Intro

Birational rigidity.

Lets us prove non-rationality of Fanos of ~~index~~ and Picard rank 1.

The story starts with Iskovskih-Manin:

Thm

Let X be a smooth quartic 3fold,
Then $Bir(X) = Aut(X)$ is finite
(trivial, for general X).

Cor X not rational.

What they really prove.

Thm Suppose $V \overset{\varphi}{\dashrightarrow} V'$ biratl map between smooth quartics. Then φ is biregular.

\begin{tikzcd} X \arrow[r, dashed, "f"] \arrow[d] & X' \arrow[d] \\ S \arrow[r, dashed, "g"] & S' \end{tikzcd}

MFS's. A morphism in the Sarkisov category is $f : X \dashrightarrow X'$ birat'l

square if g birat'l, and $f_L : X_L \dashrightarrow X_L'$ on generic fiber is biregular.

<u>Def</u> $X \to S$ Mori fiber sp is <u>birat'lly rigid</u> if given any $\varphi : X \dashrightarrow X'$ with $X' \to S'$ MFS, there exists $\alpha : X \dashrightarrow X$ s.t. $\varphi \circ \alpha : X \dashrightarrow X'$ is square.

Note: If X is Fano of Picard rk 1, ~~and~~ then $X \to pt$ is MFS. Suppose it's rigid.

Then given

\begin{tikzcd} X \arrow[r, dashed] \arrow[d] & \mathbb{P}^3 \arrow[d] \\ pt & pt \end{tikzcd}

we can hit X with birat'l auto and

\begin{tikzcd} X \arrow[r, dashed, "\alpha"] & X \arrow[r, dashed] \arrow[d] & \mathbb{P}^3 \arrow[d] \\ & pt \arrow[r] & pt \end{tikzcd}

square.

so at generic pt, hence rat'l.

~~also apply to $X \dashrightarrow X$.~~

Later, similar and more amazing results:

Thm [IP]

$X_{2,3} \subset \mathbb{P}^6$ complete intersection of cubic and quadric.

- There's a 1-param family of lines. $L \subset V_{23}$ a line $\Rightarrow \mathbb{P}^5 \dashrightarrow \mathbb{P}^3$ degree 2. $\alpha_L : V_{23} \dashrightarrow V_{23}$ birat'l invo.
- There's a 1-param family of ~~quadrics~~ conics $C \subset V$ so plane(C) contained in Q. plane$(C) \cap V_{23}$ = conic $\cup$ line. $\pi_{\text{plane}(C)} : \mathbb{P}^5 \dashrightarrow \mathbb{P}^2$ fibers V in elliptic curves. $C_S = P(S) \cap F$.

  fibers are 3-planes $S \supset P$. $S \cap Q = P \cup P(S)$

  C_S meets residual line in a pt $\rightsquigarrow$ involution.

Then $\mathrm{Bir}(X)$ is generated by α_L & β_C, with a finite set of relations.

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also get generators for large family of Fano 3-fold hypersurfaces in weighted projective space (95 families). [Corti-Pukhlikov-Reid].

Rigidity known in degree N in $\mathbb{P}^N$, $\mathrm{Bir}(X)$ trivial for any smooth [dF].

Here's the general outline.

Say we want to prove $X \to S$ is rigid.

Let $X' \to S'$ be another, $X \dashrightarrow X'$ Birat'l.

Step 1 Choose $\mathcal{H}'$ very ample on X'. Let $\mathcal{H} \subset X$ transform.

$\mathcal{H} \subset |-\mu K_X + A|$ $\mu > 0$ rat'l, A a pullback from S.

Some version of Noether-Fano-Iskovskikh ineqs

$\Rightarrow$ $\mathcal{H}$ has a base point with very big mult.

to make precise: $\boxed{K_X + \frac{1}{\mu}\mathcal{H} \text{ is not canonical.}}$

$$\Rightarrow \quad \underbrace{m_E(\mathcal{H})}_{\text{mult}} > \mu \cdot \underbrace{a_E(K_X)}_{\text{discrep.}}$$

Step 2

Use the fact that $K_X + \frac{1}{\mu}\mathcal{H}$ not canonical to show that there are restrictions on the centers of the valuation

This is the ad hoc bit, but often like: no smooth pt can be center, no curve of degree > 2 can be.

Step 3: know center of maximal mult must be singular pts or low degree curve.

Suppose

$$\begin{tikzcd} X \arrow[r, dashed, "\text{birat'l}"] \arrow[d] & X' \arrow[d] \\ S & S. \end{tikzcd}$$

any Sarkisov link must take specific form. Show first link out of X takes us back to X.

What you need to know about the Sarkisov program.

$$\begin{tikzcd} X \arrow[r, dashed, "\varphi"] \arrow[d, "\phi"', "\psi"] & X' \arrow[d, "\psi'"', "\phi'"] \\ Y & Y' \end{tikzcd}$$

φ birat'l.

$\phi : X \to Y$ a Mori Fiber space

1) $-K_X$ ϕ-ample

2) $\mathrm{rk}\, N^1(X) = \mathrm{rk}\, N^1(S) + 1$

3) $\dim S < \dim X$.

given a birat'l map between two MFS's,

Sarkisov program factors it into basic links.

$$\begin{tikzcd} X = X_0 \arrow[r, dashed] \arrow[d, "f"'] \arrow[d, shift left=4, "f_0"] & X_1 \arrow[r, dashed] \arrow[d] & X_2 \arrow[r, dashed] \arrow[d] & \cdots \arrow[r, dashed] & X' \arrow[d, "f'"] \\ Y_0 = Y_0 & Y_1 & Y_2 & & Y' \end{tikzcd}$$

all MFS.

(surface case)

All dim, Type 2.

$$\begin{tikzcd} & Z \arrow[r, dashed, "\text{log flips}"] \arrow[dl, "\text{divisorial}"'] & \cdot \arrow[dr, "\text{divisorial}"] & \\ X \arrow[d] & & & X_1 \arrow[d] \\ Y & & & Y_1 \arrow[lll, "\sim"'] \end{tikzcd}$$

PROOF. Take the graph of the birational map ϕ,

$$\Gamma \subset T \times S,$$

and its desingularization $\tau : V \to \Gamma$ as in the proof of Corollary 1-8-4.

Set

$$\psi_1 = p_1 \circ \tau \text{ and } \psi_2 = p_2 \circ \tau$$

and write down the ramification formulas

$$K_V = \psi_1^* K_T + R_{\psi_1},$$
$$K_V = \psi_2^* K_S + R_{\psi_2}.$$

If ψ_1 is not an isomorphism, then as in the proof of Theorem 1-8-2 there exists a (-1)-curve $E \subset R_{\psi_1}$ such that

$$0 > E \cdot K_V = e \cdot (\psi_2^* K_S + R_{\psi_2}).$$

Since K_S is nef, this implies that $E \cdot R_{\psi_2} < 0$ and hence

$$E \subset R_{\psi_2}$$

and that by Lemma 1-8-1 the morphisms ψ_1 and ψ_2 factors through the contraction $\mu : V = V_0 \to V_1$ of E,

$$\psi_1 = \psi_{10} = \psi_{11} \circ \mu,$$
$$\psi_2 = \psi_{20} = \psi_{21} \circ \mu.$$

If $\psi_{11} : V_1 \to T$ is an isomorphism, then $\phi = \psi_{21} \circ \psi_{11}^{-1} : T \to S$ is a morphism. If not, then we repeat the argument replacing V_0 with V_1. This procedure has to come to an end, since $\dim_{\mathbb{R}} H^2(V_i, \mathbb{R})$ drops by 1 each time, reaching the stage where ψ_{1i} is an isomorphism, verifying that $\phi = \psi_{2i} \circ \psi_{1i}^{-1}$ is actually a morphism. □

Corollary 1-8-7 (Uniqueness of the Minimal Model in Dimension 2). *There exists a unique minimal model in a fixed birational equivalence class in dimension 2. More strongly, if*

$$\phi : S_1 \dashrightarrow S_2$$

is a birational map between two minimal models in dimension 2, then ϕ is an isomorphism. □

PROOF. Apply Theorem 1-8-6 to ϕ and ϕ^{-1}.

Birational Relation Among Mori Fiber Spaces

We now focus our attention on the birational relation among Mori fiber spaces in dimension 2. We will describe the classical Castelnuovo–Noether theorem in the framework of the Sarkisov program. The grand picture of the Sarkisov program in higher ...led in Chapter 13 after we study the minimal model program in higher

Theorem 1-8-8 (Castelnuovo–Noether Theorem = Sarkisov Program in Dimension 2). *Let*

$$\begin{array}{ccc} S & \overset{\Phi}{\underset{\text{birat}}{\dashrightarrow}} & S' \\ \downarrow \phi & & \downarrow \phi' \\ W & & W' \end{array}$$

be a birational map between two Mori fiber spaces in dimension 2:

$$\phi : S \to W,$$
$$\phi' : S' \to W'.$$

Then there is an algorithm, called the **Sarkisov program** *in dimension 2, to decompose Φ into a composite of the following four types of "***links***" (elementary transformations):*

Type (I)

$$\begin{array}{ccc} & & \mathbb{F}_1 \\ & \swarrow & \\ \mathbb{P}^2 & & \downarrow \\ \downarrow & & \\ \text{pt.} & \leftarrow & \mathbb{P}^1 \end{array}$$

Type (II)

$$\begin{array}{ccccc} & & Z & & \\ & \swarrow & & \searrow & \\ S & & & & S_1 \\ \downarrow & & & & \downarrow \\ W & & \overset{\sim}{\leftarrow} & & W_1 \end{array}$$

where $S \to W$ is a $\mathbb{P}^1$ bundle over a nonsingular projective curve W, $Z \to S$ is a blowup of a point in one ruling, $Z \to S_1$ is the contraction of the strict transform of that ruling to obtain another $\mathbb{P}^1$-bundle $S_1 \to W_1 = W$.

Type (III) (*Inverse of* Type (I))

$$\begin{array}{ccc} \mathbb{F}_1 & & \\ & \searrow & \\ \downarrow & & \mathbb{P}^2 \\ & & \downarrow \\ \mathbb{P}^1 & \to & \text{pt.} \end{array}$$

$$\begin{tikzcd} \mathbb{P}^1_1 \times \mathbb{P}^1_2 \arrow[d, "p_1"] & = & \mathbb{P}^1_1 \times \mathbb{P}^1_2 \arrow[d, "p_2"] \\ \mathbb{P}^1_1 \arrow[dr] & & \mathbb{P}^1_2 \arrow[dl] \\ & \text{pt.} & \end{tikzcd}$$

PROOF. The rest of this section will be spent on the proof of the Sarkisov program in dimension 2.

Strategy for "Untwisting" Φ

The strategy for decomposing Φ into a composite of links, the operation that we call **"untwisting"** of Φ, is to set up a good invariant of Φ with reference to the fixed Mori fiber space $\phi' : S' \to W'$ (called the **Sarkisov degree**, the triplets of numbers (μ, λ, e) in lexicographical order, as will be defined below), which should tell us how far ϕ is from being an isomorphism of Mori fiber spaces.

It is the **Noether–Fano–Iskovskikh criterion** that allows us to judge precisely when Φ is an isomorphism of Mori fiber spaces in terms of the Sarkisov degree and the canonical divisor of S.

Starting with a given birational map Φ between two Mori fiber spaces, we ask whether Φ is actually an isomorphism of Mori fiber spaces via the NFI criterion. If the answer is *yes*, then there is nothing more to do, and we stop right there. If the answer is *no*, then we "untwist" Φ by an appropriate link of Type (I), (II), (III), or (IV) to obtain a new birational map Φ_1. We repeat the process with Φ_1. Each time we "untwist," the Sarkisov degree strictly drops, i.e.,

$$(\mu, \lambda, e) = (\mu_0, \lambda_0, e_0) > (\mu_1, \lambda_1, e_1) > (\mu_2, \lambda_2, e_2) > \dots,$$

where these are the Sarkisov degrees of $\Phi = \Phi_0, \Phi_1, \Phi_2, \dots$ with respect to the fixed reference $\phi' : S' \to W'$:

$$\begin{tikzcd} S = S_0 \arrow[r, dashed, "\text{link}"] \arrow[d, "\phi"] \arrow[rrrr, dashed, bend left=40, "\Phi_0"] & S_1 \arrow[r, dashed, "\text{link}"] \arrow[d, "\phi_1"] \arrow[rrr, dashed, bend left=35, "\Phi_1"] & S_2 \arrow[d, "\phi_2"] \arrow[rr, dashed, bend left=30, "\Phi_2"] & \cdots \arrow[r, dashed, "\text{link}"] & S' \arrow[d, "\phi'"] \\ W = W_0 & W_1 & W_2 & \cdots & W' \end{tikzcd}$$

Finally, by observing that the set of the Sarkisov degrees satisfies the descending chain condition, we conclude that the process must come to an end after finitely many steps, reaching the Mori fiber space $\phi' : S' \to W'$, expressing Φ as a composite of links.

Sarkisov degree. [illegible]
so that

$$H' = -\mu' K_{S'} + \phi'^* A'$$

is a very ample divisor on S'. (Note that $-K_{S'}$ is relatively ample. We refer the reader to Hartshorne [3], Chapter II, Proposition 7.10, or Iitaka [5], Theorem 7.11.)

We take a nonsingular projective surface V that dominates both S and S' by birational morphisms (which are compatible with Φ), i.e.,

$$S \overset{\sigma}{\leftarrow} V \overset{\sigma'}{\rightarrow} S' \text{ with } \sigma' \circ \sigma^{-1} = \Phi.$$

For a member

$$\mathcal{H}' \in |H'|$$

we define the **"homaloidal" transform** $\mathcal{H}$ on S of $\mathcal{H}'$ to be

$$\mathcal{H} = \sigma_* \sigma'^* \mathcal{H}'.$$

We note that the homaloidal transform does not depend on the choice of V. We are ready to define the Sarkisov degee.

(i) μ: the quasi-effective threshold.
The first of the triplet, the quasi-effective threshold μ, is defined to be a rational number (necessarily positive) such that

$$\mu K_S + \mathcal{H} \equiv_W 0,$$

that is to say,

$$(\mu K_S + \mathcal{H}) \cdot F = 0 \text{ for any curve } F \text{ in a fiber of } \phi : S \to W.$$

Observe that μ is independent of the choice of a member $\mathcal{H}$ and that since all the curves contracted by ϕ are numerically proportional, we have to check $(\mu K_S + \mathcal{H}) \cdot F = 0$ for only one curve F in a fiber of ϕ.

Note that since for a rational curve l as specified in Theorem 1-4-8, which generates the extremal ray corresponding to the Mori fiber space $\phi : S \to W$,

$$K_S \cdot l = -2 \text{ or } -3$$

(depending on whether l is of type (ii) or (iii) in the classification of extremal rays in dimension 2 in Theorem 1-4-8), we conclude by setting $F = l$ that

$$\mu \in \frac{1}{3!}\mathbb{N}.$$

(ii) λ: the maximal multiplicity.
In order to define the second of the triplet, the maximal multiplicity λ, we consider the linear system consisting of the homaloidal transforms $\mathcal{H}$ for $\mathcal{H}' \in |H'|$, which we denote by $\Phi^{-1}_{\text{homaloidal}}|H'|$. We note that this linear system may be smaller than the complete linear system $|\mathcal{H}|$.

we actually need to peek inside the black box.

How does it go? Fix $f' : X' \to Y'$, $\mu' \in \mathbb{N}$, A' on Y'.
s.t. $-\mu' K_{X'} + \varphi'^* A'$ is very ample

we'll define an invariant of $f : X \to Y$, show that every link improves it, and that once it improves enough the map is an iso.

Let H_X = transform of $H_{X'}$ on X.

$$\begin{tikzcd} & V \arrow[dl] \arrow[dr] & \\ X \arrow[rr, dashed] & & X' \end{tikzcd}$$

1) the "quasieffective threshold" μ is

s.t. $\mu K_X + H_X \underset{Y}{\equiv} 0_X$.

i.e. $H_X = -\mu K_X + f^* D$.

$(S = *,\ H_X = -\mu K_X)$

2) the "maximal multiplicity" λ is

$$\frac{1}{\lambda} = \max\{c \in \mathbb{Q}_{\geq 0} : (X, cH_X) \text{ is canonical}\}$$

$a(E; X, cH_X) \geq 0$.

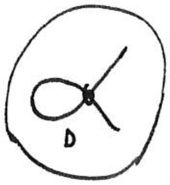
$$\begin{tikzcd} V \arrow[d, "\pi"'] \\ X \end{tikzcd}$$

$\cancel{\text{i.e. } K_V + c\, K_V + \pi^* H_X = H_V}$

$\cancel{\pi^*(K_X + H_X) = K_V + H_V}$

$$K_V + cH_V = \pi^*(K_X + cH_X) + \sum_{\text{exc}} a_i E_i$$

ex.



U
$\downarrow$
X
blow up

is (X, cD) canonical?

$$K_U + c\tilde{D} = \pi^*(K_X + cD) + \sum aE.$$

$$\pi^* K_X + E + c\tilde{D} = \pi^* K_X + c\tilde{D} + 2cE + aE.$$

$$0 = 2cE + aE - E$$

so $a = 1 - 2c$.

if $c \leq 1/2$, canonical. $\Rightarrow \boxed{\lambda = 2}$

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if D smooth, $c \leq 1$ canonical $\Rightarrow \lambda = 1$.

so bigger $\lambda \Rightarrow \mathcal{H}_X$ has worse sings somewhere.

3) $e = \#$ crepant divisors $= \# E : a(E; X, \frac{1}{\lambda}\mathcal{H}_X) = 0.$

Noether-Fano-Iskovskikh:

If $\lambda \leq \mu$ and $K_X + \frac{1}{\mu}\mathcal{H}$ is nef

(0, when $S = pt$)

then φ is an iso. Otherwise, the magic of Sarkisov kicks in.

Suppose that X is a Fano threefold,

$$\begin{array}{ccc} X & \dashrightarrow & X' \\ \downarrow & & \downarrow \\ * & & * \end{array}$$

is a Sarkisov link.

What case are we in?

Always either:

1) φ is an iso
2) $K_X + \frac{1}{\mu} H$ not canonical
3) $K_X + \frac{1}{\mu} H$ canonical, not nef.

but X Fano of Picard rk 1, $K_X + \frac{1}{\mu} H \equiv 0$.

so must be 2).

Main case, for us: $\lambda > \mu$.

in other words, $\mu < \frac{1}{c}$

$c < \frac{1}{\mu}$.

so $(X, \frac{1}{\mu} H_X)$ is <u>not</u> canonical.

here's the game in that case.

There's a <u>maximal divisorial blow-up</u> $p : Z \to X$

so $\rho(Z/X)=1$, exc'l locus is prime divisor,

p is $K+\frac{1}{\lambda}H$ crepant: $K_Z + \frac{1}{\lambda}H_Z = p^*(K_X + \frac{1}{\lambda}H_X)$.

(existence: resolve, run the $(K+\frac{1}{\lambda}H)$-mmp (over X) until minimal model Z'. run K-mmp for Z' over X, eventually spits out divisorial $p: Z \to X$.)

Then run the $K_Z + \frac{1}{\lambda}H_Z$ mmp over Y. $\rho(Z/Y)=2$.

Either ends with a div cont. or MFS

$$\begin{tikzcd}
 & Z \arrow[ld] \arrow[r, dashed] & Z' \arrow[rd] & \\
X \arrow[d] & & & X_1 \arrow[d] \\
Y & & & Y' \arrow[lll, "="']
\end{tikzcd}
\qquad
\begin{tikzcd}
 & Z \arrow[ld] \arrow[r, dashed] & X_1 \arrow[d] \\
X \arrow[d] & & Y_1 \\
Y & &
\end{tikzcd}$$

Last time

To show X birationally rigid:

1. Assume we have $\varphi: X \dashrightarrow X'$ X' another MFS,

2. NFI $\Rightarrow$ $\mathcal{H}$ on $|X|$ has $(X, \mathcal{H})$ not canonical

3. Factor φ via Sarkisov. First step is a maximal divisorial blow-up, then 2-ray game.

we'll try to figure out possible centers, and see what links arise. The hope is either: all lead back to X, or there aren't any.

<u>new example</u>

$X_5 \subset \mathbb{P}(1,1,1,1,2)$ $\quad Q=(0,0,0,0,1)$.

$F(y,x_i) = y^2 x_0 + y f_3(x_i) + f_5(x_i)$. quasismooth

there's $\mathbb{P}(1,1,1,1,2) \dashrightarrow \mathbb{P}^3$; forget last.

$\pi : Y \to X$ blow up sing. $\quad K_Y = \pi^* K_X + \frac{1}{2}E$. ~~nef~~.

$-K_Y$ is nef.

anticanonical model is

$$\overline{Y} = \mathrm{Proj} \bigoplus_{n \geq 0} H^0(-nK_Y).$$

$\varphi : Y \longrightarrow \overline{Y}$ $\quad \varphi : \overline{Y} \to \overline{Y}$ involution.

is 15 lines $\Gamma_i \subset X$, each with

$-K_X \cdot \Gamma = 1/2$.

$\overline{Y} \to \mathbb{P}^3$ is a double cover, so there's

$\sigma : \overline{Y} \to \overline{Y}$ involution. Gives involution on /

$$\begin{tikzcd}
Y \arrow[rr, dashed, "\varphi^{-1}\sigma\varphi"] \arrow[d, "\pi_Q"'] \arrow[dr, "\varphi"] & & Y \arrow[dl, "\varphi"'] \arrow[d, "\pi_G"] \\
X & \overline{Y} & X
\end{tikzcd}$$

this is a link!

Thm

Let $X_5 \subseteq \mathbb{P}(1,1,1,1,2)$ be quasismooth (smooth away from sing pt of $\mathbb{P}$). (It's a $\frac{1}{r}(1,-1,b)$ quotient terminal sing) Then X is birationally rigid and $\mathrm{Bir}\,X/\mathrm{Aut}\,X$ has order 2.

Pf. Let $X' \to S'$ a MFS. $\varphi: X \dashrightarrow X'$ birat'l.

Choose $\mathcal{H}'$ very ample on X', and write $\mathcal{H}' = |-\mu' K_{X'} + A'|$. A' pullback of ample from the base. $\mathcal{H} \subset |-\mu K_X|$ strict transform.

Part 1: Classify possible maximal centers of $\mathcal{H}$.

1) No curve is a maximal center. Let $\boxed{c = \mathrm{mult}_C \mathcal{H} > \mu}$ maximal center.

can't go through Q.

If it were, by Kawamata, blow-up of curve isn't terminal. so C not through Q.

Let $Z = \mathcal{H}_1 \cdot \mathcal{H}_2$. 1-cycle. maximal. intersecting with $S \in |-K_X|$

shows

$\mathrm{mult}_C \mathcal{H} \geq c$ $\quad -K_X = \mathcal{O}(1)$

$\deg C = \mathcal{O}_X \cdot C$

$$\underbrace{\frac{5}{2}\mu^2}_{\text{all known.}} = Z \cdot S \geq c^2 \deg C > \mu^2 \deg C.$$

$C \subset \mathrm{sm}\, X$

so $\deg C \in \mathbb{Z}$.

so $\deg C = 1$ or 2.

$\pi: Y \to X$ blow up C.

$\deg C = 1$: let $M = -2K - E$. one checks $|\mathcal{I}_C(-2K)| = C$

so M is nef. let $Z = H_1 \cdot H_2$.

$$M \cdot Z = (-2K - E)(\underbrace{-\mu K}_{\mathcal{H}} - \underbrace{cE}_{\text{mult}})^2$$

$= 5\mu^2 - 2c^2 - 2\mu c - c^2 < 0.$

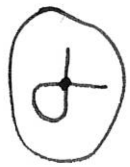
since $c > \mu$. no good!

How to control maximal sings?

The key lemmas:

<u>Lemma 1</u> $P \in S$ surface germ. $\mathcal{L}$ linear system, no fixed cpts. Suppose $K_S + \frac{1}{\mu}\mathcal{L}$ is not log canonical for some μ.

Then $(L_1 \cdot L_2)_P > 4\mu^2$.

example:  $\mu = 1-\epsilon$. then $K_S + \frac{1}{\mu}\mathcal{L}$ not lc.

$$\Rightarrow \underset{4}{\underset{\|}{(L_1 \cdot L_2)_P}} > 4(1-\epsilon)^2$$

<u>Cor</u> $P \in \mathbb{A}^3$ smooth 3-fold germ, $\mathcal{H}$ movable

Let S be a general hyperplane section through P. Then $\mathcal{L} = \mathcal{H}|_S$.

If $K_X + \frac{1}{\mu}\mathcal{H}$ not canonical, then

1) $K_S + \frac{1}{\mu}\mathcal{L}$ not lc (inversion of adjunction)

2) $Z = H_1 \cap H_2$ intersection of general members of $\mathcal{H}$, then $\mathrm{mult}_P Z > 4\mu^2$.

A simple use case.

Thm

Let $X \subset \mathbb{P}^4$ quartic, p a smooth pt.

Then p is not a maximal center.

Pf. Otherwise, $K_X + \frac{1}{\mu}\mathcal{H}$ not canonical at p, with $\mathcal{H} \subset |\mathcal{O}_X(\mu)|$

Let $S \in |\mathcal{O}_X(1)|$, $\mathcal{L} = \mathcal{H}|_S$.

Then $K_S + \frac{1}{\mu}\mathcal{L}$ not log canonical.

Hence

$$L_1 \cdot L_2 = 4\mu^2.$$

$$(L_1 \cdot L_2)_p > 4\mu^2.$$

no good!

1. ~~No curve can be a maximal center.~~

2. No smooth pt can be a center of maximal order.

If p not on one of the 15 lines.

$|I_p(-K)|$ has p as isolated fixed pt.

Intersect $S \in |I_p(-K)|$ with two general
$\uparrow$ general

things in $\mathcal{H} \sim |-\mu K_X|$.

$Z = H_1 \cdot H_2$ 1-cycle.

$$\frac{5}{2}\mu^2 = Z \cdot S \geq (Z \cdot S)_p > 4\mu^2.$$

$\rightrightarrows$ since $\mathrm{mult}_p Z > 4\mu^2$ there.

~~(earlier: $\mathrm{mult}_p Z$~~ earlier result.

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so suppose p is on a line. The base locus of $|I_p(-K)|$ is the curves Γ_i (lines). Pick $S \in |I_p(-K)|$.

$\mathcal{H}|_S = c\Gamma + \mathcal{L}$, $c = \mathrm{mult}_\Gamma H$. $\mathcal{L}$ movable.

we know that

1) Two general L_i must have large int at p

2) $L_1 \cdot L_2$ can be bounded above.

and these will clash.

this needs slightly sharper versions of inequalities.
including Γ as a boundary.

What's the conclusion?

Sarkisov program says

$$\begin{tikzcd} X \arrow[r, dashed] \arrow[d] & X_1 \arrow[d] \\ * & S_1 \end{tikzcd} \qquad \deg(\varphi \varphi_1^{-1}) < \deg \varphi.$$

By Sarkisov considerations.

the link must start with a maximal extraction.

There is only one option (use classification of threefold divisorial contractions)

The link $\overset{Y}{\underset{X}{\downarrow}} \dashrightarrow X_1$ must come

from 2-ray game on

$$\begin{tikzcd} & Y \arrow[dl] \arrow[dr] & \\ X \arrow[rr, dashed, "T"] \arrow[dr] & & X'' \arrow[dl] \\ & pt & \end{tikzcd}$$

but T solves the 2-ray game.

so the link must be T.

Similarly, for $V_{2,3}$, so smooth pt is maximal center,

and curve must have degree ≥ 2.

Check that all possible links are the ones we've seen.

Quartic: there are no maximal centers!

Flowchart 13-1-9.