

Applications of a grassmannian technique in hypersurfaces

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1 Introduction

In this paper, we further develop a technique from [RY] and apply it to study the Kobayashi Conjecture, 0-cycles on hypersurfaces of general type, and Seshadri constants of very general hypersurfaces. The idea of the technique is to translate results about very general points on very general hypersurfaces to results about arbitrary points on very general hypersurfaces.

Our first application is to hyperbolicity. Recall that a complex variety is Brody hyperbolic if it admits no holomorphic maps from \mathbb{C} .

Conjecture 1.1 (Kobayashi Conjecture). *A very general hypersurface X of degree d in \mathbb{P}^n is Brody hyperbolic if d is sufficiently large. Moreover, the complement $\mathbb{P}^n \setminus X$ is also Brody hyperbolic for large enough d .*

First conjectured in 1970 [K], the Kobayashi Conjecture has been the subject of intense study, especially in recent years [S, De, B, D2]. The suspected optimal bound for d is approximately $d \geq 2n - 1$. However, the best current bound is for d greater than about $(en)^{2n+2}$ [D2]. A related conjecture is the Green-Griffiths-Lang Conjecture.

Conjecture 1.2 (Green-Griffiths-Lang Conjecture). *If X is a variety of general type, then there is a proper subvariety $Y \subset X$ containing all the entire curves of X .*

The Green-Griffiths-Lang Conjecture says that holomorphic images of \mathbb{C} under nonconstant maps do not pass through a general point of X . Conjecture 1.2 is well-studied for general hypersurfaces, as it is a natural result to prove on the way to proving Conjecture 1.1. We provide a new proof of the Kobayashi Conjecture using previous results on the Green-Griffiths-Lang Conjecture.

Theorem 1.3. *A general hypersurface in \mathbb{P}^n of degree d admits no nonconstant holomorphic maps from \mathbb{C} for $d \geq d_{2n-3}$, where $d_2 = 286$, $d_3 = 7316$ and*

$$d_n = \left\lceil \frac{n^4}{3} (n \log(n \log(24n)))^n \right\rceil.$$

Our proof appears to be substantially simpler than the previous proofs (compare with [S, De, B, D2]), and can be adapted in a straightforward way as others use jet bundles to obtain better bounds for Conjecture 1.2. Unfortunately, the bound of about $(2n \log(n \log(n)))^{2n+1}$ that we obtain is slightly worse than Demailly's bound of $(en)^{2n}$. However, assuming the optimal result on the Green-Griffiths-Lang Conjecture, our technique allows us to prove the conjectured bound of $d \geq 2n - 1$ for the Kobayashi Conjecture.

Our second application concerns the Chow equivalence of points on very general complete intersections. Chen, Lewis, and Sheng [CLS] make the following conjecture, which is inspired by work of Voisin [V1, V2, V3].

Conjecture 1.4. *Let $X \subset \mathbb{P}^n$ be a very general complete intersection of multidegree (d_1, \dots, d_k) . Then for every $p \in X$, the space of points of X rationally equivalent to p has dimension at most $2n - k - \sum_{i=1}^k d_i$. If $2n - k - \sum_{i=1}^k d_i < 0$, we understand this to mean that p is equivalent to no other points of X .*

If this Conjecture holds, then the result is sharp [CLS]. Voisin [V1, V2, V3] proves Conjecture 1.4 for hypersurfaces in the case $2n - d - 1 < -1$. Chen, Lewis, and Sheng [CLS] extend the result to $2n - d - 1 = -1$, and also prove the analog of Voisin's bound for complete intersections. Both papers use fairly involved Hodge theory arguments. Roitman [R1, R2] proves the $2n - k - \sum_{i=1}^k d_i = n - 2$ case. Using Roitman's result, we prove all but the $2n - k - \sum_{i=1}^k d_i = -1$ case of Conjecture 1.4, and in this case we prove the result holds with the exception of possibly countably many points.

Theorem 1.5. *If $X \subset \mathbb{P}^n$ is a very general complete intersection of multidegree (d_1, \dots, d_k) , then no two points of X are rationally Chow equivalent if $2n - k - \sum_{i=1}^k d_i < -1$. If $2n - k - \sum_{i=1}^k d_i = -1$, then the set of points rationally equivalent to another point of X is a countable union of points. If $2n - k - \sum_{i=1}^k d_i \geq 0$, then the space of points of X rationally equivalent to a fixed point $p \in X$ has dimension at most $2n - k - \sum_{i=1}^k d_i$ in X .*

Our method appears substantially simpler than the previous work of Voisin [V1, V2, V3] and Chen, Lewis, and Sheng [CLS], although in the case of hypersurfaces, we do not recover the full strength of Chen, Lewis, and Sheng's result.

The third result relates to Seshadri constants. Let $\epsilon(p, X)$ be the Seshadri constant of X at the point p , defined to be the infimum of $\frac{\deg C}{\text{mult}_p C}$ over all curves C in X passing through p . Let $\epsilon(X)$ be the Seshadri constant of X , defined to be the infimum of the $\epsilon(p, X)$ as p varies over the hypersurface.

Theorem 1.6. *Let $r > 0$ be a real number. If for a very general hypersurface $X_0 \subset \mathbb{P}^{2n-1}$ of degree d the Seshadri constant $\epsilon(p, X_0)$ of X_0 at a general point p is at least r , then for a very general $X \subset \mathbb{P}^n$ of degree d , the Seshadri constant $\epsilon(X)$ of X is at least r .*

The layout of the paper is as follows. In Section 2, we lay out our general technique, and immediately use it to prove Theorem 1.6. In Section 3, we discuss

how to use the results of Section 2 to prove hyperbolicity results. In Section 4, we discuss how to prove Theorem 1.5.

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2 The Technique

We set some notation. Let B be the moduli space of complete intersections of multidegree (d_1, \dots, d_k) in \mathbb{P}^{n+k} and $\mathcal{U}_{n,\underline{d}} \subset \mathbb{P}^{n+k} \times B$ be the variety of pairs $([p], [X])$ with $[X] \in B$ and $p \in X$. We refer to elements of $\mathcal{U}_{n,\underline{d}}$ as pointed complete intersections. When we talk about the codimension of a countable union of subvarieties of $\mathcal{U}_{n,\underline{d}}$, we mean the minimum of the codimensions of each component.

We need the following result from [RY].

Proposition 2.1. *Let $C \subset \mathbb{G}(r-1, m)$ be a nonempty family of $r-1$ -planes of codimension $\epsilon > 0$, and let $B \subset \mathbb{G}(r, m)$ be the space of r planes that contain some $r-1$ -plane c with $[c] \in C$. Then $\text{codim}(B \subset \mathbb{G}(r, m)) \leq \epsilon - 1$.*

Proof. For the reader's convenience, we sketch the proof. Consider the incidence-correspondence $I = \{([b], [c]) \mid [c] \in C, [b] \in B\} \subset \mathbb{G}(r-1, m) \times \mathbb{G}(r, m)$. The fibers of π_1 are all \mathbb{P}^{m-r} 's, while for a general $[b] \in B$, the fiber $\pi_2^{-1}([b])$ has codimension at least 1 in the \mathbb{P}^r of $r-1$ -planes contained in b (since otherwise it can be shown that $C = \mathbb{G}(r, m)$). The result follows by a dimension count. \square

We need a few other notions for the proof. A parameterized r -plane in \mathbb{P}^m is a degree one map $\Lambda : \mathbb{P}^r \rightarrow \mathbb{P}^m$. Let $G_{r,m,p}$ be the space of parameterized r -planes in \mathbb{P}^m that pass through p . If (p, X) is a pointed hypersurface in \mathbb{P}^m , a pointed, parameterized r -plane section of (p, X) is a pair $(\Lambda^{-1}(p), \Lambda^{-1}(X)) =: \phi^*(p, X)$, where $\Lambda : \mathbb{P}^r \rightarrow \mathbb{P}^m$ is a parameterized r -plane whose image does not lie entirely in X . We say that $\Lambda : \mathbb{P}^r \rightarrow \mathbb{P}^m$ contains $\Lambda' : \mathbb{P}^{r-1} \rightarrow \mathbb{P}^m$ if $\Lambda(\mathbb{P}^r)$ contains $\Lambda'(\mathbb{P}^{r-1})$.

Corollary 2.2. *If $C \subset G_{r-1,m,p}$ is a nonempty subvariety of codimension $\epsilon > 0$ and $B \subset G_{r,m,p}$ is the subvariety of parameterized r -planes that contain some $r-1$ -plane $[c] \in C$, then $\text{codim}(B \subset G_{r,m,p}) \leq \epsilon - 1$.*

Let $\mathcal{X}_{n,\underline{d}} \subset \mathcal{U}_{n,\underline{d}}$ be an open subset. For instance, $\mathcal{X}_{n,\underline{d}}$ might be equal to $\mathcal{U}_{n,\underline{d}}$ or it might be the universal complete intersection over the space of smooth complete intersections. Our main technical tool is the following.

Theorem 2.3. *Suppose we have an integer m and for each $n \leq n_0$ we have $Z_{n,\underline{d}} \subset \mathcal{X}_{n,\underline{d}}$ a countable union of locally closed varieties satisfying:*

1. If $(p, X) \in Z_{n,\underline{d}}$ and is a parameterized hyperplane section of (p', X') , then $(p', X') \in Z_{n+1,\underline{d}}$.
2. $Z_{m-1,\underline{d}}$ has codimension at least 1 in $\mathcal{X}_{m-1,d}$.

Then the codimension of $Z_{m-c,\underline{d}}$ in $\mathcal{X}_{m-c,\underline{d}}$ is at least c .

Proof of Theorem 2.3. We adopt the method from [RY]. In order to prove our result, we prove that for a very general point (p_0, X_0) of a component of $Z_{m-c,\underline{d}}$, there is a variety \mathcal{F}_{m-c} and a map $\phi : \mathcal{F}_{m-c} \rightarrow \mathcal{U}_{m-c,\underline{d}}$ with $(p_0, X_0) \in \phi(\mathcal{F}_{m-c})$ and $\text{codim}(\phi^{-1}(Z_{m-c,\underline{d}}) \subset \mathcal{F}_{m-c}) \geq c$. This suffices to prove the result.

So, let (p_0, X_0) be a general point of a component of $Z_{m-c,\underline{d}}$, and let $(p_1, X_1) \in \mathcal{X}_{m-1,\underline{d}}$ be very general, so that (p_1, X_1) is not in the closure of any component of $Z_{m-1,\underline{d}}$ by hypothesis 2. Choose $(p, Y) \in \mathcal{X}_{N,\underline{d}}$ for some sufficiently large N where (p_0, X_0) and (p_1, X_1) are pointed, parameterized linear sections of (p, Y) . Then for all $n < N$, let \mathcal{F}_n be the space of all parameterized n -planes in \mathbb{P}^N passing through p such that for $\Lambda \in \mathcal{F}_n$, $\Lambda^*(p, Y)$ is in $\mathcal{X}_{n,\underline{d}}$. This means that \mathcal{F}_n is an open subset of $G_{n,N,p}$. Let $\phi : \mathcal{F}_n \rightarrow \mathcal{X}_{n,\underline{d}}$ be the map sending $\Lambda : \mathbb{P}^n \rightarrow \mathbb{P}^N$ to $\Lambda^*(p, Y)$.

We prove that $\text{codim}(\phi^{-1}(Z_{m-c,\underline{d}}) \subset \mathcal{F}_{m-c}) \geq c$ by induction on c . For the $c = 1$ case, we see by construction that $\phi^{-1}(Z_{m-1,\underline{d}})$ has codimension at least 1 in \mathcal{F}_{m-1} , since (p_1, X_1) is a parameterized $m-1$ -plane section of (p, Y) but is not in the closure of any component of $Z_{m-1,\underline{d}}$. Now suppose we know that $\text{codim}(\phi^{-1}(Z_{m-c,\underline{d}}) \subset \mathcal{F}_{m-c}) \geq c$. We use Corollary 2.2 with C equal to $\phi^{-1}(Z_{m-c-1,\underline{d}})$. By hypothesis 1, we see that B is contained in $\phi^{-1}(Z_{m-c,\underline{d}})$. It follows from this that

$$\begin{aligned} c \leq \text{codim}(\phi^{-1}(Z_{m-c,\underline{d}}) \subset \mathcal{F}_{m-c}) &\leq \text{codim}(B \subset \mathcal{F}_{m-c}) \\ &\leq \text{codim}(C \subset \mathcal{F}_{m-c-1}) - 1. \end{aligned}$$

Rearranging, we see that

$$\text{codim}(\phi^{-1}(Z_{m-c-1,\underline{d}}) \subset \mathcal{F}_{m-c-1}) = \text{codim}(C \subset \mathcal{F}_{m-c-1}) \geq c + 1.$$

The result follows. \square

As an immediate application, we prove Theorem 1.6.

Proof of Theorem 1.6. Let r be given. Let $Z_{m,d} \subset \mathcal{U}_{m,d}$ be the set of pairs (p, X) where $\epsilon(p, X) < r$. We apply Theorem 2.3 to $Z_{m,d}$. We see that $Z_{m,d}$ is a countable union of algebraic varieties, and by hypothesis, $Z_{2n-1,d} \subset \mathcal{U}_{2n-1,d}$ has codimension at least 1. Now suppose that $(p_0, X_0) \in Z_{m,d}$. Then there is some curve C in X_0 with $\frac{\deg C}{\text{mult}_{p_0} C} < r$. Thus, for any X containing p , we see that the Seshadri constant of X at p is at most $\frac{\deg C}{\text{mult}_{p_0} C} < r$. This shows that the $Z_{m,d}$ satisfy the conditions of Theorem 2.3, which shows that $Z_{n,d} \subset \mathcal{U}_{n,d}$ has codimension at least n . By dimension reasons, this means that $\mathcal{U}_{n,d}$ cannot dominate the space of hypersurfaces, so the result follows. \square

3 Hyperbolicity

Let $\mathcal{X}_{n,d}$ be the universal hypersurface in \mathbb{P}^n over the open subset U in the moduli space of all degree d hypersurfaces in \mathbb{P}^n consisting of all smooth hypersurfaces. Many people have developed a technique for bounding the entire curves contained in a fiber of the map $\pi_2 : \mathcal{X}_{n,d} \rightarrow U$. See the article of Demailly for a detailed description of many of the techniques [D2]. For a variety X , let $\text{ev} : J_k(X) \rightarrow X$ be the space of k -jets of X . Then, if $X \subset \mathbb{P}^n$ is a smooth degree d hypersurface, there is a vector bundle $E_{k,m}^{GG}T_X^*$ whose sections act on $J_k(X)$. Global sections of $E_{k,m}^{GG}T_X^* \otimes \mathcal{O}(-H)$ vanish on the k -jets of entire curves. This means that sections of $E_{k,m}^{GG}T_X^* \otimes \mathcal{O}(-H)$ cut out a closed subvariety $S'_{k,m}(X) \subset J_k(X)$ such that any entire curve is contained in $\text{ev}(S'_{k,m}(X))$. In fact, it can be shown that any entire curve is contained in the closure of $\text{ev}(S_{k,m}(X))$, where $S_{k,m}(X) \subset J_k(X)$ is $S'_{k,m}(X)$ minus the space of singular k -jets.

The construction is functorial. In particular, if V is the relative tangent bundle of the map π_2 , they construct a vector bundle $E_{k,m}^{GG}V^*$ whose restriction to each fiber of π_2 is $E_{k,m}^{GG}T_X^*$. Let $\mathcal{Y}_{n,d} \subset \mathcal{X}_{n,d}$ be the locus of $(p, X) \in \mathcal{X}_{n,d}$ such that $p \in \text{ev}(S_{k,m}(X))$. Then by functoriality, $\mathcal{Y}_{n,d}$ is a finite union of locally closed varieties.

Theorem 3.1. *Suppose that $\mathcal{Y}_{r-1,d} \subset \mathcal{X}_{r-1,d}$ is codimension at least 1. Then $\mathcal{Y}_{r-c,d} \subset \mathcal{X}_{r-c,d}$ is codimension at least c . In particular, if $\mathcal{Y}_{2n-3,d}$ is codimension at least 1 in $\mathcal{X}_{2n-3,d}$ and $d \geq 2n - 1$, then a very general $X \subset \mathbb{P}^n$ of degree d_n is hyperbolic.*

Proof. We check that $\mathcal{Y}_{r-1,d}$ satisfies both conditions of Theorem 2.3. Condition 2 is a hypothesis. Condition 1 follows by the functoriality of Demailly's construction. Namely, if (p, X_0) is a parameterized linear section of (p, X) , then the natural map $X_0 \rightarrow X$ induces a pullback map on sections

$$H^0(E_{k,m}^{GG}T_X^* \otimes \mathcal{O}(-H)) \rightarrow H^0(E_{k,m}^{GG}T_{X_0}^* \otimes \mathcal{O}(-H)),$$

compatible with the natural inclusion of jets $J_k(X_0) \rightarrow J_k(X)$. In particular, if some section of $H^0(E_{k,m}^{GG}T_X^* \otimes \mathcal{O}(-H))$ takes a nonzero value on a jet in the image $J_k(X_0) \rightarrow J_k(X)$, then its restriction to X_0 takes a nonzero value on the original jet in $J_k(X_0)$. Thus, if X_0 has a nonsingular k -jet at p which is annihilated by every section in $H^0(E_{k,m}^{GG}T_{X_0}^* \otimes \mathcal{O}(-H))$, X has such a k -jet as well.

To see the second statement, observe that by Theorem 2.3, $\mathcal{Y}_{n,d}$ has codimension in $\mathcal{X}_{n,d}$ at least $2n - 3 - n + 1 = n - 2$. It follows that a general X of degree d in \mathbb{P}^n satisfies that the image of any entire curve is contained in an algebraic curve. Since $d \geq 2n - 1$, by a theorem of Voisin [V2, V3], any algebraic curve in X is of general type. The result follows. \square

The current best bound for the Green-Griffiths-Lang Conjecture is from Demailly [D1, D2]. The version we use comes out of Demailly's proof, instead of the statement of any particular theorem.

Theorem 3.2 ([D2], Section 10). *We have that $\mathcal{Y}_{n,d} \subset \mathcal{X}_{n,d}$ is codimension at least 1 for $d \geq d_n$, where $d_2 = 286, d_3 = 7316$ and*

$$d_n = \left\lceil \frac{n^4}{3} (n \log(n \log(24n)))^n \right\rceil.$$

Using this bound, we obtain the following.

Corollary 3.3. *The Kobayashi Conjecture holds for $d \geq d_{2n-3}$, where $d_2 = 286, d_3 = 7316$ and*

$$d_n = \left\lceil \frac{n^4}{3} (n \log(n \log(24n)))^n \right\rceil.$$

This bound of about $(2n \log(n \log n))^{2n+1}$ is slightly worse than the best current bound for the Kobayashi Conjecture from [D2], which is about $(en)^{2n+2}$. However, our technique is strong enough to allow us to prove the optimal bound from the Kobayashi Conjecture, provided one could prove the optimal result for the Green-Griffiths-Lang Conjecture.

Corollary 3.4. *If $\mathcal{Y}_{d-2,d}$ has codimension at least 1 in $\mathcal{X}_{d-2,d}$ (as we would expect from the Green-Griffiths-Lang Conjecture), then a very general hypersurface of degree $d \geq 2n - 1$ in \mathbb{P}^n is hyperbolic.*

Proof. We apply Theorem 3.1. We know that if $\mathcal{Y}_{2n-3,d}$ is codimension at least 1 in $\mathcal{X}_{2n-3,d}$, then the Kobayashi Conjecture holds for hypersurfaces in \mathbb{P}^n of degree d . We apply this result with $d = 2n - 1$. \square

4 0-cycles

Let $R_{\mathbb{P}^1, X, p} = \{q \in X \mid Nq \sim Np \text{ for some integer } N\}$, where the relation \sim means Chow equivalent. The goal of this section is to prove all but the $2n - \sum_{i=1}^k (d_i - 1) = -1$ case of the following conjecture of Chen, Lewis and Sheng [CLS].

Conjecture 4.1. *Let $X \subset \mathbb{P}^n$ be a very general complete intersection of multi-degree (d_1, \dots, d_k) . Then for every $p \in X$, $\dim R_{\mathbb{P}^1, X, p} \leq 2n - k - \sum_{i=1}^k d_i$.*

Here, we adopt the convention that $\dim R_{\mathbb{P}^1, X, p}$ is negative if $R_{\mathbb{P}^1, X, p} = \{p\}$. Chen, Lewis and Sheng consider the more general notion of Γ equivalence, although we are unable to prove the Γ equivalence version here. The special case $\sum_i d_i = n + 1$ is a theorem of Roitman [R1, R2] and the case $2n - k - \sum_{i=1}^k d_i \leq -2$ is a theorem of Chen, Lewis and Sheng [CLS] building on work of Voisin [V1, V2, V3], who proves the result only for hypersurfaces. Chen, Lewis and Sheng prove Conjecture 4.1 for hypersurfaces and for arbitrary Γ in the case $2n - k - \sum_{i=1}^k d_i = -1$ in [CLS]. The case $2n - k - \sum_{i=1}^k d_i = -1$ appears to be the most difficult, and is the only one we cannot completely address with our technique.

We provide an independent proof of all but the $2n - k - \sum_{i=1}^k d_i = -1$ case of Conjecture 4.1. Aside from Roitman's result, this is the first result we are aware of addressing the case $2n - k - \sum_{i=1}^k d_i \geq 0$. We rely on the result of Roitman in our proof, but not the results of Voisin [V1, V2] or Chen, Lewis, and Sheng [CLS].

Let $E_{n,\underline{d}} \subset \mathcal{U}_{n,\underline{d}}$ be the set of (p, X) such that $R_{\mathbb{P}^1, X, p}$ has dimension at least 1. Let $G_{n,\underline{d}} \subset \mathcal{U}_{n,\underline{d}}$ be the set of (p, X) such that $R_{\mathbb{P}^1, X, p}$ is not equal to $\{p\}$. Both $E_{n,\underline{d}}$ and $G_{n,\underline{d}}$ are countable unions of closed subvarieties of $\mathcal{U}_{n,\underline{d}}$. When we talk about the codimension of $E_{n,\underline{d}}$ or $G_{n,\underline{d}}$ in $\mathcal{U}_{n,\underline{d}}$, we mean the minimum of the codimensions of each component. We prove Conjecture 4.1 by proving the following theorem.

Theorem 4.2. *The codimension of $E_{n,\underline{d}}$ in $\mathcal{U}_{n,\underline{d}}$ is at least $-n + \sum_i d_i$ and the codimension of $G_{n,\underline{d}}$ in $\mathcal{U}_{n,\underline{d}}$ is at least $-n - 1 + \sum_i d_i$.*

Corollary 4.3. *Conjecture 4.1 holds for $2n - k - \sum_{i=1}^k d_i \neq -1$. In the special case $2n - k - \sum_{i=1}^k d_i = -1$, the space of $p \in X$ Chow-equivalent to some other point of X has dimension 0 (i.e., is a countable union of points) but might not be empty as Conjecture 4.1 predicts.*

Proof. First we consider the case $2n - k - \sum_i d_i \geq 0$. Let $\pi_1 : \mathcal{U}_{n,\underline{d}} \rightarrow B$ be the projection map. If $\pi_1|_{E_{n,\underline{d}}}$ is not dominant, then the result holds trivially. Thus, we may assume that the very general fiber of $\pi_1|_{E_{n,\underline{d}}}$ has dimension $n - k - \text{codim}(E_{n,\underline{d}} \subset \mathcal{U}_{n,\underline{d}})$. If the bound on $E_{n,\underline{d}}$ from Theorem 4.2 holds, then the space of points p of X with positive-dimensional $R_{\mathbb{P}^1, X, p}$ has dimension at most $2n - k - \sum_i d_i$, which implies Conjecture 4.1 in the case $2n - k - \sum_i d_i \geq 0$.

Now we consider the situation for $2n - k - \sum_i d_i \leq -1$. Conjecture 4.1 states that $\pi_1|_{G_{n,\underline{d}}}$ is not dominant for this range. By Theorem 4.2, we see that the dimension of $G_{n,\underline{d}}$ is less than the dimension of B if $2n - k - \sum_i d_i \leq -2$, proving Conjecture 4.1. In the case $2n - k - \sum_i d_i = -1$, the dimension of $G_{n,\underline{d}}$ is at most that of B , which shows that there are at most finitely many points of X equivalent to another point of X . This proves the result. \square

Our technique would prove Conjecture 4.1 in all cases if we knew that $G_{n,\underline{d}}$ had codimension $-n + \sum_{i=1}^k d_i$ in $\mathcal{U}_{n,\underline{d}}$. However, this is not true for Calabi-Yau hypersurfaces.

Proposition 4.4. *A general point of a very general Calabi-Yau hypersurface X is rationally equivalent to at least one other point of the hypersurface.*

Proof. Let X be a very general Calabi-Yau hypersurface. Then we claim that a general point of X is Chow equivalent to another point of X . To see this, observe that any point p of X has finitely many lines meeting X to order $d-1$ at p . Such a line meets X in a single other point. Moreover, every point of X has a line passing through it that meets X at another point of X with multiplicity $d-1$. Thus, let q_1 be a general point of X , let ℓ_1 be a line through p meeting X at a second point, p to order $d-1$, and ℓ_2 be a different line meeting X at p

to order $d - 1$. Let q_2 be the residual intersections of ℓ_2 with X . Then $q_1 \sim q_2$, and since q_1 can be taken to be general, this proves the result. \square

Proof of Theorem 4.2. First consider the bound on $E_{n,d}$. By Roitman's Theorem plus the fact that $R_{\mathbb{P}^1, X, p}$ is a countable union of closed varieties, we see that $E_{-1+\sum_{i=1}^k d_i, d}$ has codimension at least one in $\mathcal{U}_{-1+\sum_{i=1}^k d_i, d}$. We note that if $p \sim q$ as points of Y , and $Y \subset Y'$, then $p \sim q$ as points of Y' as well. The rest of the result follows from Theorem 2.3 using $Z_{n,d} = E_{n,d}$.

Now consider $G_{n,d}$. From Roitman's theorem, it follows that a very general point of a Calabi-Yau complete intersection X is equivalent to at most countably many other points of X . Thus, a very general hyperplane section of such an X satisfies the property that the very general point is equivalent to no other points of X . From this, we see that $G_{-2+\sum_{i=1}^k d_i, d}$ has codimension at least 1 in $\mathcal{U}_{-2+\sum_{i=1}^k d_i, d}$. Together with Theorem 2.3, this implies the result. \square

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