① **Rationality:**

/ $\mathbb{C}$

$\times$ A variety is **Rational** if $X \simeq_k \mathbb{P}^n_k$;

$\times$ **Stably Rational** if $X \times \mathbb{P}^m \simeq_k \mathbb{P}^n_k$

? **Unirational** if $\mathbb{P}^n \overset{k}{\dashrightarrow} X$

Rationally Connected if $x, y \in X \implies$

$\exists\ f : \mathbb{P}^1 \longrightarrow X$

dim ≥ 3 $\quad f(0) = x \quad f(\infty) = y$

(dim $X = 1, 2$ all same)

Clemens - Griffiths: X 3fold w/ $H^{1,0} = H^{3,0} = 0$

then X rational $\Rightarrow IJ(X) \cong \mathrm{Jac}(C_1) \times \dots \times \mathrm{Jac}(C_m)$ for some $C_1, \dots, C_m$

Manin - Iskovskih: $\mathrm{BirAut}(X) = \{1\} \Rightarrow X$ cannot be rational

"birational rigidity"

Artin - Mumford: Topological obstructions to rationality

X rationally connected 3fold / $\mathbb{C}$

$\pi_1(X) = 0 \Rightarrow H_1(X) = 0$

$H^0(X, \mathbb{Z}) = \mathbb{Z} \quad H^1(X, \mathbb{Z}) = 0 \quad H^2(X) \cong \mathbb{Z}^b$

$H^3(X) = \mathbb{Z}^c \oplus T \quad H^4(X) = \mathbb{Z}^b \oplus T \quad H^5(X) = 0$

$H^6(X) = \mathbb{Z}$

(2)

Question: If X is (rc, unirat, stab, rat) what are the constraints on b, c, T?

<u>Prop</u>: The torsion gp $T \subseteq H^3(X,\mathbb{Z})$ is a birational invariant of a nonsing $\mathbb{C}$-proj var.

(N.B. Can interpret as Brauer group ...)

$\Downarrow$

Since for $\mathbb{P}^3$ have $T=0$, for any rational var must have $T=0$. In fact, for stably rational var $T=0$ (Künneth).

There exists a unirational variety with $T \neq 0$.

$$X' \xrightarrow{f} X \quad \text{birational morphism}$$

$$\begin{tikzcd} H^q(X',\mathbb{Z}) \arrow[dr] & H^q(X,\mathbb{Z}) \arrow[l, "f^*"'] \arrow[d, "id"] \\ & H^q(X,\mathbb{Z}) \end{tikzcd}$$

$f_* f^* 1 = 1$

$f_*(x \cdot f^*(y)) = f_*(x) \cdot y$

$$H^q(X,\mathbb{Z}) \hookrightarrow H^q(X',\mathbb{Z})$$

$$T_X \subseteq T_{X'}$$

$\therefore T_X = T_{X'}$

$$H^0(Y) \xrightarrow{0} H^3(X) \hookrightarrow H^3(X')$$

But if $X' = Bl_Y X$,

$$\to H^1(Y) \xrightarrow{0} H^4(X) \hookrightarrow H^4(X$$

3

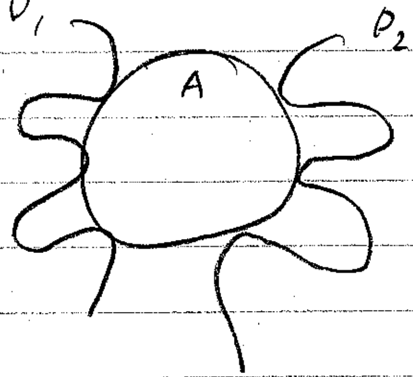
$$\begin{tikzcd} & X_n \arrow[dl, dotted] \arrow[dr] & \\ X \arrow[rr, dashed] & & X' \end{tikzcd}$$

$T_X = T_{X_n} \subseteq T_{X'}$

$\therefore \quad T_X = T_{X'}$ ✓

Artin–Mumford Example: A conic $\alpha = 0$ in $\mathbb{P}^2$

Two cubics D_1, D_2 $\quad \delta_1, \delta_2 = 0$



$A.D_i = 2(p_1^i + p_2^i + p_3^i)$

$D_1 \pitchfork D_2$

Take cubic $\beta = 0$ passing through p_j^i's and another pt on A

$\delta_1 \delta_2 = \beta^2 - 4\alpha\psi$, ψ some quartic

K3 surface

$$S = \{ \alpha(X_0, X_1, X_2) X_3^2 + \beta(X_0, X_1, X_2) X_3 + \psi(X_0, X_1, X_2) = 0 \}$$

Singular at $p = (0,0,0,1)$. Projection from p gives $S \xrightarrow{2:1} \mathbb{P}^2$ branched along $D_1 \cup D_2$

(4) K3 surface has precisely 10 A_1-sings

p_1 and $D_1 \cap D_2 \ni q_i = (x,y,z)$

$$(x,y,z) = \frac{-\beta(x,y,z)}{2\alpha(x,y,z)}$$

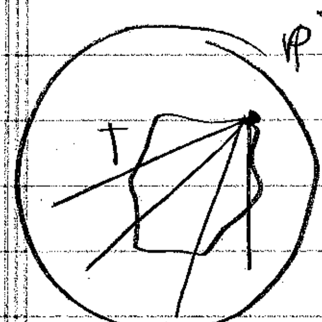
$$\left(\det\begin{bmatrix} l_{11} & l_{12} & \\ \vdots & \ddots & \\ & & l_{44}\end{bmatrix}\right)$$

gen'l linear forms

$X \xrightarrow{2:1} \mathbb{P}^3$ branched along S and blowup the 10 points

<u>Thm</u> (Artin–Mumford) X has 2-torsion in $H^3(X,\mathbb{Z})$, so X unirational, but not stably rational.

<u>Rmk:</u> $Y \xrightarrow{2:1} \mathbb{P}^3$ quartic double solid branched along T w/ T having a node



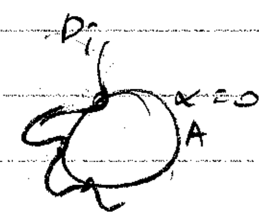
Take lines thru double point.

Conic bundle structure on Y w/ a bisection.

$$X \longrightarrow X' \xrightarrow{\pi} \mathbb{P}^2$$

double cover of $\mathbb{P}^3$ branched along S + blowup sing pts

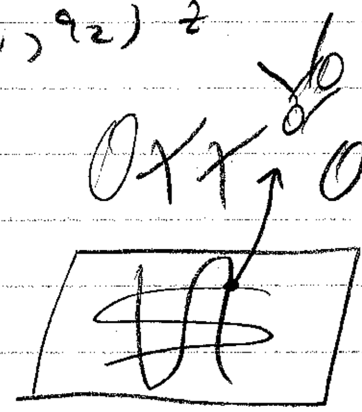
only blow up $(0,0,0,1)$



$$\delta_1\delta_2 = \beta^2 - 4\alpha\gamma$$

(8)

$$X_4^2 = \alpha(a_0, a_1, a_2) X_3^3 + \beta(a_0, a_1, a_2) X_3 Z + \gamma(a_0, a_1, a_2) Z^2$$



conic is reducible

$\Longleftrightarrow$

$\alpha \in D_1 \cup D_2$

and never non-reduced

If $a \notin A = \{\alpha = 0\}$

$$X_4 = X_3 \sqrt{\alpha(a_0, a_1, a_2)} \qquad Z = 0$$

$$X_4 = -X_3 \sqrt{\alpha(a_0, a_1, a_2)} \qquad Z = 0$$

$D_1' \xrightarrow[\text{2:1 cover}]{\text{unram}} D_1$ can distinguish the pairs of lines

$D_2' \longrightarrow D_2$

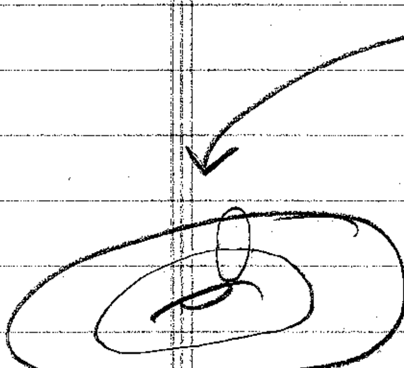
Make a 3-cycle U such that

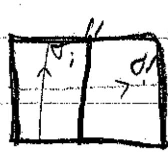
$2U \sim 0$,

and another V w/

$U \cdot V$ is 1

transverse point.





getting around one way σ_i''
preserves the lines, and
the other swaps them σ_i'

$m_a' \Longleftrightarrow m_a''$

$T = m_{a_1}' - m_{a_2}'$ $\quad a_1 \in D_1 \quad a_2 \in D_2$

(6)

If you move around σ_i', $m_{q_i}' \sim m_{q_i}''$

$$2m_{q_i}' \sim m_{q_i}' + m_{q_i}'' = m_{q_i}$$

$$2T \sim 2m_{q_1}' - 2m_{q_2}' \sim m_{q_1} - m_{q_2} \sim 0$$

$$U = \bigcup_{a \in \sigma_1''} m_a'' \qquad V = \bigcup_{a \in \sigma_1'} m_a'$$

$$U \cdot V = 1$$

$2U \sim 0$:

$$U^* = \bigcup_{a \in \sigma_1''} m_a'$$

$$2U \sim U + U^* = \bigcup_{a \in \sigma_1''} m_a \sim 0$$

$$U^* \sim U$$

✓

Unirational not stably rational quartic double solid

Voisin's Idea: X admits an integral cohomological decomp of the diagonal

$$\left[\begin{array}{l} \mathbb{P}^n: \\ \mathbb{P}^n \underset{\Delta}{\subset} \mathbb{P}^n_1 \times \mathbb{P}^n_2 \qquad Z_1 + Z_2 \; (\mathbb{P}^n \times p) \cup (D \times \mathbb{P}^n) \\ [\Delta] = \sum_{i=0}^{n} (h_1^i \otimes h_2^{n-i}) \end{array}\right]$$

⑦

$\Delta_X = Z_1 + Z_2$ $\quad A^*(X \times X)$

$\qquad\qquad\qquad\quad H^*(X \times X)$

st Z_1 is

$X \times p$ and Z_2 is $D \times X$

$D \subsetneq X$

[In particular, $\mathbb{P}^n$ admits both]

Bloch–Srinivas: Y $\quad CH_0(Y)$ supported on W

There is an equality $N\Delta_Y = Z_1 + Z_2$

$\operatorname{Supp} Z_1 \subseteq D \times Y$

$\operatorname{Supp} Z_2 \subseteq Y \times W$

If Y rat connected, $N\Delta_Y = Z_1 + Z_2$

analyze how small you can make this

Thm (Voisin)

Y smooth projective 3fold. If $N\Delta_Y = Z + Z'$ w/

$Z \subseteq Y \times \underset{\text{curve}}{W}$ $\quad Z' \subseteq D \times Y$

Then N annihilates the torsion in $H^{p}(Y, \mathbb{Z})$. (3)

In particular, if Y has an integral decomp of the diagonal, then $H^p(Y, \mathbb{Z})$ is torsion free.

For 3-folds

$$\begin{array}{c} \text{3 cycle} \\ W \subseteq Y \times Y \end{array}$$

$$\begin{tikzcd} & W \arrow[dl, "f"'] \arrow[dr, "g"] & \\ Y & & Y \end{tikzcd}$$

$g_*((f^*\alpha) \cup W)$

$W^*: H^*(Y) \longrightarrow H^*(Y)$

$(N\Delta_Y)^* = N\,\mathrm{Id} : H^p(Y,\mathbb{Z}) \longrightarrow H^p(Y,\mathbb{Z})$

$\quad \shortparallel$

$Z^* + Z'^*$ <u>Pretend D smooth</u> (or resolve $\tilde{D}$)

$$H^p(Y,\mathbb{Z}) \longrightarrow H^{p-2}(\tilde{D},\mathbb{Z})$$

factors thru

$$H^3(Y,\mathbb{Z}) \longrightarrow H^1(\tilde{D},\mathbb{Z}) \longleftarrow \text{never no torsion}$$

$\Longrightarrow$ mult kills torsion.

Cor: If Y has torsion in $H^3(Y,\mathbb{Z})$, then Y cannot have an integral decomp of the diagonal.

<u>Thm</u> $\begin{array}{c} X \\ \downarrow \pi \\ B \end{array}$ flat, rel dim $n \geq 2$ X_t sm $t \neq 0$

X_0 mild sings (rational?) A_1?

(1) If X_t admits a Chow theoretic decomp of the diagonal $\forall\, t \neq 0$, same is true for any resolution of X_0

(2) similar statement with cohom.

⑨

$X_+ \rightsquigarrow X \Leftarrow$ Artin–Mumford example

$\Rightarrow X_+$ does not admit a Chow theoretic decomp

Conclusion: If X is a very general quartic double solid w/ ≤ 7 nodes, then X is not stably rational

(≥ 1 node $\Rightarrow$ unirational)

Corollary: A general quartic 3fold is not stably rational.

$$\alpha z_3^2 + \beta z_3 + \varphi + \underbrace{\delta(z_0, z_1, z_2)}_{\substack{\text{general conic} \\ (z_0^2 ?)}} z_4^2$$