NORMAL BUNDLES OF RATIONAL CURVES IN PROJECTIVE SPACE

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Abstract. Let $b_\bullet$ be a sequence of integers $1 < b_1 \leq b_2 \leq \cdots \leq b_{n-1}$. Let $\mathrm{M}_e(b_\bullet)$ be the space parameterizing nondegenerate, rational curves of degree e in $\mathbb{P}^n$ with ordinary singularities such that the normal bundle has the splitting type $\bigoplus_{i=1}^{n-1} \mathcal{O}(e+b_i)$. When $n=3$, celebrated results of Eisenbud, Van de Ven, Ghione and Sacchiero show that $\mathrm{M}_e(b_\bullet)$ is irreducible of the expected dimension. We show that when $n \geq 5$, these loci are generally reducible with components of higher than the expected dimension. We give examples where the number of components grows linearly with n. These generalize an example of Alzati and Re.

1. Introduction

Rational curves play a central role in the birational and arithmetic geometry of projective varieties. Consequently, understanding the geometry of the space of rational curves is of fundamental importance. The local structure of this space is governed by the normal bundle. In this paper, we study the dimensions and irreducible components of the loci in the space of rational curves in $\mathbb{P}^n$ parameterizing curves whose normal bundles have a specified splitting type. We work over an algebraically closed field of characteristic zero.

We first set some notation. Let $f : \mathbb{P}^1 \to \mathbb{P}^n$ be a nondegenerate, birational map of degree e. If f has ordinary singularities, then the normal bundle N_f defined by

$$0 \longrightarrow T_{\mathbb{P}^1} \xrightarrow{\mathrm{d}f} f^*T_{\mathbb{P}^n} \longrightarrow N_f \longrightarrow 0$$

is a vector bundle of rank $n-1$ and degree $e(n+1)-2$. By Grothendieck's theorem, N_f is isomorphic to a direct sum of line bundles. Let $\mathrm{Mor}_e(\mathbb{P}^1, \mathbb{P}^n)$ denote the morphism scheme parameterizing degree e morphisms $f : \mathbb{P}^1 \to \mathbb{P}^n$. Given an increasing sequence of integers

$$b_\bullet = 1 < b_1 \leq b_2 \leq \cdots \leq b_{n-1}$$

such that $\sum_{i=1}^{n-1} b_i = 2e-2$, let $\mathrm{M}_e(b_\bullet)$ denote the locally closed locus in $\mathrm{Mor}_e(\mathbb{P}^1, \mathbb{P}^n)$ parameterizing nondegenerate rational curves of degree e with ordinary singularities such that

$$N_f \cong \bigoplus_{i=1}^{n-1} \mathcal{O}_{\mathbb{P}^1}(e+b_i).$$

The scheme $\mathrm{Mor}_e(\mathbb{P}^1, \mathbb{P}^n)$ is an irreducible scheme of dimension $(n+1)(e+1)-1$. The codimension of the locus of vector bundles E on $\mathbb{P}^1$ with a specified splitting type in the versal deformation space is given by $h^1(\mathbb{P}^1, \mathrm{End}(E))$. In analogy, we say that the expected codimension of $\mathrm{M}_e(b_\bullet)$ is $h^1(\mathbb{P}^1, \mathrm{End}(N_f))$. Equivalently, the expected dimension is

$$(e+1)(n+1) - 1 - h^1(\mathbb{P}^1, \mathrm{End}(N_f)).$$

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In this paper, we systematically construct examples where $\mathrm{M}_e(b_\bullet)$ has many components, some of larger than expected dimension.

The study of the geometry of $\mathrm{M}_e(b_\bullet)$ has a long history. Celebrated results of Eisenbud, Van de Ven [**?**], [**?**], Ghione and Sacchiero [**?**], [**?**], [**?**] characterize the possible splitting types of the normal bundles of rational curves in $\mathbb{P}^3$ and show that the locus of rational space curves whose normal bundles have a specified splitting type is irreducible of the expected dimension. Similarly, results of Ramella [**?**], [**?**] show that the locus of nondegenerate rational curves with ordinary singularities in $\mathrm{Mor}_e(\mathbb{P}^1, \mathbb{P}^n)$ with a specified splitting type for $f^*T_{\mathbb{P}^n}$ is irreducible of codimension $h^1(\mathbb{P}^1, \mathrm{End}(f^*T_{\mathbb{P}^n}))$ for all $n \geq 3$. The behavior of $\mathrm{M}_e(b_\bullet)$ for $n \geq 5$ is in stark contrast to these results.

Recently, Alzati and Re [**?**] showed that the locus of rational curves of degree 11 in $\mathbb{P}^8$ whose normal bundles have the splitting type $\mathcal{O}(13)^3 \oplus \mathcal{O}(14)^2 \oplus \mathcal{O}(15)^2$ is reducible. This was the first indication that the geometry of $\mathrm{M}_e(b_\bullet)$ is much more complicated for large n. This paper grew out of our attempt to generalize their example. We construct reducible examples in $\mathbb{P}^5$ of lower degree, we find splitting types with arbitrarily many components, and show that the difference between the expected dimension and actual dimension can grow arbitrarily large.

We now summarize our results in greater detail. First, following Sacchiero [**?**] we explain that $\mathrm{M}_e(b_\bullet)$ is nonempty provided that $b_1 \geq 2$ and $e \geq n$ (see Theorem **??**). This already shows that the loci $\mathrm{M}_e(b_\bullet)$ in general do not have the expected dimension (see Proposition **??**).

Before stating the rest of the results, we need some notation. Assume $2e \geq (n-1)d + n - k + 1$. Let q and r be the quotient and remainder in

$$2e - 2 - dk = q(n-1-k) + r.$$

Let $b_\bullet(d^k)$ denote the sequence

$$b_1 = \cdots = b_k = d, \quad b_{k+1} = \cdots = b_{n-r-1} = q, \quad b_{n-r} = \cdots = b_{n-1} = q+1.$$

Miret [**?**] has shown that the locus $\mathrm{M}_e(b_\bullet(d))$ is irreducible of the expected dimension. In contrast, we show the following.

Theorem (Theorem **??**)**.** *Let $k \geq 2$ be an even integer. Let $n \geq 3k-1$ and assume that e is sufficiently large. Then $\mathrm{M}_e(b_\bullet(d^k))$ has at least $\frac{k}{2}+1$ irreducible components.*

When $d = 2$, we obtain sharper bounds. We classify the components of $\mathrm{M}_e(b_\bullet(2^2))$ in detail. We find that it has two components, one of the expected dimension and the other of larger than expected dimension provided e is sufficiently large (see Theorem **??**). More generally, we study $\mathrm{M}_e(b_\bullet(2^k))$ in greater detail.

Theorem (Theorem **??**)**.** *Let $3k-1 \leq n$, and $e > 2kn - 2n - 2$. Then $\mathrm{M}_e(b_\bullet(2^k))$ has at least k components.*

As a source of examples, we determine the splitting type of the normal bundle to monomial rational curves with ordinary singularities (see Theorem **??**). There has been recent interest in computing these normal bundles (see [**?**]). Our methods allow us to compute these normal bundles easily.

Organization of the paper. In §**??**, we collect basic facts concerning normal bundles of rational curves and summarize results of Sacchiero, Ramella and Miret on the stratification of the space of rational curves according to the splitting types of the normal or restricted tangent bundles. In §**??**, we discuss the normal bundles of rational curves defined by monomials. In §**??**, we study the spaces $\mathrm{M}_e(b_\bullet(2^k))$ and show that the number of their components grows linearly with k provided e and n are sufficiently large. We also show that if $n \geq 5$ and e is sufficiently large, $\mathrm{M}_e(b_\bullet(2^2))$ has two irreducible components and describe the components. In §**??**, we study loci $\mathrm{M}_e(b_\bullet(d^k))$ for $d > 2$. Finally, in §**??**, we give some examples.

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2. Preliminaries

In this section, we recall basic facts concerning the geometry of the space of rational curves in $\mathbb{P}^n$. We also review Ramella's results on the splitting of $f^*T_{\mathbb{P}^n}$ [**?**], [**?**], Sacchiero's results showing that all possible splittings for the normal bundle occur [**?**] and Miret's result [**?**] on the irreducibility of $\mathrm{M}_e(b_\bullet(d))$.

2.1. **Basic facts.** Let E be a vector bundle of rank r on $\mathbb{P}^1$. By Grothendieck's theorem, every vector bundle on $\mathbb{P}^1$ is a direct sum of line bundles. Hence, there are uniquely determined integers $a_1 \leq a_2 \leq \cdots \leq a_r$ such that $E \cong \oplus_{i=1}^r \mathcal{O}(a_i)$. These integers are called the *splitting type* of E. The vector bundle is called *balanced* if $a_j - a_i \leq 1$ for $1 \leq i < j \leq r$. The *expected codimension of a splitting type* is equal to the codimension of the splitting type in the versal deformation space and is given by

$$h^1(\mathrm{End}(E)) = h^1(E^* \otimes E) = \sum_{\{i,j|a_i - a_j \leq -2\}} (a_j - a_i - 1).$$

A rational curve C of degree e in $\mathbb{P}^n$ is the image of a morphism $f : \mathbb{P}^1 \to \mathbb{P}^n$, where

$$f = (f_0 : \cdots : f_n)$$

is defined by homogeneous polynomials $f_i(s,t)$ of degree e without common factors. We will always assume that f is *birational onto its image* and that the image is *nondegenerate*. The curve C has *ordinary singularities* if the natural map $f^*\Omega_{\mathbb{P}^n} \to \Omega_{\mathbb{P}^1}$ is surjective. In this case, the kernel is identified with the conormal bundle $N_f^* = \mathcal{H}om(N_f, \mathcal{O}_{\mathbb{P}^1})$, where N_f is the normal sheaf. We conclude that N_f is a vector bundle of rank $n-1$ and degree $e(n+1)-2$.

Let

$$\partial f = \begin{pmatrix} \partial_s f_0 & \cdots & \partial_s f_n \\ \partial_t f_0 & \cdots & \partial_t f_n \end{pmatrix}$$

denote the transpose of the Jacobian matrix. For a rational curve with ordinary singularities, the Euler sequences for $\Omega_{\mathbb{P}^n}$ and $\Omega_{\mathbb{P}^1}$ induce a surjective map

$$\mathcal{O}_{\mathbb{P}^1}(-e)^{n+1} \xrightarrow{\partial f} \mathcal{O}_{\mathbb{P}^1}(-1)^2$$

and identify the conormal bundle N_f^* with the kernel of ∂f [**?**] [**?**]. Thus, the normal bundle N_f has splitting type $\bigoplus_{i=1}^{n-1} \mathcal{O}(e+b_i)$ if and only if the kernel of the map ∂f has splitting type $\bigoplus_{i=1}^{n-1} \mathcal{O}(-e-b_i)$. In other words, the space of relations among the columns of ∂f is generated by forms of degree b_i for $1 \leq i \leq n-1$. We may view a relation of degree b_i among the columns of ∂f as a rational curve of degree b_i in $\mathbb{P}^{n*}$. We will frequently discuss the geometry of the rational curves defined by these relations.

We will need to use the following basic observation.

Lemma 2.1. *Let $(f_0(s,t), \ldots, f_n(s,t))$ be an $(n+1)$-tuple of homogeneous polynomials of degree e in s,t. Let $(a_0, \ldots, a_n)$ be an $(n+1)$-tuple of homogeneous polynomials of degree b in s,t. If $\sum_{i=0}^n a_i \partial_s f_i = \sum_{i=0}^n a_i \partial_t f_i = 0$, then $\sum_{i=0}^n f_i \partial_s a_i = \sum_{i=0}^n f_i \partial_t a_i = 0$.*

Proof. By Euler's relation, the equalities

$$\sum_{i=0}^n a_i \partial_j f_i = 0, \quad j \in \{s,t\}$$

imply

$$\sum_{i=0}^{n} a_i f_i = 0.$$

Differentiating this relation, we see

$$\sum_i f_i \partial_j a_i + \sum_i a_i \partial_j f_i = \sum_i f_i \partial_j a_i = 0.$$

$\square$

We can write

$$N_f = \bigoplus_{i=1}^{n-1} \mathcal{O}(e + b_i),$$

where $2 \leq b_1 \leq \cdots \leq b_{n-1}$ and $\sum_{i=1}^{n-1} b_i = 2e - 2$. To see that $b_1 \geq 2$, we can argue as follows. If $b_1 = 1$, the map $\mathcal{O}(-e-1) \to \mathcal{O}(-e)^{n+1}$ gives a linear relation among the partial derivatives of f_i. By Lemma **??**, we obtain a scalar relation among the f_i. Hence, the map f is degenerate, contrary to assumption.

2.2. **The splitting type of the restricted tangent bundle.** The Euler sequence

$$0 \to f^*\Omega_{\mathbb{P}^n} \to \mathcal{O}(-e)^{n+1} \to \mathcal{O} \to 0$$

identifies $f^*\Omega_{\mathbb{P}^n}$ as the kernel of the homomorphism induced by f. Consider the family of homomorphisms $\mathsf{Hom}(\mathcal{O}(-e)^{n+1}, \mathcal{O})$. The Kodaira-Spencer map

$$\kappa : \mathsf{Hom}(\mathcal{O}(-e)^{n+1}, \mathcal{O}) \to \mathrm{Ext}^1(f^*\Omega_{\mathbb{P}^n}, f^*\Omega_{\mathbb{P}^n})$$

factors through the natural morphisms

$$\mathsf{Hom}(\mathcal{O}(-e)^{n+1}, \mathcal{O}) \xrightarrow{\phi} \mathrm{Ext}^1(\mathcal{O}(-e)^{n+1}, f^*\Omega_{\mathbb{P}^n}) \xrightarrow{\psi} \mathrm{Ext}^1(f^*\Omega_{\mathbb{P}^n}, f^*\Omega_{\mathbb{P}^n}),$$

where ϕ and ψ are maps in the long exact sequence obtained by applying $\mathsf{Hom}(\mathcal{O}(-e)^{n+1}, -)$ and $\mathsf{Hom}(-, f^*\Omega_{\mathbb{P}^n})$, respectively. Since $\mathrm{Ext}^1(\mathcal{O}(-e)^{n+1}, \mathcal{O}(-e)^{n+1}) = 0$ and $\mathrm{Ext}^2(\mathcal{O}, f^*\Omega_{\mathbb{P}^n}) = 0$, we conclude that both ϕ and ψ are surjective. Therefore, the Kodaira-Spencer map is surjective for f with ordinary singularities. Using this calculation, one deduces Ramella's theorem.

Theorem 2.2. [**?**] *The locally closed locus in* $\mathrm{Mor}_e(\mathbb{P}^1, \mathbb{P}^n)$ *parameterizing rational curves with ordinary singularities where* $f^*T_{\mathbb{P}^n}$ *has a specified splitting type is irreducible of the expected dimension* $(e+1)(n+1) - 1 - h^1(\mathrm{End}(f^*T_{\mathbb{P}^n}))$.

2.3. **The possible splitting types of the normal bundle.** In this subsection, we recall Sacchiero's construction of a rational curve f that has ordinary singularities and a specified splitting type for its normal bundle (see [**?**]). We will use this construction throughout the paper. For our purposes, the relations among the columns of ∂f will be especially important.

Let $\delta_1 = 1$, $\delta_i = b_{i-1} - \delta_{i-1}$. Let $c = 1 + \sum_{i=1}^{n-1} \delta_i$. Let $p(s,t)$ and $q(s,t)$ be general polynomials of degree $e - c$ (it is enough to assume that p and q do not have common roots or multiple roots and are not divisible by s or t). Let $k_i = c - \sum_{j=1}^{i} \delta_j$. Then let f be given by the tuple

$$f = (s^{k_0=c}p,\ s^{k_1=c-1}t^{c-k_1}p,\ s^{k_2}t^{c-k_2}p, \cdots, s^{k_{n-2}}t^{c-k_{n-2}}p,\ s^{k_{n-1}=1}t^{c-1}p,\ t^c q).$$

Lemma 2.3 (Sacchiero's Lemma [**?**])**.** *The rational curve f has ordinary singularities and* $N_f \cong \bigoplus_{i=1}^{n-1} \mathcal{O}(e + b_i)$.

Proof. We briefly sketch aspects of Sacchiero's argument that we will later invoke. A simple calculation shows that f has ordinary singularities. Computing N_f is equivalent to computing the kernel of the map

$$\partial f : \mathcal{O}(-e)^{n+1} \to \mathcal{O}(-1)^2.$$

We first describe $n-2$ relations satisfied by the columns of ∂f. Let R_i for $1 \leq i \leq n-2$ be the row vector $(a_0, \dots, a_n)$, where $a_j = 0$ for $j \neq i-1, i, i+1$ and

$$a_{i-1} = (k_i - k_{i+1})t^{k_{i-1}-k_i}, \quad a_i = -(k_{i-1} - k_{i+1})s^{k_{i-1}-k_i}t^{k_i - k_{i+1}}, \quad a_{i+1} = (k_{i-1} - k_i)s^{k_{i-1}-k_{i+1}}.$$

It is easy to see that the columns of ∂f satisfy R_i. Let R be the matrix whose rows are R_i for $1 \leq i \leq n-2$. Then R defines a map

$$R : \bigoplus_{i=1}^{n-2} \mathcal{O}(-e - b_i) \to \mathcal{O}(-e)^{n+1}.$$

Since the image of R is contained in the kernel of ∂f, the map R factors through the inclusion of $N_f^* \to \mathcal{O}(-e)^{n+1}$. An easy computation shows that $(n-2)\times(n-2)$ minor of R obtained by omitting the first two and the last columns is $\prod_{i=1}^{n-2}(k_i - k_{i+1})t^{k_{i-1}-k_{i+1}}$. Similarly, the $(n-2) \times (n-2)$ minor of R obtained by omitting the last three columns is $\prod_{i=1}^{n-2}(k_{i-1} - k_i)s^{k_{i-1}-k_{i+1}}$. Since these minors never simultaneously vanish on $\mathbb{P}^1$, we conclude that the rank of R is always equal to $n-2$. Hence, the image of R is a subbundle of N_f^*. Hence, by degree considerations, we obtain an exact sequence

$$0 \to \bigoplus_{i=1}^{n-2} \mathcal{O}(-e - b_i) \to N_f^* \to \mathcal{O}(-e - b_{n-1}) \to 0.$$

To conclude that N_f^* is isomorphic to $\bigoplus_{i=1}^{n-1} \mathcal{O}(-e - b_i)$, it suffices to observe that there are no nontrivial extensions of this form provided $b_{n-1} \geq \max_{1 \leq i \leq n-2}\{b_i\} - 1$. $\square$

Remark 2.4. Note that in the proof we did not need to use that $b_1 \leq b_2 \leq \cdots \leq b_{n-1}$. We only needed that $b_{n-1} \geq \max_{1 \leq i \leq n-2}\{b_i\} - 1$. This simplification will make certain constructions later in the paper simpler.

The next corollary follows from the proof of Lemma **??** and Remark **??**.

Corollary 2.5. *Let $1 = \delta_1, \delta_2, \cdots, \delta_{n-1}$ be a sequence of positive integers, $e \geq n$ an integer, and*

$$b_{n-1} = 2e - 2 - 2\sum_{i=1}^{n-1} \delta_i + \delta_1 + \delta_{n-1}.$$

Assume

$$b_{n-1} \geq \max_i \{\delta_i + \delta_{i+1}\} - 1.$$

Then there is a nondegenerate rational curve f of degree e in $\mathbb{P}^n$ with normal bundle $\oplus_i \mathcal{O}(e + b_i)$, where $b_i = \delta_i + \delta_{i+1}$ for $1 \leq i \leq n-2$. The columns of ∂f satisfy the relations R_i from the proof of Lemma **??**.

Sacchiero uses Lemma **??** to deduce the following theorem.

Theorem 2.6. [?] *For $1 \leq i \leq n-1$, let $b_i \geq 2$ satisfy $\sum_{i=1}^{n-1} b_i = 2e-2$. Then there exists a rational curve $f : \mathbb{P}^1 \to \mathbb{P}^n$ with ordinary singularities such that*

$$N_f = \oplus_{i=1}^{n-1} \mathcal{O}(e + b_i).$$

In particular, the general smooth rational curve in $\mathbb{P}^n$ has balanced normal bundle.

Other authors, (see [?]) have studied the generic splitting type of normal bundles of rational curves and described the loci where the splitting is not generic.

Sacchiero's Theorem implies that unlike the restricted tangent bundle, the stratification of the space of rational curves by the splitting type of the normal bundle is not well-behaved.

Proposition 2.7. *For $n \geq 6$, there are nonempty loci $\mathrm{M}_e(b_\bullet)$ where the expected codimension is larger than the dimension of $\mathrm{Mor}_e(\mathbb{P}^1, \mathbb{P}^n)$. In particular, when $(n-2)(2e-2n-1) \geq (e+1)(n+1)$, $M(b_\bullet(2^{n-2}))$ is nonempty even though its expected dimension is negative.*

Proof. We compute the expected codimension for curves with normal bundle

$$N_f = \mathcal{O}(e+2)^{n-2} \oplus \mathcal{O}(3e-2n+2).$$

The expected codimension is

$$\begin{aligned} h^1(\mathrm{End}(\mathcal{O}(e+2)^{n-2} \oplus \mathcal{O}(3e-2n+2))) &= (n-2)(3e-2n+2-(e+2)-1) \\ &= (n-2)(2e-2n-1). \end{aligned}$$

For fixed n, this expression grows like $2(n-2)e$ with e. On the other hand, the dimension of $\mathrm{Mor}_e(\mathbb{P}^1, \mathbb{P}^n)$ grows like $(n+1)e$ with e. For $n > 5$, $2(n-2)e$ grows faster than $(n+1)e$. Hence, for sufficiently large e the expected codimension of the locus $\mathrm{M}_e(b_\bullet(2^{n-2}))$ is larger than the dimension of $\mathrm{Mor}_e(\mathbb{P}^1, \mathbb{P}^n)$.

□

Finally, Miret showed that if we fix only the lowest degree factor of the normal bundle, then the resulting locus $\mathrm{M}_e(b_\bullet(d))$ is irreducible. In this case, Eisenbud and Van de Ven's and Ghione and Sacchiero's proofs of irreducibility for $\mathbb{P}^3$ generalize with little change.

Theorem 2.8. [?] *Let $2e-2 \geq d(n-1)+n-2$. Then the locus $\mathrm{M}_e(b_\bullet(d))$ is irreducible of the expected dimension.*

3. Monomial curves

In §**??**, we saw that computing the normal bundle N_f corresponds to determining the kernel of the map ∂f. In general, this is a hard linear algebra problem. However, for monomial maps there is a simple way of reading off the normal bundle from the terms in the sequence. Since monomial maps provide useful examples, we describe the procedure in detail here. Our approach appears to be easier than that of Alzati, Re and Tortora in [?].

In this paper we are primarily concerned with normal bundles of rational curves with ordinary singularities. It is easy to decide when monomial maps have ordinary singularities.

Lemma 3.1. *Let $f = (s^{k_0=e}, s^{k_1}t^{e-k_1}, \cdots, t^{e-k_n=e})$ be a rational curve. Then f has ordinary singularities if and only if $k_1 = e-1$ and $k_{n-1} = 1$, which implies that the image of f is smooth.*

Proof. First, we show that a rational monomial curve with $k_1 = e-1$ and $k_{n-1} = 1$ has ordinary singularities. Consider the matrix of partials coming from only considering f_0, f_1, f_{n-1} and f_n. We have

$$\begin{bmatrix} es^{e-1} & (e-1)s^{e-2}t & t^{e-1} & 0 \\ 0 & s^{e-1} & (e-1)st^{e-2} & et^{e-1} \end{bmatrix}.$$

We see that if $s \neq 0$, the first two columns will be independent, and if $t \neq 0$, the last two columns will be independent, so the curve will have only ordinary singularities. Moreover, the image of f is smooth, since the curve $(s^e, s^{e-1}t, st^{e-1}, t^e)$ is smooth and is a projection of the image of f.

Now suppose $k_1 \neq e-1$. We show f does not have ordinary singularities (the case $k_{n-1} \neq 1$ follows by symmetry). If $k_1 \neq e-1$, then we see that t divides $\partial_t f_i$ for every $i = 0, \cdots n$. Thus, at the point $t = 0$, the curve will have a non-ordinary singularity, since $\partial_t f$ will be identically zero. This completes the proof. □

Theorem 3.2. *Let $f = (s^{k_0=e}, s^{k_1}t^{e-k_1}, \cdots, t^{e-k_n=e})$ be a rational curve with ordinary singularities whose coordinates are given by monomials. Then*

$$N_f \cong \oplus_{i=1}^{n-1} \mathcal{O}(e+b_i)$$

where

$$b_i = k_{i-1} - k_{i+1}.$$

Proof. The proof of this theorem is very similar, in fact, easier than the proof of Lemma **??**. Computing the normal bundle is equivalent to computing the kernel of the map

$$\partial f : \mathcal{O}(-e)^{n+1} \to \mathcal{O}(-1)^2.$$

Hence, we would like to find generators for the relations among the columns of ∂f. First, we exhibit $n-1$ independent relations among the $(\partial_s f_j, \partial_t f_j)$. Each relation $R_i = (a_0, \cdots, a_n)$ will only have three nonzero terms, a_{i-1}, a_i and a_{i+1} for $1 \leq i \leq n-1$. Then

$$a_{i-1} = (k_i - k_{i+1})t^{k_{i-1}-k_{i+1}}, \quad a_i = -(k_{i-1} - k_{i+1})s^{k_{i-1}-k_i}t^{k_i-k_{i+1}}, \quad a_{i+1} = (k_{i-1} - k_i)s^{k_{i-1}-k_{i+1}},$$

and $a_j = 0$ for $j \neq i-1, i, i+1$ is a relation, since it is easily checked that

$$a_{i-1}\begin{bmatrix} k_{i-1}s^{k_{i-1}-1}t^{e-k_{i-1}} \\ (e-k_{i-1})s^{k_{i-1}}t^{e-k_{i-1}-1} \end{bmatrix} + a_i \begin{bmatrix} k_i s^{k_i-1}t^{e-k_i} \\ (e-k_i)s^{k_i}t^{e-k_i-1} \end{bmatrix} + a_{i+1}\begin{bmatrix} k_{i+1}s^{k_{i+1}-1}t^{e-k_{i+1}} \\ (e-k_{i+1})s^{k_{i+1}}t^{e-k_{i+1}-1} \end{bmatrix} = 0$$

Let R be the matrix whose rows are the relations R_i. Note that R_i consists of a row of homogeneous polynomials of degree $b_i = k_{i-1} - k_{i+1}$ for $1 \leq i \leq n-1$. Consequently, the matrix R defines a map

$$\bigoplus_{i=1}^{n-1} \mathcal{O}(-e-b_i) \xrightarrow{R} \mathcal{O}(-e)^{n+1},$$

whose image is in the kernel of ∂f. Therefore, the map R factors through the inclusion

$$0 \longrightarrow N_f^* \longrightarrow \mathcal{O}(-e)^{n+1}.$$

Next, we claim that R has rank $n-1$ at every point of $\mathbb{P}^1$, hence induces an isomorphism

$$\bigoplus_{i=1}^{n-1} \mathcal{O}(-e-b_i) \cong N_f^*.$$

The theorem easily follows. The $(n-1) \times (n-1)$ minors of R obtained by omitting the first two columns and the last two columns are easy to compute and are given by

$$\prod_{i=1}^{n-1} (k_i - k_{i+1})t^{k_{i-1}-k_{i+1}} \quad \text{and} \quad \prod_{i=1}^{n-1} (k_{i-1} - k_i)s^{k_{i-1}-k_{i+1}},$$

respectively. Since these minors do not simultaneously vanish, we conclude that R has rank $n-1$ at every point of $\mathbb{P}^1$. $\square$

Using the same technique, we can also compute the restricted tangent bundle of a curve generated by monomial ideals. The Euler sequence

$$0 \longrightarrow f^*T_{\mathbb{P}^n} \longrightarrow \mathcal{O}(e)^{n+1} \xrightarrow{f} \mathcal{O} \longrightarrow 0$$

exhibits $f^*T_{\mathbb{P}^n}$ as the kernel of the map defined by f. The columns of f satisfy the n relations given by

$$t^{k_{i-1}-k_i} f_{i-1} - s^{k_{i-1}-k_i} f_i = 0.$$

The argument in the proof of Theorem **??** allows us to conclude the following proposition.

Proposition 3.3. *Let $f = (s^{k_0=e}, s^{k_1}t^{e-k_1}, \cdots, t^{e-k_n=e})$ be a rational curve whose coordinates are given by monomials. Then*

$$f^*T_{\mathbb{P}^n} \cong \oplus_{i=1}^n \mathcal{O}(e + c_i)$$

where

$$c_i = k_{i-1} - k_i.$$

We conclude this section with a short discussion of monomial curves without ordinary singularities. If f does not have ordinary singularities, then N_f has both a torsion part and a free part. Taking duals and repeating the argument from the smooth case, we see that N_f^* is the kernel of the map $\mathcal{O}(-e)^{n+1} \to \mathcal{O}(-1)^2$ given by the partials of f, only now the map has a cokernel corresponding to the torsion sheaf $\mathrm{Ext}^1(N_f, \mathcal{O})$. Our calculation in Theorem **??** still works in this case for computing the splitting type of N_f^*.

4. Dimensions of Components

In this section we prove many of the main results of the paper. We start by working out the expected dimension of $\mathrm{M}_e(b_\bullet(d^k))$.

Lemma 4.1. *Assume that $2e \geq (d+1)(n-1)-k+2$. Then the expected codimension of $\mathrm{M}_e(b_\bullet(d^k))$ in $\mathrm{Mor}_e(\mathbb{P}^1, \mathbb{P}^n)$ is $k(2e+1+k)-(d+1)nk$. This is an upper bound for the codimension of every component of $\mathrm{M}_e(b_\bullet(d^k))$.*

Proof. The expected codimension is by definition $h^1(\mathrm{End}(N))$. Since

$$N \cong \mathcal{O}(e+d)^k \oplus \mathcal{O}(e+q+1)^r \oplus \mathcal{O}(e+q)^{n-1-k-r},$$

we see that

$$\begin{aligned} h^1(\mathrm{End}(N)) &= h^1\left(\mathcal{O}(d-q-1)^{kr} \oplus \mathcal{O}(d-q)^{k(n-1-k-r)}\right) \\ &= kr(q-d) + k(n-1-k-r)(q-d-1). \end{aligned}$$

Simplifying using the fact $2e-2-dk = q(n-1-k)+r$ yields the desired formula. The last statement follows from the fact that the loci $\mathrm{M}_e(b_\bullet(d^k))$ are determinantal loci. $\square$

4.1. Two Conics.

We begin by classifying the components of $\mathrm{M}_e(b_\bullet(2^2))$. The following definition will be central to our discussion.

Definition 4.2. Two parameterized rational curves α and β in $\mathbb{P}^n$ satisfy the *parameterized tangency condition* if for some choice of parameters s and t on $\mathbb{P}^1$, $\partial_s\alpha$ is a fixed polynomial multiple of $\partial_t\beta$.

In particular, if α and β have the same degree and satisfy the parameterized tangency condition, then $\partial_s\alpha$ is a scalar multiple of $\partial_t\beta$. If α, β are nondegenerate conics that satisfy the parameterized tangency condition, then their planes intersect in at least a line. Furthermore, if they intersect in a line ℓ, then both α and β are tangent to ℓ.

Let $\mathcal{G}$ denote the closure of the locus in $\mathrm{M}_e(b_\bullet(2^2))$ where ∂f has two independent degree two relations whose corresponding curves in $\mathbb{P}^{n*}$ lie in disjoint planes. Let $\mathcal{PT}$ denote the closure of the locus in $\mathrm{M}_e(b_\bullet(2^2))$ where ∂f has two independent degree two relations satisfying the parameterized tangency condition.

Theorem 4.3. *For $n \geq 5$, $e \geq 2n-3$, $\mathrm{M}_e(b_\bullet(2^2))$ has precisely two components, $\mathcal{G}$ and $\mathcal{PT}$. The dimension of $\mathcal{G}$ is the expected dimension $e(n-3)+7n-6$, and the dimension of $\mathcal{PT}$ is $e(n-2)+5n-3$.*

The proof of the theorem involves a detailed case-by-case analysis of the types of conic relations that can occur among the columns of ∂f. We start by showing that if f is nondegenerate with ordinary singularities, then the relations cannot define degenerate conics.

Lemma 4.4. *If ∂f satisfies a degree two relation that defines a two-to-one map to a line, then f is degenerate.*

Proof. Let a define the degree two relation. Then, up to changing coordinates, we can view a as $(s^2, t^2, 0, \cdots, 0)$. By Lemma **??**, we have the relation

$$\sum_{i=0}^{n} f_i \partial_s a_i = 0.$$

Hence, $f_0 = 0$ and f is degenerate. □

Lemma 4.5. *If ∂f satisfies a degree two relation $a = (a_0, \ldots, a_n)$ with all the a_i's having a common root, then f is degenerate.*

Proof. Change coordinates on $\mathbb{P}^1$ so that the common root is given by $s = 0$, and let $a_i = sa_i'$. Then

$$\sum_{i=0}^{n} a_i' \partial_j f_i = 0$$

for linear functions a_i', so f must be degenerate. □

Corollary 4.6. *If ∂f satisfies two degree two relations that define conics in the same plane in $\mathbb{P}^{n*}$, then f is degenerate.*

Proof. Any one dimensional family of degree two maps from $\mathbb{P}^1$ to $\mathbb{P}^2$ will necessarily contain a degenerate conic. By the previous two lemmas, f is degenerate. □

Thus, to study $M(b_\bullet(2^2))$, we need only consider f with ∂f satisfying two relations that define smooth conics not lying in the same plane. Hence, there are three possibilities: the planes spanned by the conics could be disjoint, meet in a point, or meet in a line. First, we study the case when the degree two relations on ∂f define conics with disjoint planes.

Proposition 4.7. *If $3k \leq n+1$ and $2e \geq 3(n-1)$, there is a component of $\mathrm{M}_e(b_\bullet(2^k))$ of the expected dimension such that for the general element f, the degree two relation on ∂f define k general conics in $\mathbb{P}^{n*}$.*

In our proofs, the following incidence correspondence will play a central role:

$$\mathcal{A} = \{(a_1, \cdots, a_k, f) | \, f \deg e, \; a_i = (a_{i0}, \ldots, a_{in}) \deg 2, \; \sum_{j=0}^{n} a_{ij} \partial_l f_j = 0, 1 \leq i \leq k, l \in \{s, t\}\},$$

where a_i are independent parameterized degree two maps, $f \in \mathrm{Mor}_e(\mathbb{P}^1, \mathbb{P}^n)$ is a nondegenerate, rational curve with ordinary singularities. The incidence correspondence $\mathcal{A}$ projects via π_1 to the space $\mathcal{C}^k$ of k-tuples $\{(a_1, \cdots, a_k)\}$ of independent parameterized conics in $\mathbb{P}^{n*}$ and via π_2 to $\mathrm{Mor}_e(\mathbb{P}^1, \mathbb{P}^n)$. We will estimate the dimensions of the fibers of these two projections.

Proof of Proposition **??**. We will show that there is one component Γ of the incidence correspondence $\mathcal{A}$ that dominates $\mathcal{C}^k$, and we will show that the general element of Γ maps to $\mathrm{M}_e(b_\bullet(2^k))$.

First, we compute the dimension of $\mathcal{C}^k$. A parameterized conic in $\mathbb{P}^{n*}$ is determined by specifying the plane it spans and a degree two map into that plane. The dimension of the Grassmannian $G(3, n+1)$ is $3(n-2)$ and the parameterized map is given by specifying the 9 coefficients of the three polynomials of degree 2. We conclude that $\mathcal{C}^k$ has dimension $3(n-2)k + 9k$.

We claim the general fiber of π_1 has dimension $(e+1)(n+1) - 2k(e+2) - 1$. Let $\mathcal{C}^\circ$ denote the Zariski open locus in $\mathcal{C}^k$ parameterizing k-tuples of conics that span linearly independent planes, and let $(a_1, \cdots, a_k) \in \mathcal{C}^\circ$. Choose coordinates so that a_i is the conic $x_{3i-3} = s^2, x_{3i-2} = -2st, x_{3i-1} = t^2$ in the linear space $\{x_0 = \cdots = x_{3i-4} = 0 = x_{3i} = \cdots = x_n\}$. The conic a_i imposes conditions only on f_{3i-3}, f_{3i-2} and f_{3i-1}. Hence, the number of conditions imposed by the k conics is k times the number of conditions imposed by one conic. By Lemma **??** the condition

$$\sum_{j=0}^{n} a_{ij}\partial_l f_j = 0 \quad \text{translates to} \quad \sum_{j=0}^{n} f_j \partial_l a_{ij} = 0.$$

Hence,

$$2sf_{3i-3} - 2tf_{3i-2} = 0 \quad \text{and} \quad -2sf_{3i-2} + 2tf_{3i-1} = 0.$$

This shows that $st|f_{3i-2}$, but that $\frac{f_{3i-2}}{st}$ can be any degree $e-2$ polynomial, and that $\frac{f_{3i-2}}{st}$ completely determines f_{3i-3}, f_{3i-2}, and f_{3i-1}. Therefore, each conic imposes $2(e+2)$ and the general fiber of π_1 has dimension $(e+1)(n+1) - 2k(e+2) - 1$. Notice that this dimension is positive under our assumption that $3k \leq n+1$.

Since the fibers of π_1 over $\mathcal{C}^\circ$ are irreducible of constant dimension, $\pi_1^{-1}(\mathcal{C}^\circ)$ is irreducible. Let Γ be the closure of $\pi_1^{-1}(\mathcal{C}^\circ)$ in $\mathcal{A}$. Then Γ is irreducible, dominates $\mathcal{C}^k$ and

$$\dim(\Gamma) = (e+1)(n+1) + 3kn - 2ek - k - 1.$$

We now compute the dimension of the general fiber of the map $\pi_2|_\Gamma$. In the next paragraph, we construct an example of a parameterized curve $f \in \pi_2(\Gamma) \cap M(b_\bullet(2^k))$. It follows that π_2 maps the general element of Γ into $M(b_\bullet(2^k))$. The general fiber of π_2 over $\pi_2(\Gamma) \cap M(b_\bullet(2^k))$ corresponds to a choice of k parameterized conics spanning the k-dimensional vector space of conic relations on ∂f. Hence, this fiber has dimension k^2. We conclude that $M(b_\bullet(2^k))$ has a component of dimension

$$(e+1)(n+1) - k(2e+k+1) + 3nk - 1.$$

This matches the expected dimension by Lemma **??**.

To finish, it suffices to construct an example f where ∂f satisfies k general conic relations. Using the division algorithm, write $2e - 2 - 2k = q(n-k-1) + r$ with $0 \leq r < n-k-1$. We construct a curve with

$$N_f = \mathcal{O}(e+2)^k \oplus \mathcal{O}(e+q)^{n-k-1-r} \oplus \mathcal{O}(e+q+1)^r.$$

The construction depends on whether $n-k$ is odd or even. In the odd case, we can construct a monomial example.

- If $n-k$ is odd, then define the sequence

  $$1, 1, x_1, 1, 1, x_2, \ldots, 1, 1, x_k, 1, x_{k+1}, 1, x_{k+2}, \ldots, x_{\frac{n-k-1}{2}}, 1,$$

  where $x_1 = \cdots = x_{\frac{r}{2}} = q$ and $x_{\frac{r}{2}+1} = \cdots = x_{\frac{n-k-1}{2}} = q-1$. Let f be the monomial map

  $$f = (s^{k_0=e}, s^{k_1}t^{e-k_1}, s^{k_2}t^{e-k_2}, \cdots, s^{k_{n-1}}t^{e-k_{n-1}}, t^{e=e-k_n})$$

  Set $k_0 = e$ and $k_{i-1} - k_i$ equal to the ith entry of the sequence above depending on the parities of $k-n$ and r. Write $N_f \cong \bigoplus_{i=1}^{n-1} \mathcal{O}(e+b_i)$. By Theorem **??**, the b_i are given by $k_{i-1} - k_{i+1} = k_{i-1} - k_i + k_i - k_{i+1}$, which is the sum of the ith and $(i+1)$st entries in the sequence. Hence the b_i have the required form.
- If $n-k$ is even, then define the sequence

  $$1, 1, x_1, 1, 1, x_2, \ldots, 1, 1, x_k, 1, x_{k+1}, 1, x_{k+2}, \ldots, x_{\frac{n-k}{2}-1}, 1,$$

  where $x_1 = \cdots = x_{\lfloor \frac{r}{2} \rfloor} = q$ and $x_{\lfloor \frac{r}{2} \rfloor + 1} = \cdots = x_{\frac{n-k}{2}-1} = q-1$. Since $n-k$ is even, then either r or $n-k-r-1$ is even. Then by Corollary **??**, there will be a curve with the required normal bundle and relations.

$\square$

Now we consider the case when the degree two relations on ∂f define conic curves in $\mathbb{P}^{n*}$ whose planes intersect in a single point. Our eventual goal is Corollary **??**, which shows that these maps do not give a new component of $\mathrm{M}_e(b_\bullet(2^2))$. Let $\mathcal{P}$ be the locus in $\mathcal{C}^2$ parameterizing two conics whose planes intersect in a single point.

Lemma 4.8. *Let $(c_1, c_2) \in \mathcal{P}$. Then either $\pi_1^{-1}(c_1, c_2)$ consists entirely of degenerate f or the dimension of $\pi_1^{-1}(c_1, c_2)$ is at most $(e+1)(n+1) - 4e - 7$.*

Proof. By Lemmas **??** and **??**, we may assume that the two conics are smooth and nondegenerate. In suitable coordinates, we may write them as

$$\begin{matrix} (g_1, & g_2, & g_3, & 0, & 0, & 0, & \cdots, & 0) \\ (0, & 0, & g_4, & g_5, & g_6, & 0, & \cdots, & 0) \end{matrix}$$

where the planes of the conics intersect at the point $(0, 0, 1, 0, \ldots, 0)$. Let M_{ij} denote the matrix

$$M_{i,j} = \begin{bmatrix} \partial_s g_i & \partial_s g_j \\ \partial_t g_i & \partial_t g_j \end{bmatrix}$$

We start by showing that if f is nondegenerate, then $\det M_{i,j}$ is not identically zero for $1 \leq i < j \leq 3$. Write $g_i = a_i s^2 + b_i st + c_i t^2$, and notice that

$$\det M_{i,j} = 2(a_i b_j - a_j b_i)s^2 + 2(b_i c_j - b_j c_i)t^2 + 4(a_i c_j - a_j c_i)st.$$

If the determinant were 0, then we would have that the 2 by 2 minors of

$$\begin{bmatrix} a_i & b_i & c_i \\ a_j & b_j & c_j \end{bmatrix}$$

would vanish, which shows that g_i and g_j are linearly dependent. This forces the first conic to be degenerate contrary to Lemma **??**.

Then we see that for any element $(f_0, \cdots, f_n)$ in the fiber of π_1, that

$$\begin{bmatrix} M_{1,2} & \begin{matrix} \partial_s g_3 \\ \partial_t g_3 \end{matrix} \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \end{bmatrix} = 0.$$

Multiplying by $M_{1,2}^{-1}$ and solving for f_0 and f_1, we get

$$\begin{bmatrix} f_0 \\ f_1 \end{bmatrix} = -M_{1,2}^{-1} \begin{bmatrix} \partial_s g_3 \\ \partial_t g_3 \end{bmatrix} f_2$$

Expanding out $M_{1,2}^{-1}$ in terms of the partials of the g_i, we finally see that

$$f_0 = -\frac{\det M_{2,3}}{\det M_{1,2}} f_2 \quad \text{and} \quad f_1 = -\frac{\det M_{1,3}}{\det M_{1,2}} f_2.$$

Unless there is cancellation, we see that $\det M_{1,2}$ must divide f_2. If $\det M_{1,2}$ does not divide f_2, we see that the 2 by 2 minors of

$$\begin{bmatrix} \partial_s g_1 & \partial_s g_2 & \partial_s g_3 \\ \partial_t g_1 & \partial_t g_2 & \partial_t g_3 \end{bmatrix}$$

must all vanish at the same point, which means that the conic must be degenerate.

We can obtain similar expressions for f_3 and f_4 in terms of f_2, and similarly can see that f_2 must be divisible by $\det M_{5,6}$. Thus, there are between $e+1-2 = e-1$ and $e+1-4 = e-3$ choices for f_2 (depending on what factors, if any, $\det M_{1,2}$ and $\det M_{5,6}$ have in common). Given f_2, we see that f_0, f_1, f_3 and f_4 are completely determined, so the fiber dimension is between $(e+1)(n+1) - 4e - 9$ and $(e+1)(n+1) - 4e - 7$, completing the proof. $\square$

Remark 4.9. The last paragraph of the proof of Lemma **??** implies the following. The locus in $\mathcal{P}$ where $\det M_{1,2}$ and $\det M_{5,6}$ have a common factor has codimension one in $\mathcal{P}$. Similarly, the locus where $\det M_{1,2}$ and $\det M_{5,6}$ has codimension two in $\mathcal{P}$. Hence, the dimension of the fibers of π_1 over the general point in $\mathcal{P}$ is $(e+1)(n+1)-4e-9$ equal to the expected dimension. The fiber dimension increases by one over a codimension one locus in $\mathcal{P}$ and increases by two over a codimension two locus in $\mathcal{P}$.

Corollary 4.10. *Let $n \geq 5$ and $2e \geq 3(n-1)$. Then $\pi_2(\pi_1^{-1}(\mathcal{P})) \cap \mathrm{M}_e(b_\bullet(2^2))$ is contained in the component $\pi_2(\Gamma)$.*

Proof. By Lemma **??** and Remark **??**, every component of $\pi_1^{-1}(\mathcal{P}) \subset \mathcal{A}$ whose general point corresponds to a nondegenerate f has dimension at most $\dim(\mathcal{P}) + (e+1)(n+1) - 4e - 9$. Observe that the fiber dimension of π_2 over a point in $\pi_2(\pi_1^{-1}(\mathcal{P}))$ is at least 4. Hence, the image of any such component is at most $\dim(\mathcal{P}) + (e+1)(n+1) - 4e - 13$. On the other hand, $\dim(\mathcal{P}) = \dim(\mathcal{C}^2) - n + 4$. Since $n \geq 5$, we conclude that $\pi_2(\pi_1^{-1}(\mathcal{P}))$ cannot contain any irreducible components of $\mathrm{M}_e(b_\bullet(2^2))$. Hence, $\pi_2(\pi_1^{-1}(\mathcal{P})) \cap \mathrm{M}_e(b_\bullet(2^2))$ is contained in $\pi_2(\Gamma)$. ☐

Let $\mathcal{L}$ denote the locus in $\mathcal{C}^2$ where the planes of the two conics intersect in a line.

Lemma 4.11. *Let $(q_1, q_2) \in \mathcal{L}$. Then either q_1 and q_2 satisfy the parameterized tangency condition and the fiber $\pi_1^{-1}(q_1, q_2)$ has dimension $(e+1)(n+1) - 3e - 7$ or $\pi_1^{-1}(q_1, q_2)$ contains no points corresponding to nondegenerate f.*

Proof. We choose coordinates on $\mathbb{P}^{n*}$ so that the two planes have the form

$$(*, *, *, 0, 0, \cdots, 0)$$
$$(0, *, *, *, 0, \cdots, 0)$$

First, we show that each conic must be tangent to the line of intersection ℓ. To get a contradiction, suppose the first conic intersects ℓ in two distinct points. Up to reparameterizing $\mathbb{P}^1$, we can express the conic as

$$(st, s^2, t^2, 0, \cdots, 0).$$

Let the other conic be

$$(0, g_1, g_2, g_3, 0 \cdots, 0),$$

where $g_i = a_i s^2 + b_i st + c_i t^2$. Then we see that the fibers will be tuples $(f_0, f_1, f_2, f_3, f_4, \cdots, f_n)$ with

$$\begin{bmatrix} t & 2s & 0 & 0 \\ s & 0 & 2t & 0 \\ 0 & 2a_1 s + b_1 t & 2a_2 s + b_2 t & 2a_3 s + b_3 t \\ 0 & b_1 s + 2c_1 t & b_2 s + 2c_2 t & b_3 s + 2c_3 t \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

If the f_i are not all zero, we see that this implies that the determinant of the 4 by 4 matrix must be 0, giving us

$$4((a_3b_2 - a_2b_3)s^4 + 2(a_3c_2 - a_2c_3)s^3t + (a_3b_1 - a_1b_3 + b_3c_2 - b_2c_3)s^2t^2 + 2(a_3c_1 - a_1c_3)st^3 + (b_3c_1 - b_1c_3)t^4) = 0$$

or

$$a_3b_2 - a_2b_3 = 0 \tag{1}$$
$$a_3c_2 - a_2c_3 = 0 \tag{2}$$
$$a_3b_1 - a_1b_3 + b_3c_2 - b_2c_3 = 0 \tag{3}$$
$$a_3c_1 - a_1c_3 = 0 \tag{4}$$
$$b_3c_1 - b_1c_3 = 0 \tag{5}$$

We consider the implications of this on the matrix with rows given by the coefficients of the g_i:

$$G = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

Equations (1) and (2) are precisely the vanishing of two of the subminors of the bottom two rows of the matrix G. If $a_2 = a_3 = 0$, then by equation (4) $a_1 c_3 = 0$. Since the a_i cannot all vanish (otherwise, the g_i would all have a common factor), we see that $c_3 = 0$. By equation (5), this shows that $b_3 c_1 = 0$. Since $b_3 \neq 0$ (otherwise we would have $g_3 = 0$ and one of our conics would be degenerate), this gives that $c_1 = 0$. Thus, our two conics have the form

$$(st, s^2, t^2, 0, 0, \cdots, 0)$$
$$(0, a_1 s^2 + b_1 st, b_2 st + a_1 t^2, b_3 st, 0, \cdots, 0)$$

We see that a_1 times the first row minus the second row will consist of a conic whose terms all have common factors, which is impossible. Therefore, our original assumption that $a_2 = a_3 = 0$ was wrong.

If a_2 and a_3 are not both 0, then equations (1) and (2) give $b_2 = \lambda a_2$, $c_2 = \nu a_2$, $b_3 = \lambda a_3$, $c_3 = \nu a_3$. If $a_3 = 0$, then we see that both b_3 and c_3 are 0, which means that $g_3 = 0$, which means that the second conic is a double cover of a line, which is impossible, so we see that $a_3 \neq 0$. Combining our expressions for c_3 with equation (4), we see that $a_3(c_1 - \nu a_1) = 0$, which means $c_1 = \nu a_1$. Thus, the determinant of G is 0 since the last column is a multiple of the first, which means that there is a linear relation among the rows, which means that the conic is degenerate. Thus, this case is impossible, which shows that both of the two conics must be tangent to the line of intersection of the two planes.

So, suppose that the two conics are both tangent to the line of intersection of the two planes. Up to a choice of coordinates on $\mathbb{P}^n$, we can assume our two conics have the form

$$(s^2, st, t^2, 0, 0, \cdots, 0)$$

and

$$(0, g_1, g_2, g_3, 0, \cdots, 0).$$

As before, write $g_i = a_i s^2 + b_i st + c_i t^2$. Since the second conic is also tangent to the line of intersection of the two planes, we see that g_3 is a square, i.e., $g_3 = (us + vt)^2$. Then we see that the fibers will be tuples $(f_0, f_1, \cdots, f_n)$ where

$$\begin{bmatrix} 2s & t & 0 & 0 \\ 0 & s & 2t & 0 \\ 0 & 2a_1 s + b_1 t & 2a_2 s + b_2 t & 2u(us+vt) \\ 0 & b_1 s + 2c_1 t & b_2 s + 2c_2 t & 2v(us+vt) \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Taking $-v$ times the third row of the matrix plus u times the fourth row of the matrix gives

$$\ell_1 f_1 + \ell_2 f_2 = 0$$

where $\ell_1 = -v(2a_1 s + b_1 t) + u(b_1 s + 2c_1 t)$ and $\ell_2 = -v(2a_2 s + b_2 t) + u(b_2 s + 2c_2 t)$. Since we also have

$$sf_1 + 2tf_2 = 0$$

we see that either $f_2 = 0$ or the two conics satisfy the parameterized tangency relation.

To compute the fiber dimension of π_1, observe that f_0 determines f_1, f_2, f_3 and must be divisible by t^3. Hence, in total there are $3e + 6$ conditions on the fiber of π_1. This concludes the proof. $\square$

Proof. (Theorem **??**) We have already seen that the component $\mathcal{G} = \pi_2(\Gamma)$ is an irreducible component. It follows easily from Lemma **??** and Corollary **??** that there is at most one more component $\mathcal{PT}$ corresponding to conics satisfying the parameterized tangency condition. We later exhibit an element of $\mathcal{PT}$ to show that this component is nonempty. If its dimension is at least as large as the dimension of $\mathcal{G}$ it will have to be a separate component, and we now work out the dimension of this component. There are $3(n-2)+9$ dimensions of choice for the first conic. Then, there is a 1-dimensional family of choices of tangent line, and an $n-2$ dimensional family of 2-planes containing this line. Finally, there is a 5-dimensional family of conics satisfying the parameterized tangency condition with the first conic, since there is a 1-dimensional choice of coordinate where the conic is tangent to the given line, a 1-dimensional choice of scaling, and a 3-dimensional family of lines satisfying the parameterized tangency condition subject to those choices of parameters. This gives a $3(n-2)+9+1+n-2+5 = 4n+7$ dimensional family of pairs of conics satisfying the parameterized tangency condition. The dimension of the fibers of π_1 over this locus is $(e+1)(n+1)-3e-7$, and the dimension of the fibers of π_2 over this locus in $\mathcal{A}$ is 4, showing that $\mathcal{PT}$ has dimension $4n+7+(e+1)(n+1)-3e-7-4 = e(n-2)+5n-3$. This will be at least the dimension of $\mathcal{G}$ when $e \geq 2n-3$ by Proposition **??**.

Finally we exhibit an example of a curve in $\mathcal{PT}$. Express $2e-6 = (n-3)q+r$. As in the proof of Proposition **??** the construction depends on the parity of n and r.

- If n is odd, then r is even. Consider the sequence
$$1, 1, 1, x_1, 1, x_2, 1, \dots, x_{\frac{n-3}{2}}, 1,$$
where $x_1 = \cdots = x_{\frac{r}{2}} = q$ and $x_{\frac{r}{2}+1} = \cdots = x_{\frac{n-3}{2}} = q-1$. Then let
$$f = (s^{k_0=e}, s^{k_1}t^{e-k_1}, s^{k_2}t^{e-k_2}, \cdots, s^{k_{n-1}}t^{e-k_{n-1}}, t^{e=e-k_n})$$
be the monomial curve, with $k_0 = e$ and $k_{i-1}-k_i$ equal to the ith entry of the sequence above. Write $N_f \cong \bigoplus_{i=1}^{n-1} \mathcal{O}(e+b_i)$. By Theorem **??**, the b_i are given by $k_{i-1}-k_{i+1} = k_{i-1}-k_i+k_i-k_{i+1}$, which is the sum of the ith and $(i+1)$st entries in the sequence. Hence the b_i have the required form.
- If n is even, then let the sequence be
$$1, 1, 1, x_1, 1, x_2, 1, \dots, x_{\frac{n-2}{2}-1}, 1,$$
where $x_1 = \cdots = x_{\lfloor \frac{r}{2} \rfloor} = q$ and $x_{\lfloor \frac{r}{2} \rfloor + 1} = \cdots = x_{\frac{n-2}{2}-1} = q-1$. Since $n-k$ is even, then either r or $n-k-r-1$ is even. Then by Corollary **??** there exists a curve of the required form with the required relations.

□

4.2. **More conics.** In this section, by considering chains of conic relations that satisfy the parameterized tangency condition, we will show that the number of components of $\mathrm{M}_e(b_\bullet(2^k))$ grows at least linearly with k for sufficiently large e and n.

Let B_j^k denote the collection of k parameterized conics $(C_1, \dots, C_k)$, where C_i with $i \leq j$ is the parameterized conic $(s^2, -2st, t^2)$ contained in the plane $x_h = 0$ for $h \neq i-1, i, i+1$ and C_i with $i > j$ is the parameterized conic $(s^2, -2st, t^2)$ contained in the plane $x_h = 0$ for $h \neq 3i-2j-1, 3i-2j, 3i-2j+1$. For example, B_3^5 is the following set of conics

$$\begin{array}{llllllllllllll}
(s^2, & -2st, & t^2, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & \cdots & 0) \\
(0, & s^2, & -2st, & t^2, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & \cdots & 0) \\
(0, & 0, & s^2, & -2st, & t^2, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & \cdots & 0) \\
(0, & 0, & 0, & 0, & 0, & s^2, & -2st, & t^2, & 0, & 0, & 0, & 0, & \cdots & 0) \\
(0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & s^2, & -2st, & t^2, & 0, & \cdots & 0)
\end{array}$$

The conics C_i and C_{i+1} in B_j^k satisfy the parameterized tangency condition for $1 \leq i < j$ and the rest of the conics are general.

Theorem 4.12. *Let $n \geq 3k-1$ and $e > 2kn - 2n - 2$. Then $\mathrm{M}_e(b_\bullet(2^k))$ has at least k components.*

Proof. Let $\mathcal{C}_j^k$ denote the locus of parameterized conics $(C_1, \dots, C_k)$ in $\mathcal{C}^k$ such that C_i and C_{i+1} satisfy the parameterized tangency condition for $1 \leq i < j$ and C_i are general for $i > j$. In particular, $B_j^k \in \mathcal{C}_j^k$. We first compute the dimension of the locus $\mathcal{C}_j^k$. As in the proof of Proposition **??**, a general parameterized conic depends on $3(n-2)+9$ parameters and given a conic C_1 there is an $n+4$ parameter family of conics satisfying the parameterized tangency condition with respect to C_1. Consequently,

$$\begin{aligned}\dim(\mathcal{C}_j^k) &= (k-j+1)(3(n-2)+9) + (j-1)(n+4) \\ &= k(3n+3) - (j-1)(2n-1).\end{aligned}$$

For a general point $x = (C_1, \dots, C_k) \in \mathcal{C}_j^k$, the set of partial derivatives of the C_i span a linear space of dimension $2k-j+1$. Let $p = (p_0, \dots, p_n)$ be a point of the span of these partial derivatives. Then the relation $\sum p_i f_i = 0$ is a polynomial in s, t of degree $e+1$, hence imposes at most $(e+2)$ conditions on the fibers of π_1. We conclude that the dimension of $\pi_1^{-1}(x)$ is at least $(e+1)(n+1) - 1 - (2k-j+1)(e+2)$.

On the other hand, consider the dimension of $\pi_1^{-1}(B_j^k)$. The f_i satisfy the relations

$$f_i = \left(\frac{s}{t}\right)^i f_0 \quad \text{for} \quad 1 \leq i \leq j+1$$

and

$$f_{3i-2j} = \frac{s}{t} f_{3i-2j-1}, \quad f_{3i-2j+1} = \left(\frac{s}{t}\right)^2 f_{3i-2j-1} \quad \text{for } j < i \leq k.$$

Hence, f_0 can be chosen freely subject to the condition that it is divisible by t^{j+1}. This determines f_i for $1 \leq i \leq j+1$. Then for $j < i \leq k$, the entry $f_{3i-2j-1}$ can be chosen freely subject to the condition that it is divisible by t^2. This determines f_{3i-2j} and $f_{3i-2j+1}$. All remaining f_i are free. We conclude that

$$\dim(\pi_1^{-1}(B_j^k)) = (e+1)(n+1) - 1 - (2k-j+1)(e+2).$$

Hence, the general fiber of π_1 over $\mathcal{C}_j^k$ is irreducible of dimension $(e+1)(n+1)-1-(2k-j+1)(e+2)$. We conclude that there is a component $\mathcal{V}_j$ of $\pi_1^{-1}(\mathcal{C}_j^k)$ with

$$\dim(\mathcal{V}_j) = (e+1)(n+1) - 1 + k(3n-2e-1) + (j-1)(e-2n+3).$$

We warn the reader that $\mathcal{V}_j$ will typically not be a component of the incidence correspondence. However, we will shortly show that each $\mathcal{V}_j$ has to be contained in a distinct irreducible component $\mathcal{P}_j$ of the incidence correspondence $\pi_1^{-1}(\mathcal{C}^k)$.

Suppose there exists an irreducible component U containing $\mathcal{V}_{j_1}$ and $\mathcal{V}_{j_2}$ for $j_1 < j_2$. Then $\pi_1(U)$ contains $B_{j_1}^k$. Hence, the general fiber dimension of π_1 restricted to U is at most $(e+1)(n+1)-1-(2k-j_1+1)(e+2)$. Hence, the dimension of U is at most $\dim(\mathcal{C}^k)+(e+1)(n+1)-1-(2k-j_1+1)(e+2)$. However, the dimension of $\mathcal{V}_{j_2}$ is $(e+1)(n+1) - 1 + k(3n-2e-1) + (j_2-1)(e-2n+3)$. We bound

$$\dim(\mathcal{V}_{j_2}) - \dim(U) \geq (j_2 - j_1)e + 3j_2 - 2j_1 - 2j_2 n + 2n - 1 \geq e + 2n + 2 - 2kn.$$

By our assumption on e, this number is positive. This is a contradiction. We conclude that $\mathcal{V}_j$ belong to different components for each $1 \leq j \leq k$.

Now consider the projection $\pi_2(\mathcal{V}_j)$. We will shortly see that the general member of $\mathcal{V}_j$ has the desired normal bundle. Consequently, there are exactly k independent conic relations among the rows of ∂f and the general fiber dimension of π_2 restricted to $\mathcal{P}_j$ is k^2. In fact, π_2 is generically a $\mathrm{GL}(k)$-bundle corresponding to choices of bases for the conic relations among the columns of ∂f. Consequently, $\pi_2(\mathcal{P}_j)$ is a distinct irreducible component of $\mathrm{M}_e(b_\bullet(2^k))$ for each $1 \leq j \leq k$.

Finally, using Sacchiero's construction, we see that there are smooth rational curves in these loci that lie in $M(b_\bullet(2^k))$. Using the division algorithm, write $2e-2-2k = q(n-k-1)+r$ with $0 \leq r < n-k-1$. We construct a curve with

$$N_f = \mathcal{O}(e+2)^k \oplus \mathcal{O}(e+q)^{n-k-1-r} \oplus \mathcal{O}(e+q+1)^r.$$

The construction depends on whether $n-k$ is odd or even.

- If $n-k$ is odd, then define the sequence

  $$1, 1, \cdots, 1, 1, x_1, 1, 1, x_2, \ldots, 1, 1, x_{k-j}, 1, x_{k-j+1}, 1, x_{k-j+2}, \ldots, x_{\frac{n-k-1}{2}}, 1,$$

  where there are $j+1$ 1's at the beginning, $x_1 = \cdots = x_{\frac{r}{2}} = q$ and $x_{\frac{r}{2}+1} = \cdots = x_{\frac{n-k-1}{2}} = q-1$. Let f be the monomial map

  $$f = (s^{k_0=e}, s^{k_1}t^{e-k_1}, s^{k_2}t^{e-k_2}, \cdots, s^{k_{n-1}}t^{e-k_{n-1}}, t^{e=e-k_n})$$

  Set $k_0 = e$ and $k_{i-1} - k_i$ equal to the ith entry of the sequence above depending on the parities of $k-n$ and r. Write $N_f \cong \bigoplus_{i=1}^{n-1} \mathcal{O}(e+b_i)$. By Theorem **??**, the b_i are given by $k_{i-1} - k_{i+1} = k_{i-1} - k_i + k_i - k_{i+1}$, which is the sum of the ith and $(i+1)$st entries in the sequence. Hence the b_i have the required form.
- If $n-k$ is even, then define the sequence

  $$1, 1, \cdots, 1, 1, x_1, 1, 1, x_2, \ldots, 1, 1, x_{k-j}, 1, x_{k-j+1}, 1, x_{k-j+2}, \ldots, x_{\frac{n-k}{2}-1}, 1,$$

  where there are $j+1$ 1's at the beginning, $x_1 = \cdots = x_{\lfloor \frac{r}{2} \rfloor} = q$ and $x_{\lfloor \frac{r}{2} \rfloor + 1} = \cdots = x_{\frac{n-k}{2}-1} = q-1$. Since $n-k$ is even, then either r or $n-k-r-1$ is even. Then by Corollary **??**, there is a curve with the required form and relations.

□

Let μ be a partition of k with h parts $k = k_1 + k_2 \cdots + k_h$. Let $\mathcal{C}_\mu$ be the locus of parameterized conics $(C_1, \ldots, C_k)$, where C_i and C_{i+1} satisfy the parameterized tangency condition for any index i with $\sum_{j=1}^{l} k_j < i < \sum_{j=1}^{l+1} k_j$ for some $0 \leq l < h$. Let $\mathcal{P}_\mu = \pi_1^{-1}(\mathcal{C}_\lambda^k)$. By an argument identical to that of Theorem **??**, the loci $\mathcal{P}_\mu$ and $\mathcal{P}_\nu$ belong to different components if e is sufficiently large and μ and ν have different number of parts. We pose the following natural question.

Question 4.13. Let μ and ν be two different partitions of k. Do $\mathcal{P}_\mu$ and $\mathcal{P}_\nu$ belong to different irreducible components of the incidence correspondence?

Remark 4.14. Since it is possible to construct elements in $\mathcal{P}_\mu$ that map to $\mathrm{M}_e(b_\bullet(2^k))$ under π_2, a positive answer to the question would imply that the number of irreducible components of $\mathrm{M}_e(b_\bullet(2^k))$ is at least the number of partitions of k provided that $n \geq 3k-1$ and e is sufficiently large. This would provide superpolynomial growth for the number of components.

5. Higher degree relations

In this section, we generalize the discussion for $\mathrm{M}_e(b_\bullet(2^k))$ to $\mathrm{M}_e(b_\bullet(d^k))$ and for $n \geq 5$ exhibit multiple irreducible components of $\mathrm{M}_e(b_\bullet(d^k))$.

Theorem 5.1. *For $n \geq 3k-1$ and $k \geq 2$ even, $\mathrm{M}_e(b_\bullet(d^k))$ has at least $\frac{k}{2}+1$ components for $e \geq k(d+1)(n+1)$.*

Proof. Let $\mathcal{D}^k$ be the space of k-tuples of independent parameterized degree d rational curves. Let $I_{k,e}$ be the incidence correspondence parameterizing pairs (D, f), where f is a degree e rational curve and $D \in \mathcal{D}^k$ is a set of k independent degree d relations among the columns of ∂f. Let π_1 and π_2 denote the two projections to $\mathcal{D}^k$ and $\mathrm{Mor}_e(\mathbb{P}^1, \mathbb{P}^n)$, respectively. We find $\frac{k}{2}+1$ components of $\mathrm{M}_e(b_\bullet(d^k))$.

First, for $0 \le j \le \frac{k}{2}$, we construct an element $B_j = (B_{j,1}, \cdots, B_{j,k}) \in \mathcal{D}^k$, where for $1 \le i \le j$, $B_{j,2i-1}$ and $B_{j,2i}$ are given by

$$\begin{matrix} (\cdots 0, & (d-1)t^d, & -dst^{d-1}, & s^d, & 0, & 0\cdots) \\ (\cdots 0, & 0, & t^d, & -ds^{d-1}t, & (d-1)s^d, & 0\cdots) \end{matrix},$$

where the nonzero coordinates are $x_{4i-4}, x_{4i-3}, x_{4i-2}$, and x_{4i-1}. For $i > j$, we let $B_{j,2i-1}$ and $B_{j,2i}$ be given by

$$\begin{matrix} (\cdots 0, & t^d, & -ds^{d-1}t, & (d-1)s^d, & 0, & 0, & 0, & 0\cdots) \\ (\cdots 0, & 0, & 0, & 0, & (d-1)t^d, & -dst^{d-1}, & s^d, & 0\cdots) \end{matrix},$$

where the nonzero coordinates are $x_{6i-2j-6}, x_{6i-2j-5}, x_{6i-2j-4}, x_{6i-2j-3}, x_{6i-2j-2}$, and $x_{6i-2j-1}$.

We work out the dimensions of the fibers of π_1 over B_j. It is clear from the definitions that for $i_1 \neq i_2$, the conditions imposed on the fibers of π_1 by the pair $B_{j,2i_1-1}, B_{j,2i_1}$ and the pair $B_{j,2i_2-1}, B_{j,2i_2}$ are independent. For $i \le j$, the matrix of partial derivatives of $B_{j,2i-1}$ and $B_{j,2i}$ is

$$A = \begin{bmatrix} d(d-1)t^{d-1} & -d(d-1)st^{d-2} & 0 & 0 \\ 0 & -dt^{d-1} & ds^{d-1} & 0 \\ 0 & dt^{d-1} & -ds^{d-1} & 0 \\ 0 & 0 & -d(d-1)s^{d-2}t & d(d-1)s^{d-1} \end{bmatrix}.$$

From the relations

$$A \begin{bmatrix} f_{4i-4} \\ f_{4i-3} \\ f_{4i-2} \\ f_{4i-1} \end{bmatrix} = 0$$

we see that s^{d+1} divides f_{4i-4}, but that subject to that condition, f_{4i-4} can be chosen freely and this choice completely determines f_{4i-3}, f_{4i-2}, and f_{4i-1}. This makes for $4(e+1)-(e-d) = 3e+d+4$ conditions. For $i > j$, we see by a similar calculation that $B_{j,2i-1}$ and $B_{j,2i}$ impose $2(e+d+e+2) = 4e+2d+4$ conditions. Thus, the total number of conditions imposed is $j(3e+d+4)+(k-2j)(2e+d+2) = k(2e+d+2)-j(e+d)$.

For future use, we construct an example of a curve in the fiber $\pi_1^{-1}(B_j)$ corresponding to a nondegenerate curve with ordinary singularities that lies in $\mathrm{M}_e(b_\bullet(d^k))$. Let q and r be defined by $2e-2-kd = q(n-1-k)+r$ with $0 \le r < q$. Consider the following sequence $\delta_1, \cdots, \delta_{n-1}$. For $1 \le i \le 4j$ the sequence repeats the length four pattern $1, d-1, 1, x_l$ and looks like

$$1,\ d-1,\ 1,\ x_1,\ 1,\ d-1,\ 1,\ x_2,\ \ldots\ 1,\ d-1,\ 1,\ x_j.$$

For $4j+1 \le i \le 3k-2j$ the sequence repeats the length six pattern

$$1,\ d-1,\ x_l-(d-2),\ d-1,\ 1,\ x_{l+1}$$

and looks like

$$1,\ d-1,\ x_{j+1}-(d-2),\ d-1,\ 1,\ x_{j+2}, \ldots,\ 1,\ d-1,\ x_{k-j-1}-(d-2),\ d-1,\ 1,\ x_{k-j}.$$

Finally, for $3k-2j+1 \le i \le n-1$, the sequence repeats the length two pattern $1, x_l$ and looks like

$$1,\ x_{k-j+1},\ 1,\ x_{k-j+2}, \ldots$$

Here, $x_1 = \cdots = x_{\lfloor \frac{r}{2} \rfloor} = q$ and $x_{\lfloor \frac{r}{2} \rfloor + 1} = \cdots = x_{\lfloor \frac{n-k-1}{2} \rfloor} = q - 1$. For example, if $n = 19$, $e = 41$, $d = 3$, $k = 6$, and $j = 2$, we have $q = 5$ and $r = 2$ and we have the sequence

$$1, 2, 1, 5, 1, 2, 1, 4, 1, 2, 3, 2, 1, 4, 1, 4, 1, 4.$$

By Corollary **??**, there exists a curve with the required normal bundle and relations.

We now argue that for $j_1 < j_2$, $\pi_1^{-1}(B_{j_1})$ and $\pi_1^{-1}(B_{j_2})$ lie in different irreducible components of $\pi_1^{-1}(\mathcal{D}^k)$. To get a contradiction, suppose that some component U of $\pi_1^{-1}(\mathcal{D}^k)$ contains both $\pi_1^{-1}(B_{j_1})$ and $\pi_1^{-1}(B_{j_2})$. Then since $\pi_1^{-1}(B_{j_1})$ lies in U, the general fiber of π_1 restricted to U has dimension at most $(e+1)(n+1) - 1 - k(2e+d+2) + j_1(e+d)$, which shows that the dimension of U is at most $\dim \mathcal{D}^k + (e+1)(n+1) - 1 - k(2e+d+2) + j_1(e+d)$. However, the dimension of $\pi_1^{-1}(B_{j_2})$ is at least $(e+1)(n+1) - 1 - k(2e+d+2) + j_2(e+d)$. Bounding $\pi_1^{-1}(B_{j_2}) - \dim U$, we get

$$\pi_1^{-1}(B_{j_2}) - \dim U \geq (j_2 - j_1)(e+d) - k(d+1)(n+1) \geq e + d - k(d+1)(n+1) > 0$$

by our assumption on e.

The examples show that any irreducible component of the incidence correspondence containing a B_j is generically a $\mathrm{GL}(k)$ bundle over its image in π_2, so $\pi_2(\pi_1^{-1}(B_j))$ all lie in different components for each $0 \leq j \leq \frac{k}{2}$. The result follows.

□

6. Examples

In this section, we discuss some basic examples of strata of rational curves with fixed normal bundle. In particular, we construct an example of a reducible stratum in $\mathbb{P}^4$ of rational curves with fixed normal bundle, and we look more carefully at the example of Alzati and Re [**?**], showing among other things that it has at least three reducible components.

6.1. Conics in $\mathbb{P}^4$.

Many of the results in section **??** were only for $n \geq 5$. In this section we completely describe $\mathrm{M}_e(b_\bullet(2^2))$ for degree e curves in $\mathbb{P}^4$.

Proposition 6.1. *For $e \geq 5$ and $n = 4$, $\mathrm{M}_e(b_\bullet(2^2))$ is irreducible of dimension $2e + 18$. This is larger than the expected dimension for $e \geq 6$.*

Proof. First we show that if f is a degree e rational curve in $\mathbb{P}^4$ such that the relations among the columns of ∂f correspond to two conics in the dual space whose planes meet in a point, then f is degenerate. To see this, note that for two such conics, their partial derivatives will span a 4-dimensional vector space of degree 1 maps to $\mathbb{P}^{4*}$. By Lemma **??** the partial derivatives will give a 4-dimensional space of linear forms a_i such that $\sum_j a_{ij} f_j = 0$. This shows that if f were nondegenerate, the restricted tangent bundle $f^* T_{\mathbb{P}^n}$ would be $\mathcal{O}(1)^4$, which is impossible by degree considerations. Thus, any such f must be degenerate.

There is another component, however, corresponding to pairs of conics satisfying the parameterized tangency condition. We work this dimension count out carefully. The dimension of the space of conics in $\mathbb{P}^4$ is $3(n-2) + 9 = 15$. Given the first conic, there is a 1-dimensional choice of tangent lines, then an 2-dimensional family of planes containing this tangent line, followed by a 5-dimensional family of conics satisfying the parameterized tangency condition, for a total of 8 dimensions. Thus, this will correspond to a 23-dimensional locus in $\mathcal{C}^2$. The fiber of π_1 over this locus is $(e+1)(n+1) - 3(e+2) = 2e - 1$-dimensional. The fibers of π_2 over this locus are 4-dimensional, so the dimension of this family is $2e - 1 + 23 - 4 = 2e + 18$. The expected dimension of $\mathrm{M}_e(b_\bullet(2^2))$ is $5(e+1) - (4e - 18) = e + 23$, so we see that for $e \geq 6$, this will have larger than expected dimension. □

6.2. **An example in $\mathbb{P}^5$.** In this section, we find the smallest example where $\mathrm{M}_e(b_\bullet(2^2))$ has two components. In particular, note that both e and n are smaller than the $e = 11$, $n = 8$ example discovered by Alzati and Re.

Corollary 6.2. *The space $\mathrm{M}_e(b_\bullet(2^2))$ has two components for $e \geq 2n-3$, $n \geq 5$. In particular for $n = 5$, $e = 7$, $\mathrm{M}_e(b_\bullet(2^2))$ is reducible.*

Proof. This follows directly from Theorem **??**. $\square$

Remark 6.3. The normal bundle of curves in $\mathrm{M}_e(b_\bullet(2^2))$ for $e = 7$, $n = 5$ is $\mathcal{O}(9)^2 \oplus \mathcal{O}(11)^2$, so the expected codimension is 4.

Thus, we see that as soon as we leave $\mathbb{P}^4$, we immediately start getting reducible strata.

6.3. **Alzati and Re's example.** Alzati and Re [**?**] exhibit two distinct irreducible components of the locus in $\mathrm{Mor}_{11}(\mathbb{P}^1, \mathbb{P}^8)$ where the normal bundle is $\mathcal{O}(13)^3 \oplus \mathcal{O}(14)^2 \oplus \mathcal{O}(15)^2$. Theorem **??** also implies the existence of these components. In their example, in one component the conic relations are general. In the other component, two of the conic relations satisfy the parameterized tangency condition. In fact, by increasing the degree ($e > 30$ certainly suffices), one can obtain examples with more than two components.

6.4. **An example without duplicate lowest factors.** As a final example of how widespread having multiple components appears to be, we work out examples of reducible $M_e(b_\bullet)$ where the two lowest b_i are distinct arbitrary integers.

Theorem 6.4. *Let $d_1 \leq d_2$ arbitrary, $e \geq (d_2+1)(n+1) + (d_1+1)(n+1)$ and $n \geq 5$. Let q and r be defined by $2e - 2 - d_1 - d_2 = q(n-3) + r$, and let $b_1 = d_1$, $b_2 = d_2$, $b_3 = \cdots = b_{n-r-1} = q$, $b_{n-r} = \cdots = b_{n-1} = q+1$. Then $\mathrm{M}_e(b_\bullet)$ is reducible.*

Proof. We show that there are two components of the incidence correspondence

$$\mathcal{A} = \{(a_1, a_2, f) | \sum_{i=0}^{n} a_{ji} \partial_\ell f_i = 0 \ j \in \{1,2\} \ \ell \in \{s,t\}\}.$$

First, note that the plane curve

$$(s^d, s^{d-2}t + st^{d-1}, t^d, 0, \cdots, 0)$$

has nondegenerate curves with at worst ordinary singularities in its fiber. Moreover, it imposes $2(d+e)$ conditions on the fiber, since we see that any $(f_0, \cdots, f_5)$ in the kernel will have to satisfy

$$\begin{bmatrix} ds^{d-1} & (d-2)s^{d-3}t + t^{d-1} & 0 \\ 0 & s^{d-2} + (d-1)st^{d-2} & dt^{d-1} \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \end{bmatrix} = 0.$$

Thus, we see that $s^{d-1}t^{d-1}$ has to divide f_1, but that otherwise f_1 can be chosen freely and will completely determine f_0 and f_2. So it imposes $3(e+1) - (e+1-2(d-1)) = 2(e+d)$ conditions. Note that this requires $e + 1 \geq 2d - 2$.

From this, we see that there will be a component of the incidence correspondence dominating the space of pairs (a_1, a_2) of dimension $(n+1)(d_1 + d_2 + e + 3) - 4(e+d)$. We find an example of a curve with these relations. Let q and r be defined by

$$2e - 2 - d_1 - d_2 = q(n-3) + r.$$

Consider the following sequence

$$1, \ d_1 - 1, \ x_1, \ d_1 - 1, \ d_2 - d_1 + 1, \ x_2, \ d_2 - d_1 + 1, \ x_3, \ d_2 - d_1 + 1, \ x_4, \ \ldots$$

where

$$x_1 = q - d_1 + 1, x_2 = \cdots, x_{\lfloor \frac{r}{2} \rfloor} = q - d_2 + d_1$$

and the rest of the x_i are $q - d_2 + d_1 - 1$. By Corollary **??** there will be a rational curve with the correct relations.

Now we show there is another component of dimension at least as large. To show this, we need only find one pair of relations of degree d_1 and d_2 such that the fiber contains nondegenerate curves with ordinary singulaties and the fiber's dimension jumps by at least e (from which the result follows, since $e \geq (d_1 + d_2 + 2)(n + 1)$). To find such an example, consider the relations

$$((d_1 - 1)t^{d_1}, -d_1 s t^{d_1 - 1}, s^{d_1}, 0, \cdots, 0)$$

and

$$(0, (d_2 - d_1 + 1)t^{d_2}, -d_2 s^{d_1 - 1} t^{d_2 - d_1 + 1}, (d_1 - 1)s^{d_2}, 0 \cdots, 0).$$

We see that the conditions imposed on the fiber over these relations are

$$\begin{bmatrix} 0 & -d_1 t^{d_1 - 1} & d_1 s^{d_1 - 1} & 0 \\ d_1(d_1 - 1)t^{d_1 - 1} & -d_1(d_1 - 1)st^{d_1 - 2} & 0 & 0 \\ 0 & 0 & -d_2(d_1 - 1)s^{d_1 - 2}t^{d_2 - d_1 + 1} & d_2(d_1 - 1)t^{d_2} \\ 0 & d_2(d_2 - d_1 + 1)t^{d_2 - 1} & -d_2(d_2 - d_1 + 1)s^{d_1 - 1}t^{d_2 - d_1} & 0 \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix} = 0.$$

The first and last rows are multiples of each other. In particular, the last row imposes no new conditions on the f_i, so the fiber dimension jumps by at least $e + d_2$.

It remains to find a single example of a nondegenerate curve having ordinary singularities and these relations. Let q and r be defined by $2e - 2 - d_1 - d_2 = q(n - 3) + r$. Consider the following sequence

$$1,\ d_1 - 1,\ d_2 - d_1 + 1,\ x_1,\ d_2 - d_1 + 1,\ x_2,\ d_2 - d_1 + 1,\ x_3,\ d_2 - d_1 + 1,\ \ldots$$

where $x_1 = \cdots x_{\lfloor \frac{r}{2} \rfloor} = q - d_2 + d_1$ and $x_{\lfloor \frac{r}{2} \rfloor + 1} = \cdots = q - d_2 + d_1 - 1$. Then by Corollary **??**, there exists a curve of the required form. ☐

References

[AR] A. Alzati and R. Re, Irreducible components of Hilbert Schemes of rational curves with given normal bundle, preprint.

[ART] A. Alzati, R. Re, and A. Tortora, An algorithm for determining the normal bundle of a rational monomial curve, preprint.

[EV81] D. Eisenbud and A. Van de Ven, On the normal bundles of smooth rational space curves, Math. Ann., **256** (1981), 453–463.

[EV82] D. Eisenbud and A. Van de Ven, On the variety of smooth rational space curves with given degree and normal bundle, Invent. Math., **67** (1982), 89–100.

[GS80] F. Ghione and G. Sacchiero, Normal bundles of rational curves in $\mathbb{P}^3$, Manuscripta Math., **33** (1980), 111–128.

[M86] J. M. Miret, On the variety of rational curves in $\mathbb{P}^n$, Ann. Univ. Ferrara – Sez. VII – Sc. Mat., **XXXII** (1986), 55–65.

[R90] L. Ramella, La stratification du schéma de Hilbert des courbes rationnelles de $\mathbb{P}^n$ par le fibré tangent restreint, C. R. Acad. Sci. Paris Sr. I Math., **311** no. 3 (1990), 181–184.

[R93] L. Ramella, Sur les schémas définissant les courbes rationnelles lisses de $\mathbb{P}^3$ ayant fibré normal et fibré tangent restreint fixés, Mém. Soc. Math. France (N.S.), **54** (1993), ii+74.

[Ra07] Z. Ran, Normal bundles of rational curves in projective spaces, Asian J. Math., **11** no. 4 (2007), 567–608.

[Sa80] G. Sacchiero, Fibrati normali di curvi razionali dello spazio proiettivo, Ann. Univ. Ferrara Sez. VII, **26** (1980), 33–40.

[Sa82] G. Sacchiero, On the varieties parameterizing rational space curves with fixed normal bundle, Manuscripta Math., **37** (1982), 217–228.

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