Counterexamples and Cubic Hypersurfaces

Recall - **TFAE:** X smooth
i) $CH_0(X_F)_0 = 0 \quad \forall\ F/\mathbb{C}$
ii) $CH_0(X_K)_0 = 0 \quad K = \mathbb{C}(X)$
iii) $\Delta_X = \{x\} \times X + Z$, $\mathrm{supp}\, Z \subseteq D \times X$, $D \neq X$ (Decomp of diagonal)

Thm - If $CH_0(X_F) \neq 0$, then X is not stably rational

Hope - This will detect irrationality.

Today - Examples where this fails or turns out to be hard:
- Barlow surfaces
- cubic hypersurfaces

I Barlow surfaces

Prop - X smooth, projective. If X admits a diagonal decomp mod alg equiv, then X admits a decomp in Chow.

Cor - Let S be a general type surface w/ $CH_0(S) = \mathbb{Z}$, $\mathrm{Tors}(H^*(S, \mathbb{Z})) = 0$. Then S has universally trivial CH_0.

Pf - Hodge diamond of S:

$CH_0(S) = \mathbb{Z}$
$\Rightarrow p_g = 0 = q$

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & p & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Lefschetz 1-1 $\Rightarrow$ H^* generated by algebraic cycles

H^* no torsion $\xRightarrow{\text{Kunneth}}$ $[\Delta_S] = \sum_i [\alpha_i] \otimes [\beta_i] \in H^4(S \times S, \mathbb{Z})$,
α_i, β_i alg, $\dim \alpha_i + \dim \beta_i = 2$

$[\alpha_i]\otimes[\beta_i]=[\alpha_i\times\beta_i]$, supported over $D\times S$ w/ $D\subsetneq S$ if $\dim\alpha_i<2$

$\Rightarrow [\Delta_S]=[S\times s]+[Z]$ in $H^4(S\times S,\mathbb{Z})$, $\mathrm{Supp}\subseteq D\times S$, $D\subsetneq S$

$\xRightarrow[+CH_0(S\times S)=\mathbb{Z}]{\text{Bloch-Srinivas}} [\Delta_S]=[S\times s]+[Z]$ in CH/alg

$\Rightarrow [\Delta_S]=[S\times s]+[Z']$ in CH. $\square$

Q - Are there any nonrational such surfaces? $p_g=0=q$, H^* torsion free

A - yes, Barlow surfaces

Thm (Barlow) - $\exists$ smooth surfaces S w/ $p_g=0=q$, $\pi_1(S)=0$, S general type.

Note - $\pi_1(S)=0\Rightarrow H_1(S)=0\Rightarrow H^2_{tors}(S)=0\Rightarrow H_{2,tors}(S)=0$
$\Rightarrow H^3{}_{tors}(S)=0$, so H^* is torsion free.

Idea of Barlow - Take $S=Y/G$ for a special choice of Y,G to get the right invariants.

Starting surface (Van de Geer, Zagier) $F=\mathbb{Q}(\sqrt{21})$

$\mathcal{O}_F$ = ring of ints

$SL_2(\mathcal{O}_F)\circlearrowright \mathbb{H}\times\mathbb{H}$ by $\begin{bmatrix}a&b\\c&d\end{bmatrix}(z_1,z_2)=\left(\frac{az_1+b}{cz_1+d},\frac{az_2+b}{cz_2+d}\right)$

$\Gamma=\left\{\begin{bmatrix}a&b\\c&d\end{bmatrix}\in SL_2(\mathcal{O}_F)\mid a\equiv d\equiv 1,\ b\equiv c\equiv 0 \bmod 2\mathcal{O}_F\right\}$

Let Y be the minimal resolution of a compactification of $\mathbb{H}\times\mathbb{H}/\Gamma$

Fact - $p_g=4$, $q=0$, $K^2=10$, $\pi_1(Y)=\Gamma$ (Shvartsman)
general type 20-nodal quintic

4 0 1

Group action – By taking subgroups of $SL_2(\mathcal{O}_F)$ containing Γ,
we get $D_{10} = \langle \alpha, \beta \mid \alpha^2 = 1, \alpha\beta\alpha = \beta^4 \rangle \curvearrowright Y$
i) β has no fixed points
ii) α has finite fixed locus so that Y/D_{10} has $4\ A_1$ singularities

General Facts about quotients Y/G
a) Let $E = \langle g \in G \mid gy = y \text{ for some } y \in Y \rangle$. Then
$\pi_1(Y/G) \cong G/E$ if $\pi_1(Y)$ is trivial, Y normal

$$\begin{tikzcd} & Y \arrow[dl, "\tilde{\pi}"'] \arrow[d, "\pi"] \\ T \arrow[r, "p"'] & X \end{tikzcd} \quad \star$$

b) If S_0 has rational singularities and $S \to S_0$ is a resolution,
then $\pi_1(S) = \pi_1(S_0)$. (Van Kampen)

c) If Y smooth, G acts freely except for at $p_1, \dots, p_\ell$ where
$G_{p_i} \cong \mathbb{Z}/2$, S: min. resolution of $X = Y/G$, $|G| = n$
- $K_S^2 = K_X^2 = \frac{1}{n} K_Y^2$
- $\chi(\mathcal{O}_S) = \chi(\mathcal{O}_X) = \frac{1}{n}(\chi(\mathcal{O}_Y) + t/4)$ (Noether's formula: $\chi = \frac{K \cdot K + e}{12}$)
- If Y is mininal of general type, so is S

(Pf – ETS $D \subseteq S$ curve $\Rightarrow K_S \cdot D \geq 0$. $F \subseteq S$ exc. divs, $F \cdot K_S = 0 \Rightarrow$ WLOG $D \nsubseteq F$
$\Rightarrow nD = f^* n D_0 + B^{\text{supp on } F} \Rightarrow K_S \cdot nD = K_S \cdot f^* n D_0 = f^* K_X \cdot f^* n D_0 = K_X \cdot n D_0$
$= \frac{1}{n} \pi^* K_X \cdot \pi^* n D_0 = \frac{1}{n} K_Y \cdot \pi^* n D_0 \geq 0 \Rightarrow K_S \cdot D \geq 0$)

Let S be blow up at $4\ A_1$ sings of Y/D_{10}.
$K_S^2 = \frac{1}{10} 10 = 1$
$\pi_1(S) = 1$ (since $\alpha\beta^k$ are all conj., have non-empty fixed locus, gen. G)
$\chi(\mathcal{O}_S) = \frac{1}{10}(5 + \frac{20}{4}) = 1$, $q = 0$ since $\pi_1(S) = 1$
$\Rightarrow p_g = 0$ (since $h^0(\mathcal{O}_S) = 1$)
S minimal of general type

$\square$

II Cubic Hypersurfaces

Thm (Voisin) - Let X be a smooth cubic hypersurface, and suppose $H^*(X,\mathbb{Z})/H^*(X,\mathbb{Z})_{alg}$ has no 2-torsion (which will happen if $\dim X \leq 4$ or $\dim X$ odd, or X very general). Then X admits a Chow-theoretic diagonal decomp iff X admits a cohomological diagonal decomp.
(Pf uses $\mathrm{Hilb}^2 X$, complicated analysis of cohomology, could follow from top. thms)

Recall - Smooth cubic 3-folds are irrational. A very general smooth cubic 4-fold is conjectured to be irrational, but not a single one has been proven to be so.

Thm - Let X be a rationally connected 3-fold. Then X admits a coho decomp of the diagonal iff the following hold

1. $H^3(X,\mathbb{Z})$ torsionfree
2. $\exists$ universal codim 2 cycle in $X \times J(X)$
3. $\frac{\theta^g}{(g-1)!}$ is algebraic on $J(X)$, $\dim J(X) = g$

comes from the lattice str., $H^2(A) = \mathrm{Hom}_{\mathbb{Z}}(\Lambda^2 H_1(A), \mathbb{Z})$

Def - X r.c. 3-fold admits a universal codim 2 cycle if $\exists$ $Z \in CH^2(J(X) \times X)$ s.t. $Z|_{a \times X} \sim_{\text{homologous}} 0$ $a \in J^3(X)$ and

$$\Phi_Z : J(X) \longrightarrow J(X) \quad \text{is the identity.}$$
$$a \longmapsto AJ_X(Z_a)$$

$\left(Z_a \sim_{H_2(X)} 0 \Rightarrow Z_a = \partial\Gamma,\ \Gamma \text{ a 3-chain } AJ(Z_a) = \int_\Gamma \omega \ (\mathrm{mod}\ H^3(X,\mathbb{Z})) \right)$

Rmk - $\exists$ unirational 3-folds not satisfying 1. (Artin-Mumford) and 2 (Voisin, quartic double solids). 3 is unknown

Fact - cubic 3-folds satisfy 1 and for them, $3 \Rightarrow 2$ (Voisin)

Cor - X a smooth cubic 3-fold has universally trivial $CH_0(X) \iff \frac{\theta^4}{4!} \in J(X)$ is algebraic.

$(1+g+g^2)(2-g-g^2)$
$= 2+2g+2g^2-g-g^2-1-g^2-1-g = 0$

$H^{n-p}(X,\Omega^q) \cong R_{\tau(p)}$ $\quad \tau(p) = (n-p+1)d-(n+2)$ $\quad n=3,\ p=1,\ q=1$
$= 2\cdot 3-5 = 1$

$\alpha\cdot\beta = g^*\alpha\cdot g^*\beta$
$\pi\alpha = 0 \Leftrightarrow \alpha\cdot\beta = 0 \ \forall \beta\ g\text{-inv.}$

<u>Thm</u> (Voisin) - There is a countable union of closed codim ≤ 3 subvars of the moduli space of smooth cubic 3-folds where $\frac{\theta^4}{4!}$ is algebraic.

<u>Pf sketch</u> - First suppose $\exists$ isogeny $\mu: J(C) \to J(X)$ of odd degree, C a (possibly reducible) curve
$\mu^*\theta_X = m\theta_C$. ($m$ odd)

Since C is a curve, $\frac{\theta_C^4}{4!}$ is algebraic on $J(C)$
$\Rightarrow \mu_*\theta_C = m\frac{\theta^4}{4!}$ is algebraic on $J(X)$
$CG \Rightarrow$ $(J(X),\theta)$ is a Prym variety $\Rightarrow 2\frac{\theta^4}{4!}$ is algebraic

(very sp. is a Prym) $\Rightarrow \frac{\theta^4}{4!}$ is algebraic in this case

Consider $X = V(x_0^3+x_1^3+x_2^3+x_3^3+x_4^3)$ invariant under
$g: [x_0,x_1,x_2,x_3,x_4] \longmapsto [x_0, jx_1, j^2x_2, x_3, x_4]$ $\quad j = e^{2\pi i/3}$

<u>Recall</u> - $H^{n-p}(X,\Omega^q) \cong R_{\tau(p)}$ ← Jacobian ring, plays nice with the gp action
$\tau(p) = (n-p+1)d-(n+2)$ (since $g^*\Omega = \Omega$ for Ω canonical generator of $H^0(X,\mathcal{O}_X(1))$)
$n=3, p=2, q=2, d=3$
$\Rightarrow \tau = 2\cdot 3-5 = 1$
$R = \mathbb{C}[x_0,\dots,x_4]/(3x_0^2,\dots,3x_4^2) \Rightarrow \dim H^1(X,\Omega_X^2) = 5$
$\dim H^1(X,\Omega_X^2)^{inv} = 3$

Set $\pi = Id + g^* + (g^2)^* \in End(H^3(X,\mathbb{Z}))$
$\Rightarrow \pi$ projects $H^3(X,\mathbb{Q})$ orthogonally onto $H^3(X,\mathbb{Q})^{inv}$

$H^3(X,\mathbb{Q}) = H^3(X,\mathbb{Q})^{inv} \oplus H^3(X,\mathbb{Q})^{\#}$ $= Im(Id - \frac{\pi}{3})$

Over $\mathbb{Z}$, have $H^3(X,\mathbb{Z})^{inv}$, $H^3(X,\mathbb{Z})^{\#} = (H^3(X,\mathbb{Z})^{inv})^{\perp}$
$\Rightarrow \Gamma = H^3(X,\mathbb{Z})^{inv} \oplus H^3(X,\mathbb{Z})^{\#}$ is a sublattice.

$\text{Index}(\Gamma \subseteq H^3(X,\mathbb{Z})) = 3^m$ since
$a \in H^3(X,\mathbb{Z}) \Rightarrow 3a = \underbrace{(a + g^*a + g^{*2}a)}_{\text{inv}} + \underbrace{(3a - (a + g^*a + (g^2)^*a))}_{*} \in \Gamma$

$\Rightarrow$ have $\Gamma_1 \subseteq H^3(X,\mathbb{Z})^{\text{inv}}$, $\Gamma_2 \subseteq H^3(X,\mathbb{Z})^{\#}$
indexes a power of 3
$\Rightarrow$ get ppavs $\underset{\dim 3}{A}, \underset{\dim 2}{B}$ w/ $\mu : A \oplus B \to J(X)$,

$\mu^* \theta_X = 3^m(\theta_A, \theta_B)$ (Schottky problem trivial for $g=2,3$)

$\Rightarrow J(X)$ isog. to $J(C_A \cup_x C_B)$ $\square$

Jacobians of curves have $\frac{\theta^{g-1}}{(g-1)!}$ algebraic idea: (Smith and Varley)
, $\text{mult}_D \theta \geq h^0(C,D)$, θ stratified by

<u>Griffiths Residue Calculus</u>

$\Omega = \sum (-1)^i x_i \, dx_0 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_{n+1}$

$F^p H^n(X) \cong H^0(\Omega_{\mathbb{P}}^{n+1}((n-p+1)X)) / H^0(\Omega_{\mathbb{P}}^{n+1}((n-p)X))$

$H^0(\Omega_{\mathbb{P}^n}^{n+1}(kX)) = \{ g\frac{\Omega}{f^k} \mid g \deg kd-(n+2) \}$

$R = \mathbb{C}[x_0, \dots, x_n] / (\frac{\partial f}{\partial x_0}, \dots, \frac{\partial f}{\partial x_n})$

$\Rightarrow H^{n-p}(X, \Omega_X^q) = R_{\tau(p)}$

<u>Technical Theorem</u> - X smooth, proj, dim n, has coho diagonal decomp
$\Rightarrow (*)$ $\exists$ Z_i smooth, proj dim. $n-2$, $\Gamma_i \in CH^{n-1}(Z_i \times X)$, ints n_i
s.t. $\forall$ $\alpha, \beta \in H^n(X,\mathbb{Z})$
$\langle \alpha, \beta \rangle_X = \sum_i n_i \langle \Gamma_i^* \alpha, \Gamma_i^* \beta \rangle_{Z_i}$

Suppose $(*)$ + 1) $H^{2i}(X,\mathbb{Z})$ is alg for $2i \neq n$, $H^{2i+1}(X,\mathbb{Z}) = 0$ $2i+1 \neq n$
2) $H^*(X,\mathbb{Z})$ torsion free
Then X admits a diagonal decomp.

<u>Pf that $\frac{\theta^{g-1}}{(g-1)!}$ algebraic is necessary</u> - Let X admit coho diagonal decomp $\rightsquigarrow Z_i, \Gamma_i, n_i$ from Technical Theorem

$\Phi_X : CH^2(X)_{hom} \to J(X)$ Abel-Jacobi

$\gamma_i = \Phi_X \circ \Gamma_{i*} : Z_i \to CH^2(X)_{hom} \to J(X)$ (after choosing ref. pt in Z_i)

Set $Z_i' = \gamma_{i*} Z_i$

Have $\langle \alpha, \beta \rangle_X = \sum_i n_i \langle \Gamma_i^* \alpha, \Gamma_i^* \beta \rangle_{Z_i}$

$(\Lambda^2 H^1(J(X), \mathbb{Z}) \cong H^2(J(X), \mathbb{Z}) = H^{2g-2}(J(X), \mathbb{Z})^\vee .)$

Let $\alpha, \beta \in H^1(J(X), \mathbb{Z}) \xrightarrow{\sim} H^3(X, \mathbb{Z})$

$\alpha, \beta \longmapsto \alpha', \beta'$

$\langle \alpha', \beta' \rangle_X = \langle \frac{\theta^{g-1}}{(g-1)!} \alpha, \beta \rangle_{J(X)}$

$\parallel$

$\sum_i n_i \langle \Gamma_i^* \alpha', \Gamma_i^* \beta' \rangle_{Z_i} = \sum_i n_i \langle \gamma_i^* \alpha, \gamma_i^* \beta \rangle_{Z_i} \Leftrightarrow \sum_i n_i [Z_i'] = \frac{\theta^{g-1}}{(g-1)!}$ $\square$