Universal Domains

Let k be a field. A universal domain Ω over k is an alg. closed field such that Ω contains every function field $k(X)$ for every variety X/k.

RMK: $\mathrm{tr.deg}_k(\Omega) \geq \aleph_0$.

Idea: If X/k is a variety, put $X_\Omega := X \times_k \operatorname{Spec}\Omega$. Then $CH_0(X_\Omega)$ encodes information about ~~varieties~~ cycles $[X \times Y]$ as Y varies.

Elementary Chow Group Facts

Lemma: Let X/k var. Let L/k purely transcendental. Then $CH^*_*(X) \xrightarrow{\sim} CH^*_*(X_L)$ ~~is proof~~ induced by flat pullback.

Proof: Since CH commutes with ~~direct~~ inverse limits of ~~flat~~ schemes (w/ ~~flat~~ flat transitions maps) we reduce to the case of $L = k(T)$.

$$L = \varinjlim_{\substack{f \in k[T] \\ f \text{ prime}}} k[T]_f \Rightarrow X_L = \varprojlim_{f \text{ prime}} X \times D(f) \qquad D(f) := \operatorname{Spec}(k[T]_f).$$

By direct limit argument, suffices to show that $p^*: CH^*(X) \to CH^*(X \times D(f))$ is an iso. But since f is prime $D(f) = \operatorname{Spec}(k[T]_f)$ has a k-rat'l point. [f has at most one root in k].

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[d] & X \times D(f) \arrow[r, "p"] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} k \arrow[r, hook, "j"'] & D(f) \arrow[r] & \operatorname{Spec} k \end{tikzcd}$$

Thus i gives a section of p. As p is smooth, i is a regular embedding (LCI) and so we get a Gysin map $i^*: CH^*(X \times D(f)) \to CH^*(X)$

$pi = \mathrm{id}_X \Rightarrow i^* p^* = \mathrm{id}_{CH(X)} \Rightarrow p^*$ injective.

OTOH $X \times D(f) \underset{\text{open}}{\subset} X \times \mathbb{A}^1$ so

$$\begin{tikzcd} CH^*(X \times \mathbb{A}^1) \arrow[r, two heads] & CH^*(X \times D(f)) \arrow[r] & 0 \\ CH^*(X) \arrow[u, "\cong"] \arrow[ur, "p^*"'] & & \end{tikzcd}$$

$\mathbb{A}^1$-invariance of CH^*

$\Rightarrow p^*$ surjective. $\square$

<u>The</u> <u>Bloch-Srinivas</u> Dec<u>omposition</u> of the <u>diagonal</u>. <u>$\dim X = n$</u>

<u>Thm</u>: Let X/k be a smooth complete, connected variety. Let $U \subset X$ open. $V = X \setminus U$. Suppose $CH_0(U_\Omega) = 0$. Then there are n-dimensional cycles Γ_1, Γ_2 on $X \times X$ w/ Γ_1 supported on $V \times X$

Γ_2 —"— $X \times D$ for some divisor D and

$$N[\Delta] = [\Gamma_1] + [\Gamma_2] \quad \text{in } CH^n(X \times X)$$

<u>Proof</u>: Let L be the function field of X and fix $L \subsetneq \Omega$. Put $\eta = \operatorname{Spec} L$. The inclusion $\eta \to X$ of the generic point defines a closed point of X_L:

$$\begin{tikzcd} \eta \times \eta \arrow[r, "p"] \arrow[d] \arrow[dd, bend right, "q"'] & \eta \arrow[d] \\ X_L \arrow[r] \arrow[d] & X \arrow[d] \\ \eta \arrow[r] & \operatorname{Spec} k \end{tikzcd}$$

The diagonal $\Delta_\eta : \eta \to \eta \times \eta$ gives a section of p & q. and so ~~defines~~

$$\eta \xrightarrow{\Delta_\eta} \eta \times \eta \to X_L \quad \text{defines an } L\text{-rat'l pt of } X_L.$$

We shall denote this class in $CH_0(X_L)$ by $[\eta]$.

<u>Step 2</u>: Understanding $\ker(CH_0(U_L) \to CH_0(U_\Omega))$.

Factor $L \subset \Omega$ via $L \to E \to \Omega$

purely transcendental / algebraic.

Since we get a factorization

$$CH_0(U_L) \xrightarrow{\sim} CH_0(U_E) \xrightarrow{j} CH_0(U_\Omega)$$

we need to compute $\ker j$. For every finite ext'n F/E we get a <u>proper</u>, flat $f : U_F \to U_E$ w/ $f_* f^* = \mathrm{Id} \cdot [F:E]$.

But $\Omega = \bar{E}$ so Ω is a direct limit of finite extensions. If $j(\alpha) = 0$ then α dies after pulling back to some $CH_0(U_F)$ for F/E finite.

ie $\alpha \in \ker(f^* : CH_0(U_E) \to CH_0(U_F))$ $\Rightarrow$ $0 = f_* f^*(\alpha) = \deg(F/E) \cdot \alpha$.

$\Rightarrow$ $\ker j$ and hence $\ker(CH_0(U_L) \to CH_0(U_\Omega))$ is TORSION.

$CH_0(U_\Omega) = 0$, so $CH_0(U_L)$ is torsion.

$$CH_0(V_L) \to CH_0(X_L) \to CH_0(U_L) \to 0$$

$\exists N > 0$ st $N[\eta] = 0$ in $CH_0(U_L) \Rightarrow N[\eta] = [\beta]$ in $CH_0(X_L)$.
for some cycle $\beta \in CH_0(V_L)$. $\beta = \sum_i m_i [\delta_i]$ δ_i closed points on V_L.
Take closure of each δ_i in $V \times X$ (ie scheme theoretic image. $\delta_i \hookrightarrow V_L = V \times \mathrm{Spec}\, L \to X \times X$)
and get a cycle Γ_1 on $V \times X$.

Recall that η is embedded in $X_L = X \times \eta$ via
$\eta \xrightarrow{\Delta} \eta \times \eta \to X \times \eta$ scheme theoretic image in $X \times X$ is given by

$$\begin{array}{ccc} \eta \xrightarrow{\Delta_\eta} \eta \times \eta \to & \overset{X_L}{\overset{\|}{X \times \eta}} & \xrightarrow{g} X \times X \\ \downarrow & & \\ X & \xrightarrow{\qquad\Delta_X\qquad} & \end{array}$$

Thus $g^*: CH^*(X \times X) \to CH^*(X_L)$ gives $g^*(\Delta_X) = [\eta]$
$g^*(\Gamma_1) = \beta$.

so $N[\Delta_X] - \Gamma_1 \in \ker g^*$.

<u>OTOH</u>, $X_L = \varprojlim_{\substack{D \text{ divisor} \\ \text{on } X}} X \times (X - D)$. so $CH^*(X_L) = \varinjlim_{D} [CH^*(X \times (X-D))]$

so image of $N[\Delta_X] - \Gamma_1$ must vanish in some $CH^*(X \times (X-D))$. for some D.

$$CH^*(X \times D) \to CH^*(X \times X) \to CH^*(X \times (X-D)) \to 0$$

Hence, $N[\Delta_X] - \Gamma_1 = \Gamma_2$ for some D supported on $CH^*(X \times D)$. $\square$

<u>RMK</u>: Suppose that k is an <u>uncountable</u>, alg closed field. Let X/k smooth $U \subset X$ open w/ $CH_0(U) = 0$. Then since X is defined by finitely many equations, there is a f.g. field E over the prime field $\mathbb{F}$ ($= \mathbb{Q}$ or $\mathbb{F}_p$) over which X is defined i.e. $\exists\, Y/E$ st $Y \times_E k = X$. Since $\mathrm{tr.deg}_E\, k = 2^{\aleph_0}$ we know that the function field of every variety $/E$ embeds into k. So we can take $\Omega = k$ and thus obtain a decomposition of the diagonal for Y and hence for X.

Applications & Examples

Algebraic and Homological Equivalence Let $X/\mathbb{C}$ proper.

$Z^i_{hom}(X)$ = cycles in codim i homologous to 0.

$Z^i_{alg}(X)$ = — " — alg. equiv to 0

$Z^i_{rat}(X)$ — " — rat. equiv to 0.

$$Z^i_{rat}(X) \subset Z^i_{alg}(X) \subset Z^i_{hom}(X).$$

Classical Results: If $X/\mathbb{C}$ smooth proper, then $Z^1_{alg}(X) = Z^1_{hom}(X)$.
if $\dim X = n$ $Z^n_{alg}(X) = Z^n_{hom}(X)$.

Bloch-Srinivas: Let $X/\mathbb{C}$ smooth proper. $U \subset X$ open st $V = X \setminus U$ has $\dim \leq 2$. Then if $CH_0(U) = 0$ then $Z^2_{hom}(X) = Z^2_{alg}(X)$.

Weaker, easier-to-prove statement by Voisin: $Z^2_{hom}/Z^2_{alg} \otimes \mathbb{Q} = 0$

RMK: The hypothesis that CH_0 be supported in dim 2 is essential
Counterexamples of Clemens for 3-folds where $Z^2_{hom}/Z^2_{alg} \otimes \mathbb{Q} \neq 0$.

Pf of Voisin's Result: By assumption, we have a decomposition

$$N\Delta_X = \Gamma_1 + \Gamma_2 \quad \text{where} \quad \Gamma_1 \in CH^n(V \times X) \quad \Gamma_2 \in CH^n(X \times D)$$

D a divisor.

Let $\begin{matrix} \tilde{X}_V, & \tilde{X}_D \\ {\scriptstyle \pi_V}\downarrow & {\scriptstyle \pi_D}\downarrow \\ X & X \end{matrix}$ be embedded resolutions of V, D

proper transform $\hat{V} \hookrightarrow \tilde{X}_V$ is nonsingular etc. Under

$\tilde{X}_V \times X \xrightarrow{\pi_V \times id} X \times X$ we get a cycle $\tilde{\Gamma}_1$ supported on $\hat{V} \times X$
st $(\pi_V \times id)_* \tilde{\Gamma}_1 = \Gamma_1 \in CH(V \times X)$

Similarly get $\widetilde{\Gamma}_2$ on $X \times \widetilde{D} \hookrightarrow X \times \widetilde{X}_D \xrightarrow{Id \times \pi_D} X \times X$

Technical point: $\pi_V : \widetilde{X}_V \to X$ is a composition of blowups along nonsingular centers. Locally π_V factors as a composite of a regular embedding followed by a smooth projection. } so-called lci morphism

As such, a generalized Gysin pullback $\pi_V^! : CH^*(X) \to CH^*(\widetilde{X}_V)$ exists. [cf Fulton 6.6] These pullbacks satisfy projection formula, etc.

If $\gamma \in CH^2(X)$, how does the correspondence Γ_1 act?

$$\begin{tikzcd}
\widetilde{V} \times X \arrow[r, hook, "\text{reg. embed}"] \arrow[d, "p_1'"] \arrow[rr, bend left, "g'"] & \widetilde{X}_V \times X \arrow[r] \arrow[d] & X \times X \arrow[r, "p_2"] \arrow[d, "p_1"] & X \\
\widetilde{V} \arrow[r, hook, "\text{reg. embed.}"] \arrow[rr, bend right, "g"'] & \widetilde{X}_V \arrow[r] & X &
\end{tikzcd}$$

g is LCI:
$$(\Gamma_1^*)\gamma := (p_{2*})(p_1^*\gamma \cdot \Gamma_1)$$
$$= (p_{2*})(p_1^*\gamma \cdot g'_*\widetilde{\Gamma}_1)$$
$$= (p_{2*})\, g'_*\left(g'^{!} p_1^*(\gamma) \cdot \widetilde{\Gamma}_1\right) \quad \text{proj. formula}$$
$$= (p_{2*})\left(g'_*\left(p_1'^{*} g^!(\gamma) \cdot \Gamma_1\right)\right)$$

BUT $g^!(\gamma)$ is in $CH^2(\widetilde{V}) = CH_0(\widetilde{V})$ since V non singular surface.

Similarly, $(\Gamma_2^*)\gamma$ involves a cycle of codimension 1 supported on $\widetilde{D}$ — hence a divisor on $\widetilde{D}$. $X \times \widetilde{D}$

Since $N\Delta_X = \Gamma_1 + \Gamma_2$ we get for $\gamma \in CH^2(X)_{\mathbb{Q}}$

$N \cdot \gamma = \Gamma_1^*\gamma + \Gamma_2^*\gamma$. so if γ is homologous to 0,

we see that $\Gamma_1^*\gamma$ involves a homologously trivial pt on $\widetilde{V} \Rightarrow$ alg equiv 0.

$\Gamma_2^*\gamma$ ——||—— divisor on $\widetilde{D} \Rightarrow$ alg equiv to 0.

RMK: Characteristic p: Cycle maps exist for any Weil cohomology theory. and one can ask ~~[illegible]~~ to what extent alg. and hom. equivalence coincide.

Bloch Srinivas: ~~[illegible]~~ k alg closed char p. X/k smooth Then if $CH_0(X)$ supported on a surface $Z^2_{hom}(X)$ and $Z^2_{alg}(X)$ agree modulo p-torsion. Assuming resolution of singularities.

This was written before de-Jong, etc.

EXAMPLE. ~~Algebraic and homological equivalence can differ~~!

Notation: $A_0(X)$ = subgp of $CH_0(X)$ consisting of 0-cycles alg ~ 0

$Hom_0(X)$ = ——— // ——— homologically ~ 0.

Lemma (Bloch-Ogus) A_0 is divisible for X/k smooth.

Let V^{n-1} be the vanishing locus of $X_0^{n+1} + X_1^{n+1} + \cdots + X_n^{n+1}$ on $\mathbb{P}^n_{\mathbb{C}}$. Fix $\zeta \in \mu^{n+1}$ a generator. ~~$\boxed{n \text{ odd}}$ even~~ (n odd)

$H = \mathbb{Z}/(n+1)\mathbb{Z} \curvearrowright V^{n-1}$ $[a_0 : a_1 : \cdots : a_n] \mapsto [a_0 : \zeta a_1 : \zeta^2 a_2 : \cdots : \zeta^n a_n]$

action of generator.

$X := V^{n-1}/H$ $\pi : V^{n-1} \to X$ finite of deg. $n+1$, flat

~~constant~~ fibre dim 0, map of reg. schemes.

~~Easy check: $\pi_* A_0(V^{n-1}) \twoheadrightarrow A_0(X)$~~

~~$\pi_* \pi^* = (n+1)\,\mathrm{Id}$ $A_0(X) \to A_0(X)$~~

CLAIM: $A_0(X) = 0$. (~~But clearly $Hom_0(X) \neq 0$~~).

bserve

L#

① $A_0(Y)$ is divisible for <u>any</u> smooth variety (Bloch–Ogus) (Roitman)

② $\pi_*: A_0(V^{n-1}) \twoheadrightarrow A_0(X)$ is surjective

③ $\pi_* \circ \pi^* = (n+1)\cdot Id$ on $A_0(X)$

<u>RMK</u>: Suffices to show $\pi^* \circ \pi_*$ is <u>torsion</u> in $End(A_0(V^{n-1}))$

<u>Indeed</u>: $(\pi^* \circ \pi_*)^r$ ...

$\pi^* \circ [\,(r-1)(n+1)Id_{A_0(X)}\,] \circ \pi_*$

if $(\pi^* \circ \pi_*)^r = 0 \Rightarrow \pi_* \circ (\pi^* \circ \pi_*)^r = 0$

$$\Rightarrow (\pi_* \circ \pi^*)^r \circ \pi_* = 0$$

$$\Rightarrow r(n+1)\, Id_{A_0(X)} \circ \pi_* = 0$$

But since π_* and $r(n+1)\,Id_{A_0(X)}$ are surjective, so we have $A_0(V^{n-1}) \overset{0}{\twoheadrightarrow} A_0(X)$. ∎

hat is, we shall show that $(\pi^* \circ \pi_*)$ is 0 in $End(A_0(V^{n-1})) \otimes \mathbb{Q}$

o $\mathbb{Q} \to \overline{\mathbb{Q}}$ faithfully flat suffices to check inside $End(A_0(V^{n-1})) \otimes \overline{\mathbb{Q}}$.

$= H^{\oplus n}$ $G \curvearrowright V^{n-1}$ via $e_i: x_j \mapsto \zeta^{\delta_{ij}} x_j$

basis $e_1, \cdots e_n$

Je recover the original H action by restricting to $e_1 e_2^2 e_3^3 \cdots e_n^n$

G acts on closed points, we can identify each e_i as an element $\overline{e}_i \in End(A_0(V^{n-1})) \otimes \overline{\mathbb{Q}}$.

$\pi^* \circ \pi_* [P] = H$-orbit of P in V^{n-1} (n+1 pts)

$\pi^* \circ \pi_* = \sum_{i=0}^{n} \overline{e}_1 \overline{e}_2^2 \cdots \overline{e}_n^n$ in $End(A_0(V)) \otimes \overline{\mathbb{Q}}$.

CLAIM 1: If $1 < i_1 < \cdots < i_k < n$ then $\sum_{j=0}^{n} (\bar{e}_{i_1}\bar{e}_{i_2}\cdots\bar{e}_{i_k})^j = 0$.

Maschke Thm $\Rightarrow$

$\bar{\mathbb{Q}}[G]$ semisimple commutative [product of copies of $\bar{\mathbb{Q}}$].

$\bar{\mathbb{Q}}[G] \longrightarrow \mathrm{End}(A_0(V)) \otimes \bar{\mathbb{Q}}$ has semisimple image R.

Thus, to prove $\sum_{i=1} (\bar{e}_1 \bar{e}_2^2 \cdots \bar{e}_n^n)^i = 0$ in R we check modulo each max ideal. [b/c R has trivial Jacobson radical]

Going modulo each maximal ideal gives a map $R \xrightarrow{\varphi} \mathbb{Q}$ where each $\bar{e}_i$ is sent to some $(n+1)$st root of unity.

Thus, $\bar{e}_{i_1}^{m_1} \cdots \bar{e}_{i_k}^{m_k}$ maps to some other root of unity β.

$\Rightarrow \sum_{j=0}^{n} (\bar{e}_{i_1}^{m_1} \cdots \bar{e}_{i_k}^{m_k})^j$ maps to $1 + \beta + \beta^2 + \cdots + \beta^n$.

But $0 = \beta^{n+1} - 1 = (\beta - 1)(1 + \beta + \cdots + \beta^n)$

so $\sum_{j=0}^{n} (\bar{e}_{i_1}^{m_1} \cdots \bar{e}_{i_k}^{m_k})^j = 0 \Leftrightarrow \beta \neq 1$.

WTS: $\bar{e}_1 \bar{e}_2^2 \cdots \bar{e}_n^n$ maps to something <u>different</u> from 1.

$\mathbb{Z}/(n+1)\mathbb{Z} \xrightarrow{\sim} \mu^{n+1}$

$k \longmapsto z^k$.

say that $\bar{e}_i$ maps to $z^{y_i} \in \mu^{n+1}$

hence $y_i \in \mathbb{Z}/(n+1)\mathbb{Z}$.

We want

(9)

$$z^{y_1} z^{2y_2} z^{3y_3} \cdots z^{ny_n} \neq 1 \iff y_1 + 2y_2 + \cdots + ny_n \not\equiv 0 \mod n+1.$$

according to claim 1:

$\left\{ \sum_{i=0}^{k} y_{n-i} \right\}_{k=\{0,\dots,n-1\}}$ is a set of n distinct nonzero numbers in $\mathbb{Z}/(n+1)\mathbb{Z}$

$$y_1 + 2y_2 + \cdots + ny_n = y_n + (y_n + y_{n-1}) + (y_n + y_{n-1} + y_{n-2}) + \cdots + \left(\sum_{i=0}^{n-1} y_{n-i}\right).$$

$$\equiv 1 + 2 + \cdots + n \pmod{n+1}.$$

$$\equiv \frac{n(n+1)}{2} \mod (n+1)$$

$$\not\equiv 0 \quad \text{if } n \text{ odd}.$$