act transitively on the 16 equivalence classes of $Cov_{3,2}/\sim$. Therefore, $W_{3,2}(E)$ is an irreducible 16-sheeted cover of E.

Example 2.14. Consider the case $g = 2$ and $d = 4$. Let a cover $\pi_1\colon C_1 \to E$ correspond to an element $\alpha = \beta = (13)(24), \gamma_1 = \gamma_2 = (12)$ in $Cov_{4,2}$. The group generated by $\alpha, \beta, \gamma_1, \gamma_2$ is a proper subgroup of S_4. On the other hand, let $\pi_2\colon C_2 \to E$ correspond to another element $\alpha = \beta = (1234), \gamma_1 = \gamma_2 = (12)$ in $Cov_{4,2}$. In this case, $\alpha, \beta, \gamma_1, \gamma_2$ generate S_4. Since the conjugacy type of the group generated by $\alpha, \beta, \gamma_1, \gamma_2$ is invariant under the monodromy actions, the two covers π_1 and π_2 are contained in two different components of $W_{4,2}(E)$. Hence, $W_{4,2}(E)$ is reducible.

The conjugacy type of the subgroup generated by $\alpha, \beta, \gamma_1, \ldots, \gamma_{2g-2}$ in S_d is a *parity* that distinguishes connected components of $\mathcal{H}_{d,g}$. Since $\mathcal{H}_{d,g}$ is a finite unramified cover of $\mathcal{M}_{1,2g-2}$, it does not make a difference to consider connected or irreducible components of $\mathcal{H}_{d,g}$. For the beginning case $g = 2$, Kani studied the Hurwitz space $\mathcal{H}_{d,2}$ over a field of characteristic prime to $2d$. One of the related results is the following, cf. [Ka, Corollary 1.3].

Proposition 2.15. *If d is odd, the number of components of $\mathcal{H}_{d,2}$ equals* $\sum_{n|d,\ n<d} \sigma_1(n)$.

Therefore, $\mathcal{H}_{d,2}$ is irreducible iff d is prime. This coincides with our analysis for Example 2.13 and 2.14.

3. Geometry of $Y_{d,g,\sigma}$

3.1. **Slope.** In this part, we will prove Theorem 1.15. The Hurwitz space $\overline{\mathcal{H}}_{d,g,\sigma}$ parameterizes degree d genus g connected admissible covers of elliptic curves that have a unique branch point with the ramification class σ. The morphism $\overline{\mathcal{H}}_{d,g,\sigma} \to \overline{\mathcal{M}}_{1,1}$ is finite of degree $N_{d,g,\sigma}$ unramified away from $[C_0]$ that parameterizes the rational one-nodal curve C_0. The key observation is the following.

Proposition 3.1. *For a cover parameterized by a point of $\overline{\mathcal{H}}_{d,g,\sigma}$, the stable limit of its covering genus g curve is not contained in the boundary δ_i for $i > 0$.*

Proof. We need to show that the stable limit of an admissible cover of the rational one-nodal curve C_0 is not contained in δ_i for $i > 0$. Suppose $\pi : C \to C_0$ is such a cover. C is a nodal genus g connected curve. Let q denote the node of C_0. By [HM2, Definition 3.149], $\pi^{-1}(q)$ consists of all the nodes of C. Call a node *internal* if its removal does not disconnect the whole curve. Otherwise, call it *external.* If all the nodes of C are internal, the stable limit of C will not be contained in δ_i for $i > 0$.

Take an irreducible component A of C. Consider all the nodes of C that are contained in A. We assign those nodes in pairs as follows. If A self-intersects at a node r, π maps the two local branches of r on A to the local branches of p on C_0 with the same ramification order, respectively. Then we take r with itself as a pair (r, r). If π maps a smooth point r_1 of A to p, since A is irreducible, there is another smooth point r_2 of A that maps to p. Note that r_1 and r_2 are both nodes of C by the definition of admissible covers. Namely, A meets some other components of C at r_1 and r_2. We take r_1 and r_2 as a pair (r_1, r_2) on A.

Suppose there is node of C, where two irreducible components A_0 and A_1 meet by gluing $a_0 \in A_0$ and $a_1 \in A_1$. Consider the pair assignment on A_1. The point a_1 must be contained in a pair (a_1, a_2), where a_2 is a smooth point of A_1. Since a_2 is a node of C, there has to be another irreducible component A_2 that meets A_1 at a_2. Then a_2 has to be contained in a pair (a_2, a_3) on A_2, where a_3 is a smooth point of A_2, etc. This process has to stop for some k. Hence, we get $a_{k+1} = a_0$, namely, (a_k, a_0) is a pair on $A_k = A_0$. The components $A_1, \ldots, A_k$ form a loop along the nodes $a_1, \ldots, a_k$. Therefore, these nodes are internal.

A more intuitive argument was told to the author by McMullen. There is a local vanishing cycle $\beta \in H_1(E, \mathbb{Z})$ when a smooth elliptic curve E degenerates to the rational nodal curve C_0. Consider a cover $\pi : C \to E$. Since β is nonzero, as β shrinks to the node of C_0, any component γ of $\pi^{-1}(\beta)$ is not vanishing in $H_1(C, \mathbb{Z})$. Hence, when γ shrinks, the resulting node will be internal. □

Recall the set up for the one-parameter family $Y_{d,g,\sigma}$. We take a general pencil Z of plane cubics. Blowing up the 9 base points of the pencil, we get a smooth surface $\mathrm{Bl}_9(\mathbb{P}^2)$ as an elliptic fibration over Z. There are 12 rational nodal fibers over Z. Fix a section Γ corresponding to the blow-up of a base point. Using Γ, Z admits a finite map of degree 12 to $\overline{\mathcal{M}}_{1,1}$. We define $Y_{d,g,\sigma}$ as the fiber product of the two morphisms $Z \to \overline{\mathcal{M}}_{1,1}$ and $\overline{\mathcal{H}}_{d,g,\sigma} \to \overline{\mathcal{M}}_{1,1}$. The map $Y_{d,g,\sigma} \to Z$ is finite of degree $N_{d,g,\sigma}$, possibly branched at $b_1, \ldots, b_{12}$ that parameterize the 12 rational nodal curves in the pencil.

Proof of Theorem 1.15. Pull back $\mathrm{Bl}_9(\mathbb{P}^2)$ via the map $Y_{d,g,\sigma} \to Z$ to a surface S over $Y_{d,g,\sigma}$. As in the proof of Theorem 1.5, we make a degree n base change $\widetilde{Y} \to Y_{d,g,\sigma}$ such that there exists a universal family of admissible covers as follows:

$$\begin{array}{ccc} \widetilde{T} & \xrightarrow{\varphi} & \widetilde{S} \\ & \searrow \quad \swarrow & \\ & \widetilde{Y} & \end{array}$$

Let us first figure out the degree of δ on $\widetilde{Y}$. By Proposition 3.1, it is equal to the degree of δ_0 on $\widetilde{Y}$. Suppose over a point $b_i \in \widetilde{Y}$, the fiber of $\widetilde{S}$ is a rational nodal curve. In a neighborhood of b_i, topologically we can identify all the smooth fibers of $\widetilde{S}$ to a punctured torus (E, p). Let α and β denote the standard symplectic basis of $\pi_1(E_b, p)$ such that β is the vanishing cycle when the smooth elliptic curve degenerates to the rational nodal curve over b_i. Recall that a cover in $\overline{\mathcal{H}}_{d,g,\sigma}$ corresponds to an equivalence class in $Cov_{d,g,\sigma}/\sim$. We still use α and β to denote the monodromy images in S_d associated to the closed paths α and β, respectively. Assume that β belongs to the conjugacy class $1^{a_1}\dots d^{a_d}$. Since β is the vanishing cycle, the degenerate admissible cover will have a_i nodes where the map φ is locally given by $(u, v) \to (u^i, v^i)$. By the proof of Proposition 3.1, such a node is internal. It contributes $\frac{1}{i}$ to the degree of δ_0 on $\widetilde{Y}$. We count it with weight $\frac{1}{i}$, since i nearby sheets in $\widetilde{Y}$ corresponding to smooth covers approach together locally around this node. Therefore, the degree of δ on $\widetilde{Y}$ is given by

$$12n \sum_{1^{a_1}\dots d^{a_d}} \Big(\frac{a_1}{1} + \dots + \frac{a_d}{d}\Big) N_{d,g,\sigma}^{1^{a_1}\dots d^{a_d}} = 12nM_{d,g,\sigma}.$$

The coefficient $12n$ comes from the total number of the singular fibers of $\widetilde{S}$ over $\widetilde{Y}$.

To calculate the degree of λ on $\widetilde{Y}$, we use the equality $12\lambda = \delta + \kappa$ for the surface $\widetilde{T}$ over $\widetilde{Y}$. Let ω denote the relative dualizing sheaf. It suffices to compute $\kappa = \omega^2_{\widetilde{T}/\widetilde{Y}}$. Let the m sections $\widetilde{\Gamma}_1, \dots, \widetilde{\Gamma}_m$ of $\widetilde{T}$ and the section $\widetilde{\Gamma}$ of $\widetilde{S}$ correspond to the pull-backs of the branch section Γ from S. Since the ramification class σ is of type $(l_1)\dots(l_m)$, we have $\varphi^*\widetilde{\Gamma} = \sum_{i=1}^m l_i\widetilde{\Gamma}_i$. Since S is the pull-back of $\mathrm{Bl}_9(\mathbb{P}^2)$ from the pencil Z to $Y_{d,g,\sigma}$ and the map $Y_{d,g,\sigma} \to Z$ is of degree $N_{d,g,\sigma}$, on S we have

$$\omega^2_{S/Y} = 0,\ \Gamma^2 = -N_{d,g,\sigma},\ \Gamma.\,\omega_{S/Y} = N_{d,g,\sigma}.$$

For $\varphi: \widetilde{T} \to \widetilde{S}$, by the Riemann-Hurwitz formula, $\omega_{\widetilde{T}/\widetilde{Y}}$ and $\omega_{\widetilde{S}/\widetilde{Y}}$ satisfy the relation

$$\omega_{\widetilde{T}/\widetilde{Y}} = \varphi^*\omega_{\widetilde{S}/\widetilde{Y}} + \sum_{i=1}^m (l_i - 1)\widetilde{\Gamma}_i.$$

Moreover,

$$\varphi_*\widetilde{\Gamma}_i = \widetilde{\Gamma},\ (\varphi^*\omega_{\widetilde{S}/\widetilde{Y}})^2 = dn\omega^2_{S/Y} = 0$$

and

$$\widetilde{\Gamma}_i.\,\widetilde{\Gamma}_j = 0,\ i \neq j.$$

Hence, we get

$$l_i\widetilde{\Gamma}_i^2 = \widetilde{\Gamma}_i.\,(\varphi^*\widetilde{\Gamma}) = (\varphi_*\widetilde{\Gamma}_i).\,\widetilde{\Gamma} = \widetilde{\Gamma}^2 = n\Gamma^2 = -nN_{d,g,\sigma}$$

and

$$\widetilde{\Gamma}_i.\,(\varphi^*\omega_{\widetilde{S}/\widetilde{Y}}) = (\varphi_*\widetilde{\Gamma}_i).\,\omega_{\widetilde{S}/\widetilde{Y}} = \widetilde{\Gamma}.\,\omega_{\widetilde{S}/\widetilde{Y}} = n(\Gamma.\,\omega_{S/Y}) = nN_{d,g,\sigma}.$$

Now, a routine calculation shows that

$$\omega^2_{\widetilde{T}/\widetilde{Y}} = \left(\sum_{i=1}^m l_i - \sum_{i=1}^m \frac{1}{l_i}\right) nN_{d,g,\sigma} = \left(d - \sum_{i=1}^m \frac{1}{l_i}\right) nN_{d,g,\sigma}.$$

Using the equality $12\lambda = \delta + \kappa$, the degree of λ on $\widetilde{Y}$ equals

$$n\left(M_{d,g,\sigma} + \frac{1}{12}\Big(d - \sum_{i=1}^m \frac{1}{l_i}\Big)N_{d,g,\sigma}\right).$$

The slope of $Y_{d,g,\sigma}$ follows from that of $\widetilde{Y}$, since a finite base change does not change the slope. □

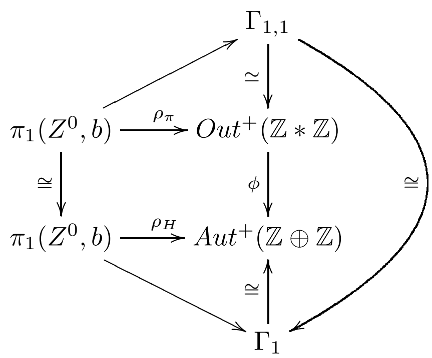
Remark 3.2. In practice, the slope formula for $Y_{d,g,\sigma}$ can be applied to its components as well. An irreducible component of $Y_{d,g,\sigma}$ corresponds to a subset of $Cov_{d,g,\sigma}/\sim$ that is an orbit under the monodromy actions in Theorem 1.18. We only need to re-enumerate the total sum $N_{d,g,\sigma}$ and the weighted sum $M_{d,g,\sigma}$ for this subset. By the proof of Theorem 1.15, the slope formula does not change its form.

3.2. **Monodromy.** In this part, we study the monodromy of the map $Y_{d,g,\sigma} \to Z$. Fix a base point b on $Z \cong \mathbb{P}^1$. Let Z^0 denote the punctured $\mathbb{P}^1$ at the points $b_1, \ldots, b_{12}$ corresponding to the 12 rational nodal fibers. Let E denote the smooth elliptic fiber over b and $p \in E$ be the marked point. Consider the π_1-monodromy map $\rho_\pi : \pi_1(Z^0, b) \to Out^+(\pi_1(E_p)) = Out^+(\mathbb{Z} * \mathbb{Z})$, where $Out^+(\mathbb{Z} * \mathbb{Z}) = Aut(\mathbb{Z} * \mathbb{Z})/Inn(\mathbb{Z} * \mathbb{Z})$ is the orientable outer automorphism group.

Proposition 3.3. *The map ρ_π is surjective. Its image group $Out^+(\mathbb{Z} * \mathbb{Z})$ is isomorphic to $SL_2(\mathbb{Z})$. Moreover, the group of the π_1-monodromy actions on $Cov_{d,g,\sigma}/\sim$ can be generated by the two actions $h_\alpha : (\alpha, \beta) \to (\alpha, \alpha\beta)$ and $h_\beta : (\alpha, \beta) \to (\alpha\beta, \beta)$.*

Proof. We give an indirect proof using the fact that the H_1-monodromy map ρ_H associated to a general pencil of plane cubics is surjective. For H_1-monodromy, the marked point does not affect the homology. Consider the map $\rho_H : \pi_1(Z^0, b) \to Out^+(H_1(E_p)) = Aut^+(\mathbb{Z} \oplus \mathbb{Z}) \cong SL_2(\mathbb{Z})$. To show that ρ_H is surjective for a general pencil Z, it suffices to exhibit a special pencil for which the claim holds. Such special pencils were already studied in [Sa], where the H_1-monodromy is surjective.

Next, we consider the following commutative diagram:



Here Γ_1 and $\Gamma_{1,1}$ are the mapping class groups for an ordinary torus and a torus with one marked point, respectively. Γ_1 and $Out^+(\mathbb{Z} * \mathbb{Z})$ are both isomorphic to $SL_2(\mathbb{Z})$. The map ϕ is induced by quotienting out the commutators, i.e. $\mathbb{Z} * \mathbb{Z}/([\alpha, \beta]) \cong \mathbb{Z} \oplus \mathbb{Z}$, where α and β are the generators of the group $\mathbb{Z} * \mathbb{Z}$. Moreover, there is an isomorphism $\Gamma_{1,1} \xrightarrow{\cong} \Gamma_1$. Hence, the surjectivity of ρ_π follows from that of ρ_H.

Finally, $Out^+(\mathbb{Z} * \mathbb{Z})$ is generated by the two actions induced from the Dehn twists along the two loops represented by α and β. Correspondingly, the monodromy actions send (α, β) to $(\alpha, \alpha\beta)$ and $(\alpha\beta, \beta)$, respectively. □

Proof of Theorem 1.18. The map $Y_{d,g,\sigma} \to Z$ is finite of degree $N_{d,g,\sigma}$ and unramified away from $b_1, \ldots, b_{12}$. Two smooth covers in $Y_{d,g,\sigma}$ are contained in the same component iff they can be sent to each other under the monodromy actions induced by closed paths in $\pi_1(Z^0, b)$. By Proposition 3.3, these actions are generated by h_α and h_β. Therefore, the two covers are contained in the same component of $Y_{d,g,\sigma}$ iff they belong to the same orbit under the actions generated by h_α and h_β. □

Note that the actions h_α and h_β are well-defined for $Cov_{d,g,\sigma}/\sim$. Let us further specify a local monodromy action, which will be used when we study the genus of $Y_{d,g,\sigma}$.

Proposition 3.4. *Suppose β corresponds to the local vanishing cycle around a nodal rational fiber over b_i. Going along a loop in a neighborhood of $b_i \in Z$, the induced monodromy action sends (α, β) to $(\alpha\beta, \beta)$.*

Proof. This is a direct consequence of the Picard-Lefschetz formula or the Kodaira's classification of elliptic fibrations. □

3.3. **Application.** In this part, we apply the general theory of $Y_{d,g,\sigma}$ obtained previously to various examples.

3.3.1. *The slope of* $Y_{d,2,\sigma}$. In the case $g = 2$, let us compute those numbers $N_{d,2,\sigma}^{1^{a_1}\ldots d^{a_d}}$ in Definition 1.14. For simplicity, we only consider the case when d is *prime*.

Recall that $Cov_{d,g,\sigma}^{1^{a_1}\ldots d^{a_d}} = \{(\alpha,\beta) \in S_d \times S_d \mid \beta^{-1}\alpha^{-1}\beta\alpha \in \sigma,\ \langle\alpha,\beta\rangle \text{ is transitive},\ \beta \in 1^{a_1}\ldots d^{a_d}\}$. The symmetric group S_d acts on $Cov_{d,g,\sigma}^{1^{a_1}\ldots d^{a_d}}$ by the conjugate actions. For an element $\tau \in S_d$, consider the subset $Cov_{d,g,\sigma}^{1^{a_1}\ldots d^{a_d}}(\tau) = \{(\alpha,\beta) \in Cov_{d,g,\sigma}^{1^{a_1}\ldots d^{a_d}} \mid \tau\alpha = \alpha\tau,\ \tau\beta = \beta\tau\}$.

Lemma 3.5. *When d is prime, the following equality holds*

$$N_{d,g,\sigma}^{1^{a_1}\ldots d^{a_d}} = \frac{|Cov_{d,g,\sigma}^{1^{a_1}\ldots d^{a_d}}|}{d!}.$$

Proof. For the conjugate actions of S_d on $Cov_{d,g,\sigma}^{1^{a_1}\ldots d^{a_d}}$, Burnside's Lemma implies that

$$N_{d,g,\sigma}^{1^{a_1}\ldots d^{a_d}} = \frac{1}{|S_d|}\sum_{\tau\in S_d} |Cov_{g,d,\sigma}^{1^{a_1}\ldots d^{a_d}}(\tau)|.$$

By Lemma 2.8, if τ commutes with all the elements of a transitive subgroup of S_d, it must belong to a conjugacy class l^m, where $lm = d$. Since d is prime, τ can be either id or a cycle of length-d. Take a cycle $(12\cdots d)$ that sends the letter i to $i+1$. We know that $(12\cdots d) = \alpha(12\cdots d)\alpha^{-1} = (\alpha(1)\alpha(2)\cdots\alpha(d))$. Hence, $\alpha(i) = i+s$ for all i, where s is an integer independent of i. Similarly, $\beta(j) = j+t$ for all j and t is independent of j. One can check that $\beta^{-1}\alpha^{-1}\beta\alpha = id$, which is not in the conjugacy class σ. Therefore, $Cov_{d,g,\sigma}^{1^{a_1}\ldots d^{a_d}}(\tau)$ is non-empty iff τ is the identity. In this case, $\tau = id$ automatically commutes with all (α,β). Hence, we get $N_{d,g,\sigma}^{1^{a_1}\ldots d^{a_d}} = \frac{1}{d!}|Cov_{d,g,\sigma}^{1^{a_1}\ldots d^{a_d}}|$. □

When $g = 2$, by the Riemann-Hurwitz formula, the ramification class σ can be of type $1^{d-3}3^1$ or $1^{d-4}2^2$. Let us consider the case $\sigma = 1^{d-3}3^1$ first. Suppose a permutation $\gamma = (abc) \in 1^{d-3}3^1$ satisfies the equality $\alpha^{-1}\beta\alpha = \beta\gamma$. Since $\alpha^{-1}\beta\alpha$ and β are in the same conjugacy class, $\beta\gamma$ is conjugate to β. Note that a, b, c cannot be contained in three different cycles of β, since $(a\cdots)(b\cdots)(c\cdots)(abc) = (a\cdots b\cdots c\cdots)$ changes the conjugacy type of β. Hence, only two cases are possible:
(1) $(a\cdots b\cdots c\cdots)(abc) = (a\cdots c\cdots b\cdots)$;
(2) $(a\cdots b\cdots)(c\cdots)(abc) = (a\cdots)(b\cdots c\cdots)$.

Using the fact that $\langle\alpha,\beta\rangle$ is transitive, in case (1), β belongs to a conjugacy class l^m, where $lm = d$. Since d is prime, β can only be a cycle of length-d. Take β to be $(12\ldots d)$. There are $\binom{d}{3}$ choices for the cycle (abc). Let us fix a choice. For α, we know that $(\alpha^{-1}(1)\alpha^{-1}(2)\cdots\alpha^{-1}(d)) = \alpha^{-1}(12\cdots d)\alpha = (a\cdots b\cdots c\cdots)(abc) = (a\cdots c\cdots b\cdots)$. Hence, there are d choices for α. Overall, by Lemma 3.5, we get

$$N_{d,2,1^{d-3}3^1}^{d^1} = \frac{1}{d!}(d-1)!\binom{d}{3}d = \binom{d}{3}.$$

For case (2), we have $(\underbrace{a\cdots \overbrace{b\cdots}^{l_2}}_{l_1})(\underbrace{c\cdots}_{l_2})(abc) = (a\cdots)(b\cdots c\cdots)$. Hence, β belongs to the conjugacy class $l_1^{a_1}l_2^{a_2}$, where $l_1 > l_2$ and $a_1l_1 + a_2l_2 = d$. There are $\frac{d!}{\prod_{i=1}^{2}(l_i^{a_i})(a_i!)}$ choices for β. Let us fix a choice. There are $a_1l_1a_2l_2$ choices for the cycle (abc). Fix one of these. By the transitivity condition, there are $\prod_{i=1}^{2}(l_i^{a_i})(a_i-1)!$ choices for α. Overall, by Lemma 3.5, we get

$$N_{d,2,1^{d-3}3^1}^{l_1^{a_1}l_2^{a_2}} = l_1l_2.$$

Finally, we have

$$N_{d,2,1^{d-3}3^1} = \binom{d}{3} + \sum_{\substack{a_1l_1+a_2l_2=d\\ l_1>l_2}} l_1l_2$$

and

$$M_{d,2,1^{d-3}3^1} = \frac{1}{d}\binom{d}{3} + \sum_{\substack{a_1l_1+a_2l_2=d\\ l_1>l_2}} (\frac{a_1}{l_1}+\frac{a_2}{l_2})l_1l_2.$$

Proposition 3.6. *When d is prime, the total sum $N_{d,2,1^{d-3}3^1}$ equals $\frac{3}{8}(d-2)(d-1)(d+1)$ and the weighted sum $M_{d,2,1^{d-3}3^1}$ equals $\frac{5}{12}(d-2)(d-1)(d+1)$.*

Proof. Recall that $\sigma_i(n) = \sum_{k|n} k^i$. By Lemma 2.10, we have the equality

$$\sum_{k=1}^{d-1}\sigma_1(k)\sigma_1(d-k) = (\frac{1}{12}-\frac{d}{2})\sigma_1(d) + \frac{5}{12}\sigma_3(d).$$

When d is prime, the right hand side of this equality equals $\frac{1}{12}(d-1)(d+1)(5d-6)$. Moreover, we have

$$\sum_{\substack{a_1l_1+a_2l_2=d,\\ l_1>l_2}} l_1l_2 = \frac{1}{2}\left(\sum_{k=1}^{d-1}\sigma_1(k)\sigma_1(d-k) - (d-1)\right)$$

and

$$\begin{aligned}\sum_{\substack{a_1l_1+a_2l_2=d,\\ l_1>l_2}} (\frac{a_1}{l_1}+\frac{a_2}{l_2})l_1l_2 &= \frac{1}{2}\left(\Big(\sum_{a_1l_1+a_2l_2=d} a_1l_2+a_2l_1\Big) - d(d-1)\right)\\ &= \left(\sum_{a_1l_1+a_2l_2=d} l_1l_2\right) - \frac{1}{2}d(d-1).\end{aligned}$$

The equalities for $N_{d,2,1^{d-3}3^1}$ and $M_{d,2,1^{d-3}3^1}$ follow right away. □

Using Proposition 3.6 and the slope formula for $Y_{d,2,1^{d-3}3^1}$, we get the slope of $Y_{d,2,1^{d-3}3^1}$ equal to 10. This completes the proof for Corollary 1.16 when $\sigma = 1^{d-3}3^1$ and d is prime.

Next, let us consider the case $\sigma = 1^{d-4}2^2$. The analysis is similar as before. Hence, we only state the result without giving the detailed proof.

Proposition 3.7. *When d is prime, the total sum $N_{d,2,1^{d-4}2^2}$ equals $\frac{1}{6}(d-3)(d-2)(d-1)(d+1)$ and the weighted sum $M_{d,2,1^{d-4}2^2}$ equals $\frac{5}{24}(d-3)(d-2)(d-1)(d+1)$.*

Using Proposition 3.7 and the slope formula for $Y_{d,2,1^{d-4}2^2}$, we get the slope of $Y_{d,2,1^{d-4}2^2}$ equal to 10. This completes the proof for Corollary 1.16 when $\sigma = 1^{d-4}2^2$ and d is prime.

Now, we present a short but indirect proof of Corollary 1.16 for all d.

Proof of Corollary 1.16. The moduli space $\overline{\mathcal{M}}_2$ is special in that the equality $\lambda = \frac{1}{10}\delta_0 + \frac{1}{5}\delta_1$ holds. Hence, for a one-dimensional subscheme B in $\overline{\mathcal{M}}_2$ that is not contained in the boundary, its slope satisfies $s(B) \leq 10$. In particular, $s(B) = 10$ iff B does not meet the boundary component δ_1. By Proposition 3.1, the image of $Y_{d,2,\sigma}$ does not meet δ_1. Therefore, it has slope equal to 10. Even if $Y_{d,2,\sigma}$ is reducible, by the same argument, each of its components has slope 10. □

Note that 10 is the sharp lower bound for the slope of effective divisors on $\overline{\mathcal{M}}_2$.

3.3.2. *The slope of* $Y_{d,3,1^{d-5}5^1}$. Consider the case $g = 3$, d is prime and σ is the conjugacy class $1^{d-5}5^1$. We have the following enumerative numbers:

$$N^{d^1}_{d,3,1^{d-5}5^1} = 8\binom{d}{5}, \ N^{2^{a_2}1^{a_1}}_{d,3,1^{d-5}5^1} = 8(a_2 - 1),$$

$$N^{l_1^{a_1} l_2^{a_2}}_{d,3,1^{d-5}5^1} = \frac{l_1 l_2}{2}(3l_1^2 + 3l_2^2 - 19l_1 - 11l_2 + 4d + 22), \ l_1 > l_2 > 1,$$

$$N^{l_1^{a_1} l_2^{a_2} l_3^{a_3}}_{d,3,1^{d-5}5^1} = \begin{cases} 11 l_1 l_2 l_3, \ l_1 \neq l_2 + l_3 > l_2 > l_3; \\ 7 l_1 l_2 l_3, \ l_1 = l_2 + l_3 > l_2 > l_3. \end{cases}$$

Now the slope formula says that

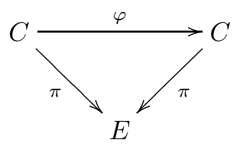
$$s(Y_{d,3,1^{d-5}5^1}) = \frac{12 M_{d,3,1^{d-5}5^1}}{M_{d,3,1^{d-5}5^1} + \frac{2}{5} N_{d,3,1^{d-5}5^1}}.$$

The slope $s(Y_{d,3,1^{d-5}5^1})$ in general depends on d. Nevertheless, a computer check implies that the slope decreases to 9 when d approaches infinity. Note that 9 is the sharp lower bound s_3 for the slope of effective divisors on $\overline{\mathcal{M}}_3$.

Next, we study in detail a beginning example.

Example 3.8. For $g = 3$, $d = 5$ and $\sigma = 5^1$, a direct enumeration shows that there are 40 equivalence classes in $Cov_{5,3,5^1}/\sim$. Under the monodromy actions in Theorem 1.18, those classes fall into 4 orbits. Therefore, $Y_{5,3,5^1}$ has 4 irreducible components. The slope formula can be applied in the same form to the components of $Y_{5,3,5^1}$. One can check that two of the components have slope 9 and the other two have slope $9\frac{1}{3}$.

3.3.3. *Counting weighted connected covers.* In this part, we provide a method that systematically calculates $N^{1^{a_1}\dots d^{a_d}}_{d,g,\sigma}$. For simplicity, we focus on the case when d is prime and σ is the conjugacy class $1^{d-2k}2^k$, $k = 2g - 2$. We also denote $1^{d-2k}2^k$ by $\tau_{d,k}$. For a cover $\pi : C \to E$, an automorphism φ of this cover is given by the following commutative diagram:



If π corresponds to the data $(\alpha, \beta) \in S_d \times S_d$, the automorphism φ corresponds to an element $\tau \in S_d$ such that $(\tau\alpha\tau^{-1}, \tau\beta\tau^{-1}) = (\alpha, \beta)$. Hence, the automorphism group $Aut(C, \pi)$ can be identified as the set $Stab(\alpha, \beta)$ of stablizers under the S_d conjugate actions.

People are often interested in the weighted Hurwitz numbers, i.e. counting a cover (C, π) with weight $\frac{1}{|Aut(C,\pi)|}$. Hence, we define a weighted number

$$\widetilde{N}^{\wp}_{d,k} = \sum_{(\alpha,\beta) \in Cov^{\wp}_{d,g,\tau_{d,k}}/\sim} \frac{1}{|Stab(\alpha, \beta)|} = \frac{1}{d!}|Cov^{\wp}_{d,g,\tau_{d,k}}|,$$

where $\wp$ denotes the conjugacy class $1^{a_1} \dots d^{a_d}$. When d is prime, a cover does not have non-trivial automorphisms, cf. the proof of Lemma 3.5. Therefore, counting covers with weight does not make a difference in this situation.

Let us first get rid of the transitivity condition imposed on $\langle \alpha, \beta \rangle$. Define a set

$$\widehat{Cov}^{\wp}_{d,k} := \{(\alpha, \beta) \in S_d \times S_d \mid \beta \in \wp, \ \beta^{-1}\alpha^{-1}\beta\alpha \in \tau_{d,k}\}.$$

Denote its cardinality by

$$\widehat{N}^{\wp}_{d,k} = |\widehat{Cov}^{\wp}_{d,k}|.$$

We fix an element $\tau \in \tau_{d,k}$ and consider pairs $(\gamma, \beta) \in \wp \times \wp$ such that $\beta\gamma = \tau$. For such a pair, there are $\frac{|S_d|}{|\wp|}$ choices for α that satisfy the equality $\alpha^{-1}\beta\alpha = \gamma$. Hence,

$$\widehat{N}^{\wp}_{d,k} = |\tau_{d,k}| \cdot \frac{|S_d|}{|\wp|} \cdot |\{(\gamma, \beta) \in \wp \times \wp : \beta\gamma = \tau\}|.$$

By [St, 7.68 a], we know that

$$|\{(\gamma, \beta) \in \wp \times \wp : \beta\gamma = \tau\}| = \frac{|\wp|^2}{|S_d|} \cdot \left(\sum_{\chi} \frac{1}{\deg(\chi)} |\chi(\wp)|^2 \chi(\tau) \right),$$

where χ runs over all the irreducible characters of S_d. Therefore, we get

$$\widehat{N}^{\wp}_{d,k} = |\wp| \cdot |\tau_{d,k}| \cdot \left(\sum_{\chi} \frac{1}{\deg(\chi)} |\chi(\wp)|^2 \chi(\tau_{d,k}) \right).$$

Next, let us derive $\widetilde{N}^{\wp}_{d,k}$ from $\widehat{N}^{\wp}_{d,k}$. Take a pair $(\alpha, \beta) \in \widehat{Cov}^{\wp}_{d,k}$. The subgroup $\langle \alpha, \beta \rangle$ of S_d may not be transitive. Consider its orbits and the actions of α, β on them. They correspond to the following data:

$$\{(\alpha_i, \beta_i), i = 1, \ldots, m \mid \beta^{-1}\alpha^{-1}\beta\alpha \in 1^{d_i - 2k_i} 2^{k_i} = \tau_{d_i,k_i},$$
$$\beta_i \in \wp_i \text{ a conjugacy class of } S_{d_i}, \ \bigcup_{i=1}^m \wp_i = \wp, \ \sum_{i=1}^m k_i = k,$$
$$\sum_{i=1}^m d_i = d, \ \langle \alpha_i, \beta_i \rangle \text{ is a transitive subgroup of } S_{d_i}\}.$$

Two data $(\wp_i, k_i, d_i)$ and $(\wp_j, k_j, d_j)$ are of the same type iff $\wp_i \sim \wp_j$, $k_i = k_j$ and $d_i = d_j$. Hence, we get the following equality

$$\widehat{N}^{\wp}_{d,k} = \sum \binom{d}{\underbrace{d_1, \ldots, d_1}_{p_1}, \ldots, \underbrace{d_m, \ldots, d_m}_{p_m}} \frac{(d_1!)^{p_1} \cdots (d_m!)^{p_m}}{(p_1!) \cdots (p_m!)} (\widetilde{N}^{\wp_1}_{d_1,k_1})^{p_1} \cdots (\widetilde{N}^{\wp_m}_{d_m,k_m})^{p_m},$$

where the condition on the summation is that $\sum_{i=1}^m p_i d_i = d$, $\sum_{i=1}^m p_i k_i = k$ and $\bigcup_{i=1}^m \wp_i^{p_i} = \wp$. Simplifying the above expression, we get

$$\widehat{N}^{\wp}_{d,k} = \sum d! \prod_{i=1}^m \frac{(\widetilde{N}^{\wp_i}_{d_i,k_i})^{p_i}}{p_i!}.$$

If $\wp$ is the conjugacy class $1^{a_1} \ldots d^{a_d}$, the data $(\wp_i, k_i, d_i)$ correspond to a vector $(a_{i1}, \ldots, a_{id}, k_i)$ with integer entries, $0 \leq a_{ij} \leq a_j$, $0 \leq k_i \leq k$, $\sum_{j=1}^d j a_{ij} = d_i$. Put all the vectors in a matrix

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1d} & k_1 \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{md} & k_m \end{pmatrix}$$

Then the summation runs over all possible $(p_1, \ldots, p_m)$ satisfying

$$(p_1, \ldots, p_m) \cdot A = (a_1, \ldots, a_d, k).$$

Define an index set

$$I = \{(a_1, a_2, \cdots) \mid a_i \geq 0 \text{ and there are finitely many nonzero entries}\}.$$

For $\wp = 1^{a_1} \ldots d^{a_d}$, rewrite it as $1^{a_1} 2^{a_2} \ldots$. Then it is determined by an element $(a_1, a_2, \ldots) \in I$, where $d = a_1 + 2a_2 + \cdots$.

Finally, define two generating functions as follows:

$$\widehat{Z}(y; x_1, x_2, \ldots) = \sum_{I,k} \frac{\widehat{N}^{\wp}_{d,k}}{(a_1 + 2a_2 + \cdots)!} \cdot y^k x_1^{a_1} x_2^{a_2} \cdots$$

and
$$\widetilde{Z}(y;x_1,x_2,\ldots)=\sum_{I,k}\widetilde{N}^{\wp}_{d,k}\cdot y^k x_1^{a_1}x_2^{a_2}\cdots,$$
where $d=a_1+2a_2+\cdots$ and $\wp$ is the conjugacy class $1^{a_1}2^{a_2}\ldots$ for each term.

Therefore, we get the following relation between the generating functions for the number of connected and possibly disconnected covers:
$$\widehat{Z}=exp(\widetilde{Z})-1.$$

Note that if k is odd or $2k>d=a_1+2a_2+\cdots$, we have $\widehat{N}^{\wp}_{d,k}=\widetilde{N}^{\wp}_{d,k}=0$. The evaluation of a character χ on a conjugacy class $\wp$ can be worked out by standard formulae from the representation theory of S_d. Using the above method, we hope that one can estimate the quotient $\frac{N_{d,g,\sigma}}{M_{d,g,\sigma}}$ to reveal how the slope of $Y_{d,g,\sigma}$ looks like for large d and g.

3.3.4. *The genus of* $Y_{d,2,\sigma}$. Fix $g=2$ and a prime number d. Let us first consider the case $\sigma=1^{d-3}3^1$.

Proof of Corollary 1.19. To get a feel of the method we will use below, consider the beginning case $d=3$. There are three non-equivalent pairs (α,β) in $Cov_{3,2,3^1}/\sim$:
(1) $\alpha=(13),\beta=(12)$;
(2) $\alpha=(123),\beta=(12)$;
(3) $\alpha=(12),\beta=(123)$.

Let β correspond to the monodromy of the vanishing cycle when an elliptic curve degenerates to a rational nodal curve. In Proposition 3.4, the local monodromy action $(\alpha,\beta)\to(\alpha\beta,\beta)$ can switch sheets (1) and (2) but keep (3) unchanged. Hence, $Y_{3,2,3^1}$ is a triple cover of $Z\cong\mathbb{P}^1$ simply branched at the 12 points $b_1,\ldots,b_{12}$. By the Riemann-Hurwitz fromula, $2g(Y_{3,2,3^1})-2=3(2g(Z)-2)+12$. Therefore, we get $g(Y_{3,2,3^1})=4$.

In general, we need to classify the orbits of the $N_{d,2,1^{d-3}3^1}$ sheets of the finite map $Y_{d,2,1^{d-3}3^1}\to Z$ under the local monodromy action. This will tell us how the map $Y_{d,2,1^{d-3}3^1}\to Z$ is ramified. Then one can apply the Riemann-Hurwitz formula to compute the genus of $Y_{d,2,1^{d-3}3^1}$.

Assume that $d\geq 5$ is prime. Let $\beta^{-1}\alpha^{-1}\beta\alpha=\gamma=(abc)$ denote a fixed cycle in S_d. By the local monodromy action, we have to find the minimal positive integer k such that there exists $\tau\in S_d$ satisfying $\tau(\alpha\beta^k,\beta)\tau^{-1}=(\alpha,\beta)$. Note that such a τ must satisfy $\tau\gamma\tau^{-1}=\gamma$, since
$$\tau\gamma\tau^{-1}=\tau\beta^{k-1}(\alpha\beta^k)^{-1}\beta(\alpha\beta^k)\beta^{-k}\tau^{-1}=\beta^{-1}\alpha^{-1}\beta\alpha=\gamma.$$

If β belongs to the conjugacy class d^1, by $\tau\beta\tau^{-1}=\beta$, we get $\tau=\beta^m$ for some integer m. Hence, $\tau(abc)\tau^{-1}=(a+m\ b+m\ c+m)=(abc)$. It implies that $3m$ is divisible by d. Since $d\geq 5$ is prime, τ is id. Hence, the minimal positive integer k satisfying $\beta^k=id$ equals d. Since $N^{d^1}_{d,2,1^{d-3}3^1}=\binom{d}{3}$, we get $\frac{1}{d}\binom{d}{3}=\frac{1}{6}(d-1)(d-2)$ orbits of this form under the local monodromy action. Each orbit has cardinality d. From the perspective of the covering map $Y_{d,2,1^{d-3}3^1}\to Z$, there are d sheets in such a orbit meeting at a point that parameterizes an admissible cover of the rational nodal curve.

If β belongs to the conjugacy class $l_1^{a_1}l_2^{a_2}$, $l_1>l_2$, we know that l_1 and l_2 are coprime. Without loss of generality, assume that
$$\beta=(t_{11}t_{12}\cdots t_{1l_1})\cdots(t_{a_11}\cdots t_{a_1l_1})$$
$$\cdot(s_{11}s_{12}\cdots s_{1l_2})\cdots(s_{a_21}\cdots s_{a_2l_2})$$
and that $a=t_{1l_2}$, $b=t_{1l_1}$, $c=s_{11}$. From the condition that $\tau\beta\tau^{-1}=\beta$, $\tau\gamma\tau^{-1}=\gamma$ and d is prime, we can verify that τ fixes all the letters in the two cycles $(t_{11}t_{12}\cdots t_{1l_1})$ and $(s_{11}s_{12}\cdots s_{1l_2})$. Then we have
$$\alpha^{-1}\beta\alpha=\beta\gamma=(t_{1\ l_2+1}\cdots t_{1l_1}s_{11}\cdots s_{1l_2})(t_{21}\cdots t_{2l_1})\cdots(t_{a_11}\cdots t_{a_1l_1})$$
$$\cdot(t_{11}\cdots t_{1l_2})(s_{21}\cdots s_{2l_2})\cdots(s_{a_21}\cdots s_{a_2l_2}).$$
We can assume that α sends the cycle $(t_{a_11}\cdots t_{a_1l_1})$ and $(s_{a_21}\cdots s_{a_2l_2})$ to $(t_{1\ l_2+1}\cdots t_{1l_1}s_{11}\cdots s_{1l_2})$ and $(t_{11}\cdots t_{1l_2})$, respectively. Furthermore, assume that $\alpha(t_{ij})=t_{i+1\ j}$, $1\leq i<a_1-1$, $1\leq j\leq l_1$, $\alpha(s_{ij})=s_{i+1\ j}$, $1\leq i<a_2-1$, $1\leq j\leq l_2$, $\alpha(t_{a_1-1\ j})=t_{a_1\ j+w_1}$ and $\alpha(s_{a_2-1\ j})=s_{a_2\ j+w_2}$, where the last two actions are twisted by w_1 and w_2. α contains a cycle $(t_{11}t_{21}\cdots t_{a_1-1\ 1}t_{a_1\ 1+w_1}\alpha(t_{a_1\ 1+w_1})\cdots)$.

The corresponding cycle in $\alpha\beta^k$ is $(t_{11}t_{2\ 1+k}\cdots t_{a_1\ 1+(a_1-1)k+w_1}\alpha(t_{a_1\ 1+a_1k+w_1})\cdots)$. Since $\tau\alpha\beta^k\tau^{-1}=\alpha$ and t_{11}, $\alpha(t_{a_1\ 1+w_1})$, $\alpha(t_{a_1\ 1+a_1k+w_1})$ are all fixed by τ, we get $t_{a_1\ 1+a_1k+w_1}=t_{a_1\ 1+w_1}$, i.e. $l_1|a_1k$. Similarly, we have $l_2|a_2k$. One can check that these two conditions on k are sufficient for the existence of the desired τ. Since $N_{l_1^{a_1}l_2^{a_2}}=l_1l_2$, we get $(l_1,a_1)(l_2,a_2)$ orbits of this form under the local monodromy action. Each orbit has cardinality $\frac{l_1l_2}{(l_1,a_1)(l_2,a_2)}$.

The map $Y_{d,2,1^{d-3}3^1}\to X$ is finite of degree $N_{d,2,1^{d-3}3^1}$. By the Riemann-Hurwitz formula, we have

$$2g(Y_{d,2,1^{d-3}3^1})-2=-2N_{d,2,1^{d-3}3^1}+12\bigg(\frac{(d-1)(d-2)}{6}(d-1)$$
$$+\sum_{\substack{a_1l_1+a_2l_2=d\\ l_1>l_2}}(l_1,a_1)(l_2,a_2)\Big(\frac{l_1l_2}{(l_1,a_1)(l_2,a_2)}-1\Big)\bigg).$$

After simplifying the expression, we get the genus formula for $Y_{d,2,1^{d-3}3^1}$.

Finally, by Proposition 3.6, we know that $N_{d,2,1^{d-3}3^1}\sim\frac{3}{8}d^3$ and $\sum_{\substack{a_1l_1+a_2l_2=d\\ l_1>l_2}}l_1l_2\sim\frac{5}{24}d^3$. The fact that $\sum_{\substack{a_1l_1+a_2l_2=d\\ l_1>l_2}}(l_1,a_1)(l_2,a_2)$ has asymptotic order smaller than d^3 follows from [HL, 7.4]. This tells us $g(Y_{d,2,1^{d-3}3^1})\sim\frac{15}{8}d^3$. □

For the genus of $Y_{d,2,1^{d-4}2^2}$, the argument is similar as above. Hence, we skip the proof of Corollary 1.20.

4. Density

In this part, we will prove Theorem 1.21. Let us fix g. The closure of the image of the union $W_{d,g}$ for all d in $\overline{\mathcal{M}}_g$ contains the locus of genus g curves that admit a cover of an elliptic curve. The fact that this locus is Zariski dense in $\overline{\mathcal{M}}_g$ was established in [CP, Theorem (3)].

Next, consider $Y_{d,g,\sigma}$. The union of their images in $\overline{\mathcal{M}}_g$ for all d consists of infinitely many one-dimensional subschemes. Let σ denote the conjugacy class $(l_1)\cdots(l_m)$ in S_d, where (l_i) stands for a cycle of length-l_i. Assume that $l_1,\ldots,l_k$ are greater than 1 and l_i is equal to 1 for $i>k$. By the Riemann-Hurwitz formula, $2g-2=\sum_{i=1}^k(l_i-1)=\big(\sum_{i=1}^k l_i\big)-k$. For a cover $C\to E$ that has a unique branch point with the ramification class σ, the ramification divisor on C has the form $\sum_{i=1}^k\mu_iq_i$, where $\mu_i=l_i-1$. Hence, we can fix $l_1,\ldots,l_k$ and add arbitrarily many cycles of length-1 to σ. This makes the degree d of the cover vary from $2g-2+k$ to infinity. Meanwhile, the ramification divisor remains the same. In this sense, we say that the ramification class σ does not change. Therefore, the density result for $Y_{d,g,\sigma}$ in Theorem 1.21 is equivalent to the following statement.

Proposition 4.1. *In the above setting, the image of* $\bigcup_d^\infty Y_{d,g,\sigma}$ *is Zariski dense in* $\overline{\mathcal{M}}_g$ *iff* $k\geq g-1$.

Our strategy is to identify a genus g cover of an elliptic curve as a *lattice point* in the Hodge bundle $\mathbb{H}$ over $\mathcal{M}_g$ parameterizing a curve along with a section of its dualizing sheaf.

Definition 4.2. Let $\mu=(\mu_1,\ldots,\mu_k)$ denote a partition of $2g-2$. Define $\mathcal{H}(\mu)$ as the subvariety of $\mathbb{H}$ parameterizing $(C,\omega,q_1,\ldots,q_k)$, where $[C]\in\mathcal{M}_g$, $\omega\in H^0(K_C)$ and $(\omega)=\sum_{i=1}^k\mu_iq_i$.

The dimension n of $\mathcal{H}(\mu)$ equals $3g-3+g-\big(\sum_{i=1}^k(\mu_i-1)\big)=2g-1+k$. Take a basis $\gamma_1,\ldots,\gamma_n\in H_1(C,q_1,\ldots,q_k;\mathbb{Z})$ the relative homology of C with k marked points, such that $\gamma_1,\ldots,\gamma_{2g}$ are the standard basis of $H_1(C;\mathbb{Z})$ and γ_{2g+i} is a path connecting q_1 and q_{i+1}, $i=1,\ldots,n-2g$. The period map $\Phi:(C,\omega)\to\mathbb{C}^n$ is given by

$$\Phi(C,\omega)=\Big(\int_{\gamma_1}\omega,\ldots,\int_{\gamma_n}\omega\Big),$$

which provides a local coordinate chart for $\mathcal{H}(\mu)$, cf. [Ko] and [EO].

Proof of Proposition 4.1. For a cover $C \to E$ with a unique branch point and the ramification class $\sigma = (l_1)\cdots(l_m)$, the pull-back of a holomorphic 1-form from E becomes a holomorphic 1-form ω on C. Moreover, the divisor (ω) equals $\sum_{i=1}^k \mu_i q_i$, where $\mu_i = l_i - 1$ and $\sum_{i=1}^k \mu_i = 2g-2$. Take a torus T given by $\mathbb{C}/\Lambda$, where $\Lambda = \langle 1, \tau\rangle$ is a lattice. Consider the local coordinates $\phi = \Phi(C,\omega) = (\phi_1,\ldots,\phi_n) \in \mathbb{C}^n$ for the space $\mathcal{H}(\mu)$. By [EO, Lemma 3.1], we know that $\phi_i \in \Lambda$, $i = 1,\ldots,2g$ iff the following conditions hold:
(1) there exists a holomorphic map $f : C \to T$;
(2) $\omega = f^{-1}(dz)$;
(3) f is ramified at q_i with the ramification order $\mu_i + 1 = l_i$, $i = 1,\ldots,k$;
(4) $f(q_{i+1}) - f(q_1) = \phi_{2g+i} \mod \Lambda$, $i = 1,\ldots,k-1$.

To get a cover $C \to T$ branched at a point with the ramification class $(l_1)\ldots(l_k)$, it is equivalent to find a point ϕ in $\mathcal{H}(\mu)$ that has local coordinates $\phi_i = 0$ modulo Λ, $i = 1,\ldots,n$. If we vary τ, the union of such lattice points ϕ is Zariski dense in $\mathcal{H}(\mu)$. Consider the following diagram:

$$\begin{array}{ccccc} \bigcup_d^\infty \mathcal{H}_{d,g,\sigma} & \longrightarrow & \mathcal{H}(\mu) & \longrightarrow & \mathbb{H} \\ & \searrow & & & \downarrow \pi \\ & & & & \mathcal{M}_g \end{array}$$

As long as $\mathcal{H}(\mu)$ dominates $\mathcal{M}_g$, the image of $\bigcup_d^\infty Y_{d,g,\sigma}$ will be dense in $\overline{\mathcal{M}}_g$. Hence, we need to check that when $k \geq g-1$, for a general $[C] \in \mathcal{M}_g$ there exists $\omega \in H^0(K_C)$ such that $(\omega) = \sum_{i=1}^k \mu_i q_i$.

The case $k > g-1$ can be reduced to $k = g-1$, since there is a natural stratification among all the moduli spaces $\mathcal{H}(\mu)$. When $k = g-1$, we apply the De Jonquières' Formula from [ACGH, VIII §5]. Suppose that $a_1,\ldots,a_m$ are distinct integers, where a_i appears n_i times in the partition μ of $2g-2$. We have $\sum_{i=1}^m n_i = g-1$ and $\sum_{i=1}^m n_i a_i = 2g-2$. Define $R(t) = 1 + \sum_{i=1}^m a_i^2 t_i$ and $P(t) = 1 + \sum_{i=1}^m a_i t_i$. On a genus g curve, the virtual number of the canonical divisors that have n_i points of multiplicity a_i equals

$$\left[\frac{R(t)^g}{P(t)}\right]_{t_1^{n_1}\cdots t_m^{n_m}}.$$

If this number is nonzero, $\mathcal{H}(\mu)$ dominates $\mathcal{M}_g$. Define $A = \sum_{i=1}^m a_i^2 t_i$ and $B = \sum_{i=1}^m a_i t_i$. Indeed, we have

$$\begin{aligned} & \left[\frac{(1+A)^g}{1+B}\right]_{t_1^{n_1}\cdots t_m^{n_m}} \\ = & \left[\left(1 + \binom{g}{1}A + \cdots + \binom{g}{g-1}A^{g-1}\right)\left(1 - B + B^2 - \cdots\right)\right]_{t_1^{n_1}\cdots t_m^{n_m}} \\ = & \left[\binom{g}{g-1}A^{g-1} - \binom{g}{g-2}A^{g-2}B + \cdots + (-B)^{g-1}\right]_{t_1^{n_1}\cdots t_m^{n_m}} \\ = & \left[\frac{A^g - (A-B)^g}{B}\right]_{t_1^{n_1}\cdots t_m^{n_m}} \\ = & \left[A^{g-1} + A^{g-2}(A-B) + \cdots + (A-B)^{g-1}\right]_{t_1^{n_1}\cdots t_m^{n_m}} > 0, \end{aligned}$$

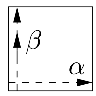
since $A - B = \sum_{i=1}^m (a_i^2 - a_i)s_i$ has nonnegative coefficients and A has positive coefficients.

On the other hand, if $k \leq g-2$, the dimension of $\mathcal{H}(\mu)$ is $2g-1+k \leq 3g-3$. Modulo a scalar, its image in $\overline{\mathcal{M}}_g$ forms a proper subvariety. In this case, the image of $\bigcup_d^\infty Y_{d,g,\sigma}$ is not dense in $\overline{\mathcal{M}}_g$. □

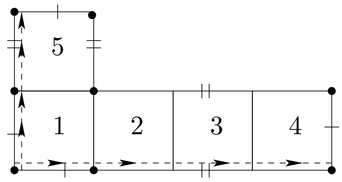
5. Related results and open problems

First, let us compare the slopes of the two families $W_{d,g}$ and $Y_{d,g,\sigma}$. For $g = 2$, by Corollary 1.7 and 1.16, the slope of $Y_{d,2,\sigma}$ yields the sharp lower bound $s_2 = 10$ on $\overline{\mathcal{M}}_2$. But the slope of $W_{d,2}$ converges to 5 as d goes to infinity. For general g, since we do not have closed formulae for their slopes, it is unclear whether one should use $Y_{d,g,\sigma}$ other than $W_{d,g}$ for the purpose of bounding slopes of effective divisors on $\overline{\mathcal{M}}_g$. Nevertheless, $Y_{d,g,\sigma}$ is special in that it can be regarded as a Teichmüller curve. In particular, $Y_{d,g,\sigma} \to \overline{\mathcal{M}}_g$ is rigid, cf. [CTM3, Theorem 1]. From this perspective, $Y_{d,g,\sigma}$ remains one of our best hopes to pin down the slope problem.

In [HL], those covers in $Y_{d,g,\sigma}$ were studied from the viewpoint of square-tiled surfaces. Take a standard torus E. If C is a cover of E, C can be realized as a possibly degenerate lattice polygon with some edges and vertices identified. If the degree of the map is d, this lattice polygon has area d. There is a natural correspondence between our description and that in [HL]. The following example illustrates the idea. Take two loops α and β of the torus E as in the picture below:



Consider the following octagon:



It parameterizes a degree 5 cover of E. All of its vertices are identified as the unique ramification point marked with a $\bullet$. Mark the unit squares covered by the octagon by $1, \ldots, 5$. Consider the monodromy actions of S_5 induced by the two paths α and β. It is easy to see $\alpha = (1234)$ and $\beta = (15)$. Then we get $\beta^{-1}\alpha^{-1}\beta\alpha = (154)$, which belongs to the ramification class $1^2 3^1$. Using such square-tiled surfaces, the genus and the number of components for the Hurwitz space $\mathcal{H}_{d,2,1^{d-3}3^1}$ were computed in [HL] for prime d. In [CTM1], the number of components of $\mathcal{H}_{d,2,1^{d-3}3^1}$ was studied for general d. The result is the following.

Proposition 5.1. *$\mathcal{H}_{d,2,1^{d-3}3^1}$ is irreducible when d is even or d equals 3. It has two components when $d > 3$ is odd.*

Note that for a cover with non-trivial automorphisms, it corresponds to an orbifold point of the Hurwitz space $\mathcal{H}_{d,g,\sigma}$. The orbifold Euler characteristic of $\mathcal{H}_{d,2,1^{d-3}3^1}$ was calculated in [B].

We conclude with several open problems that are important for a deeper understanding of these families of covers.

Question 5.2. *What are the limits for the slopes of $W_{d,g}$ and $Y_{d,g,\sigma}$ when d approaches infinity?*

Question 5.3. *How many irreducible components does $\mathcal{H}_{d,g}$ have? In particular, does the conjugacy type of $\langle \alpha, \beta, \gamma_1, \ldots, \gamma_{2g-2} \rangle$ uniquely determine a component of $\mathcal{H}_{d,g}$?*

Question 5.4. *How many irreducible components does $\mathcal{H}_{d,g,\sigma}$ have?*

When $g \gg d$ with the full monodromy group, the irreducibility of the Hurwitz space has been established, cf. [GHS], [Kan] and [V] for related results.

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