Recovering Nodal Plane Curves from their Bitangents

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Abstract

We show that a general irreducible nodal plane curve is uniquely determined by its bitangents, the collection of lines which are tangent to it in two distinct points. Caporaso and Sernesi showed this result for curves of some curves of degree 4, in particular for the curves with singularities no worse than nodes. We extend their argument to all such curves of arbitrary degree. In particular, we give a proof that the 6-nodal quintic is determined by its bitangents, and we then use this fact to prove that all general nodal plane quintics are determined by their bitangents. We proceed analogously for nodal sextics, and then show that all other higher-degree cases are determined trivially.

1 Introduction

It is a clear fact that the collection of tangent lines of a plane curve determines the curve; this is the statement that projective duality is reflexive. What is perhaps more interesting is whether or not a finite subsets of distinguished tangent lines determines a plane curve. A natural way to choose these tangent lines (for curves of degree four or higher) is to consider the lines which are tangent to a curve in more than one point. We informally refer to these lines as bitangents.

The Plücker formulas, a collection of equations relating invariants of a curve to those of its dual, give a way of systematically computing the number of lines bitangent to a smooth plane curve of degree greater than three. It is easy to show by Bézout's theorem that a smooth curve of degree greater than four is necessarily determined by its bitangents. Indeed, we notice that bitangents of a smooth curve X correspond to nodes of its dual curve $\check{X}$, so two curves with the same bitangents must have corresponding dual curves sharing the same nodes. Now, by the Plücker formulas, we see that two curves of degree greater than four which share the same bitangent lines must have the same dual curves, as the number of intersection points given by the nodes of the dual exceeds the square of the degree of the dual curve. We will show this result explicitly in the next section.

For curves of degree four, this argument no longer holds. The Plücker formulas give that a smooth plane quartic has 28 bitangents, too few to apply Bézout's theorem. However, smooth plane curves are still determined by their bitangents. In their 2001 paper [1], Caporaso and Sernesi prove that the general smooth quartic is determined by its bitangent lines. A natural next step is to generalize this result for the singular curves. In this paper, we examine the curves with the simplest singularities: nodal curves. The argument above is no longer well-defined for curves which fail to be smooth. For one, consider a line which passes through two nodes of a curve. Each of these intersections has multiplicity 2, so there is a sense in which it is a "bitangent". On the other hand, it does not correspond to a node of the dual curve as we have come to expect. To resolve this conundrum, we introduce a few definitions. We say that a line l is a theta line of a curve C if it intersect the curve with multiplicity higher than 1 at two distinct points. l is said to be of type k if it passes through k nodes of the curve. Thus, our true bitangents, the lines corresponding to nodes of the dual curve, are the type 0 theta lines, distinct from the type 2 theta line example given above. The collection of all theta lines of a plane curve is called its theta curve.

With this structure, we can finally articulate the main claim of this paper.

Theorem 1.1. *A general plane curve of degree d with δ-nodes is determined by its theta curve.*

2 Preliminaries

One of the gems of classical projective geometry is the notion of projective duality. Two distinct points in the projective plane uniquely determine a line which contains them, and two lines in $\mathbb{P}^2$ uniquely determine a point of intersection. This gives the collection of distinct lines in the projective plane $\check{\mathbb{P}}^2$ an induced geometry: the "points" of $\check{\mathbb{P}}^2$ are the lines of $\mathbb{P}^2$, and the "lines" of $\check{\mathbb{P}}^2$ are the points of $\mathbb{P}^2$. In particular, if $p_1, p_2 \in \mathbb{P}^2$ correspond to $l_1, l_2 \in \check{\mathbb{P}}^2$, then the line $\overline{p_1 p_2}$ in $\mathbb{P}^2$ corresponds to the point $l_1 \cap l_2$ in $\check{\mathbb{P}}^2$.

Another way of expressing this duality is by working directly with vector spaces. Let V be a vector space of dimension three. The projectivization of V, denoted $\mathbb{P}V$, is the collection of one-dimensional subspaces of V. But V also determines another vector space of dimension three: its dual space $\check{V}$. Thus, by the same construction, we can consider $\mathbb{P}\check{V}$. A line in $\mathbb{P}V$ corresponds to a two dimensional subspace of V; we naturally consider the set of all linear functionals $\check{v} \in \check{V}$ which kill this subspace. This is clearly a one dimensional subspace of $\check{V}$, defining a unique point $[\check{v}] \in \mathbb{P}\check{V}$. Thus, by duality of vector spaces, we have a natural association between points and lines of a projective space and its dual.

The first real example of projective duality is in the case of plane curves of $\mathbb{P}^2$. Thus, it makes sense to consider how a plane curve X transforms under projective duality. To any general point on X we assign the tangent line at that point. The closure of the collection of all such lines a variety in $\check{\mathbb{P}}^2$, which we call the dual curve of X, and denote by $\check{X}$.

What would we expect the degree of X to be? If X is smooth, we need to compute the number of intersection points of $\check{X}$ with a general line in $\check{\mathbb{P}}^2$. Equivalently, we want to compute the number of tangent lines of X passing through a general point p in $\mathbb{P}^2$. Without loss of generality, let $p = [0 : 0 : 1]$. Any line passing through p which is tangent to X satisfies the two equations

$$\frac{\partial F}{\partial Z_2} = 0, \ \ F = 0 \tag{1}$$

where $F(Z_0, Z_1, Z_2)$ is the polynomial which cuts out X. If F has degree d, then the number of tangents of X passing through p is the intersection of F with a polynomial of degree $d-1$. Therefore by Bézout's theorem, the degree of $\check{X}$ is $d(d-1)$.

For our purposes, we need to consider a few singular cases. This paper deals mainly with nodal curves, but for the sake of the Plücker formulas, we require some understanding of cusps as well. Let X be a plane curve of degree d with δ nodes and κ cusps. If we choose p to be as above, then we expect to get $d(d-1)$ tangents of X passing through p. However, some of those so-called tangents are not as we generally imagine them. Instead, they pass through the singular points of X, thereby acquiring a higher intersection multiplicity. Indeed, a line through p intersects a node with multiplicity 2, and intersects a cusp with multiplicity 3. Thus, the true number of tangent lines of X corresponding to points on the dual curve $\check{X}$ is

$$d^* = d(d-1) - 2\delta - 3\kappa. \tag{2}$$

The duality of curves is reflexive, such that X is the dual curve of $\check{X}$. Thus, by the reasoning above, we get the formula

$$d = d^*(d^* - 1) - 2\delta^* - 3\kappa^*, \tag{3}$$

where δ^* is the number of nodes of $\check{X}$ and κ^* is the number of cusps of $\check{X}$. But what exactly are these singularities of $\check{X}$? A node of $\check{X}$ corresponds to a line in $\mathbb{P}^2$ which is tangent to X in two points. These are the bitangents of X. A cusp of $\check{X}$ corresponds to a flex line of X; we do not deal with them in this paper. Because we have quite a few numbers which are not necessarily determined a priori, we need to produce another formula for counting the singularities of X and $\check{X}$.

Given a smooth plane curve X over $\mathbb{C}$, the underlying set of complex points $U(X)$ has the structure of a Riemann surface. In particular, the genus of X is exactly the genus of this Riemann surface. Consider the projection from a point p not contained in X,

$$\pi_p : X \longrightarrow \mathbb{P}^1, \quad x \mapsto \overline{px} \cap \mathbb{P}^1. \tag{4}$$

This gives a cover of $\mathbb{P}^1$ with branch points corresponding to the tangents to X passing through p. The Riemann-Hurwitz formula, relating the Euler characteristic of a surface that of a ramified covering, implies that $\chi(X) = d \cdot \chi(\mathbb{P}^1) - d(d-1)$. Equivalently, if g is the genus of X, we know that $2g - 2 = d(d-1) - 2d$, so we get the following formula for the genus of X:

$$g = \frac{1}{2}(d-1)(d-2). \tag{5}$$

For the case of a singular X, the argument above must be modified. Singular curves are not manifolds of any kind, so we give the following definition to generalize the notion of genus to singular curves.

Definition 2.1. *The geometric genus of a plane curve X is the genus of its normalization X^ν.*

We can now extend our construction above to the case of a plane curve X of degree d with δ nodes and κ cusps. We project from a point p not on X; away from the singular locus, the ramification is $d(d-1) - 2\delta - 3\kappa$ by our computation above. The nodes are separated in the normalization, so they are ramified; the cusps, on the other hand, correspond to a unique tangent line through p in the normalization. Thus, the total ramification of this covering is $d(d-1) - 2\delta - 2\kappa$. By the Riemann-Hurwitz formula,

$$\chi(X^\nu) = d \cdot \chi(\mathbb{P}^1) - d(d-1) + 2\delta + 2\kappa. \tag{6}$$

Equivalently, $2g - 2 = d(d-1) - 2\delta - 2\kappa - 2d$, and we get the following formula for the geometric genus:

$$g = \frac{1}{2}(d-1)(d-2) - \delta - \kappa. \tag{7}$$

It turns out that the geometric genus of $\check{X}$ is equal to that of X. Indeed, by the Riemann-Hurwitz formula, there are no (non-trivial) rational maps from a curve of lower genus to a curve of higher genus, thus $g(X) \leqslant g(\check{X}) \leqslant g(X)$. Therefore the following formula comes from duality:

$$g = \frac{1}{2}(d^*-1)(d^*-2) - \delta^* - \kappa^*. \tag{8}$$

Thus, we get two equations which are linear in δ^* and κ^*. Solving them gives the Plücker formulas.

Theorem 2.1 (Plücker Formulas)**.** *Let X be a plane curve of degree d with δ nodes and κ cusps. Let δ^* and κ^* denotethe number of nodes and cusps of $\check{X}$. We have the following formulas:*

$$\kappa^* = 3d(d-2) - 6\delta - 8\kappa \tag{9}$$

$$\delta^* = \frac{1}{2}d^4 - d^3 - \left(\frac{9}{2} + 2\delta + 3\kappa\right)d^2 + (9 + 2\delta + 3\kappa)d + 2\delta^2 + 3\delta\kappa + \frac{9}{2}\kappa^2 + 10\delta + \frac{23}{2}\kappa \tag{10}$$

Recall that δ^* is the number of bitangents of X. We see that if we fix the number of nodes and cusps of X, the number of bitangents gets very large as the degree d increases. For smooth curves X of degree d, we give bitangent counts in Table 1.

The number of bitangents grows very quickly as the degree of the curve increases. We expect this from the Plücker formula (10), but this gives us a sense of scale. Bitangents of smooth curves manifest as distinguished points on the dual curves, so we expect that such large quantities of bitangents should completely determine a curve. After all, how many curves can possibly share the same 1320 bitangents? The following well-known theorem partially confirms this reasoning.

Theorem 2.2. *A general plane curve of degree greater than four is determined by its bitangents.*

Table 1: Bitangent Counts of Smooth Curves

Degree d	Genus g	Number of Bitangents δ^*
4	3	28
5	6	120
6	10	324
7	15	700
8	21	1320
$\vdots$	$\vdots$	$\vdots$
d	$\frac{1}{2}(d-1)(d-2)$	$\frac{1}{2}d(d-2)(d^2-9)$

Proof. Let X, Y be two general plane curves of degree d with the same bitangent lines, and denote the curve which is the union of these lines by $\Theta \subset \mathbb{P}^2$. Consider the images of X and Y in dual projective space $\check{\mathbb{P}}^2$; we get two new plane curves $\check{X}, \check{Y}$ of degree $d(d-1)$. Each of the bitangent lines in Θ is sent to a common node of $\check{X}$ and $\check{Y}$. Consequently, the two curves $\check{X}, \check{Y}$ have $\frac{1}{2}d(d-2)(d^2-9)$ common nodes. Each nodal intersection is an intersection of multiplicity at least 4, so counting with multiplicity, $\check{X}$ intersects $\check{Y}$ in at least $2d(d-2)(d^2-9)$ points.

Bézout's Theorem gives us an upper bound on the number of intersections between two distinct plane curves of degrees r, r': namely, the product of their respective degrees rr'. In our case, the upper bound on the intersections of $\check{X}$ and $\check{Y}$ is $d^2(d-1)^2$. But $d^2(d-1)^2$ is less than $2d(d-2)(d^2-9)$, the number of node intersections of $\check{X}$ and $\check{Y}$, for all values of d greater than four. Therefore $\check{X} = \check{Y}$ and $X = Y$, as required. $\square$

For plane quartics, the final inequality $d^2(d-1)^2 < 2d(d-2)(d^2-9)$ fails to hold, so we must adopt much more clever arguments. It turns out, however, that general plane quartics are also determined by their bitangents, as proven in Caporaso and Sernesi's paper [1]. Their argument is that if a general quartic can be deformed to one which is determined by its bitangents, it too must be determined. In particular, this gives us the following theorem:

Theorem 2.3. *A general plane curve of degree greater than three is determined by its bitangents.*

Having answered this question for general plane curves, we now turn our attention to nodal plane curves. It is shown in [1] that general one-nodal, two-nodal, and three-nodal quartics are determined by their bitangents. In this paper, we seek to replicate their argument to show that general nodal plane curves are determined by their bitangents.

3 General Structure

Let X be a (smooth or nodal) plane curve of degree d. Let V be the set of irreducible plane curves of degree d with singularities no worse than nodes, and let $V^0 \subset V$ be the set of smooth plane curves of degree d.

Definition 3.1. *A line $L \subset \mathbb{P}^2$ is said to be a theta line of X if the scheme $X \cap L$ is everywhere non-reduced. If L contains i nodes of X, we say that L is a theta line of type i.*

We also define the notion of a theta curve of X. Let $n = \frac{1}{2}d(d-2)(d^2-9)$. The theta map

$$\theta : V^0 \longrightarrow \mathrm{Sym}^n(\check{\mathbb{P}}^2) \tag{11}$$

sends every smooth $S \in V^0$ to its set of n bitangents $\theta(S)$. We claim that θ extends to a proper morphism of V.

Lemma 3.1. *There exists an extension of θ to V such that for all $X \in V$ the components of $\theta(X)$ are theta lines of X.*

Proof. It is apparent that X has finitely many theta lines, since all points of X have multiplicity no greater than 2. Consider

$$J^0 = \{(S, \theta(S)) \mid S \in V^0\} \subset V^0 \times \mathrm{Sym}^n(\check{\mathbb{P}}^2); \tag{12}$$

Let $J = \mathrm{Cl}(J^0)$ in $V \times \mathrm{Sym}^n(\check{\mathbb{P}}^2)$, equipped with the canonical projections

$$\pi_1 : V \times \mathrm{Sym}^n(\check{\mathbb{P}}^2) \to V, \quad \pi_2 : V \times \mathrm{Sym}^n(\check{\mathbb{P}}^2) \to \mathrm{Sym}^n(\check{\mathbb{P}}^2). \tag{13}$$

Then any to any X in V we can assign a set of n lines, not necessarily distinct, via $(l_1 \cdot l_2 \cdots l_n) \in \pi_2\left(\pi_1^{-1}(X)\right)$. It is also clear that each l_i is a theta line of X, as any deformation of elements of V_0 to X sends bitangent lines to l_i. In particular, X has only finitely many theta lines, so $\pi_2\left(\pi_1^{-1}(X)\right)$ is finite. For $S \in V_0$, π_1 is one-to-one. Moreover, J is irreducible because V_0 and J_0 are irreducible. Thus by the Zariski Connectedness Principle, the fibers of π_1 must be connected. But the only connected sets with finitely many elements are point sets, so π_1 is a bijection. Therefore

$$\pi_2 \circ \pi_1^{-1} = \theta : V \longrightarrow \mathrm{Sym}^n(\check{\mathbb{P}}^2) \tag{14}$$

is a proper morphism. □

Thus the theta curve of $X \in V$ is defined to be $\theta(X)$. For the sake of formality, we finally specify what it means for X to be determined by its bitangents.

Definition 3.2. *A curve $X \in V$ is said to have the theta property if for all $X' \in V$, $\theta(X) = \theta(X') \iff X = X'$.*

We dedicate the rest of this paper to proving Theorem 1.1. However, we can rule out some cases right away. As we discussed earlier, general irreducible nodal quartics have the theta property. We claim that curves with $d \geqslant 7$ also have the theta property. To see this, we simply rely on our former trick with Bézout's theorem on the dual curve. The smallest (real) root of the polynomial $4\delta^* - (d^*)^2$ is less than 7, regardless of which value we choose for δ. Thus, to prove this theorem, we need only consider quintics and sextics.

4 Nodal Quintics

Recall that we have separated the theta lines of singular plane curves into distinct classes based on how many singularities they contain. We will now classify the various types of nodal quintics, as well as their respective theta curves.

A smooth quintic $S \in V^0$ has only type 0 theta lines, all 120 of which are distinct. As we showed above, S has the theta property. Thus, we are most interested in δ-nodal quintics, $\delta = 1, \ldots, 6$. The following lemma will help us resolve some of these cases right away.

Lemma 4.1. *The number of type 0 theta lines of a general δ_0-nodal quintic is* $120 - 30\delta_0 + 2\delta_0^2$.

Proof. The dual curve $\check{X}$ of X has degree $d^* = d(d-1) - 2\delta_0 = 20 - 2\delta_0$. We have by the Plücker degree formula (3) that

$$d = d^*(d^* - 1) - 2b_0 - 3f = 380 - 78\delta_0 + 4\delta_0^2 - 2b_0 - 3f \tag{15}$$

where b_0 is the number of type 0 theta lines of X and f is the number of flex lines of X. f is known by the Plücker formula (9) to be

$$f = 3d(d-2) - 6\delta_0 = 45 - 6\delta_0. \tag{16}$$

Table 2: Counts of Theta Lines for Nodal Quintics

Nodes δ	Type 0 lines b_0	Type 1 lines b_1	Type 2 lines b_2
0	120	0	0
1	92	14	0
2	68	24	1
3	48	30	3
4	32	32	6
5	20	30	10
6	12	24	15

Thus, (15) becomes

$$5 = 380 - 78\delta_0 + 4\delta_0^2 - 2b_0 - 3(45 - 6\delta_0), \tag{17}$$

and we get the required form for the type 0 theta lines:

$$b_0 = 120 - 30\delta_0 + 2\delta_0^2 \tag{18}$$

□

We can now narrow down the types of quintics we look at. For sufficiently smooth quintics, there are enough type 0 theta lines to completely determine the curve via Bézout's theorem. This yields the theorem below.

Theorem 4.1. *Let X be a general δ_0-nodal quintic, $\delta_0 \leqslant 2$. Then X has the theta property.*

Proof. We proceed as before by using Bézout's theorem on the dual curve. Let X, X' be two δ_0-nodal quintics with $\theta(X) = \theta(X')$. Their dual curves $\check{X}$, $\check{X'}$ share the same $b_0 = 120 - 30\delta_0 + 2\delta_0^2$ nodes; thus they share at least $4b_0$ common points, counting multiplicities. On the other hand, the upper bound on the number of intersection points between $\check{X}$ and $\check{X'}$ is $(d^*)^2 = 400 - 80\delta_0 + 4\delta_0^2$. For $\delta_0 = 1, 2$, the number of common intersections given by the nodes exceeds this upper bound. Therefore $\check{X} = \check{X'}$ and $X = X'$, as required. □

Even if this method of proving the theta property breaks down for larger δ, Lemma 4.1 still gives us valuable information about the theta curve of $X \in V$. It is clear that $\theta(X)$ contains only theta lines of type 0, type 1, and type 2, which we will denote by b_0, b_1, and b_2 respectively. We have already computed b_0; b_2 is even more trivial, given by $\binom{\delta}{2}$. b_1 can be computed based on the Riemann-Hurwitz formula for the degree three projection to $\mathbb{P}^1$ from one of the nodes, giving $b_1 = 16\delta - 2\delta^2$. Table 2 displays the breakdown of the theta lines of all irreducible δ-nodal quintics.

Notice that $b_0 + b_1 + b_2$ is generally not equal to 120, like we may have expected. This is because type 1 and type 2 theta lines appear with higher multiplicities in $\mathrm{Sym}^{120}(\check{\mathbb{P}}^2)$. In fact, we claim that the type of the theta line completely determines its multiplicity.

Lemma 4.2. *Let $X \in V$. Type i theta lines of X have multiplicity 2^i in $\theta(X)$.*

The proof of this fact is given in Lemma 3.3.1 of [1]. Although they prove this in the case of quartics, they use local properties of the singularities to show this, so their proof is applicable for nodal quintics as well.

This lemma is crucial, in part because it implies that the nodes of a curve $X \in V$ are completely determined by $\theta(X)$. Indeed, we can right away determine the types of all the theta lines in $\theta(X)$ by counting their multiplicities; knowing the type 2 theta lines is the same as knowing the nodes of X.

Now we can finally get started on proving the non-trivial part. We begin with the worst case: the 6-nodal quintic. To complete this argument, we start with a lemma which will give us the needed bijection between 6-nodal quintics and their theta curves.

Lemma 4.3. *There are finitely many 6-nodal quintics which share the same theta lines.*

Proof. The type 2 theta lines of a nodal quintic determine its nodes, so any two 6-nodal quintics with the same theta lines share the same nodes. Let X be any 6-nodal quintic in V; label its nodes $n_1, \dots, n_6$. Now consider the blow-up of $\mathbb{P}^2$ at those six points:

$$\pi : \mathrm{Bl}_{n_1,\dots,n_6}(\mathbb{P}^2) \longrightarrow \mathbb{P}^2. \tag{19}$$

Let Bl $\pi^*\mathcal{O}(5)$ be a total transform of the pullback of $\mathcal{O}(5)$; the corresponding strict transform gives a complete linear series via

$$|\mathcal{O}(5) - 2E_1 - \cdots - 2E_6| : \mathrm{Bl}_{n_1,\dots,n_6}(\mathbb{P}^2) \longrightarrow \mathbb{P}^2. \tag{20}$$

Here $E_1, \dots, E_6$ denote the exceptional divisors corresponding to the nodes of X. This rational map sends all 6-nodal quintics in $|\mathcal{O}(5) - 2E_1 - \cdots - 2E_6|$ to lines in $\mathbb{P}^2$. Denote by L_X the image of X under (20). The type 1 theta lines of X are in $|\mathcal{O}(1) - E_i|$, and are sent to cubics in $|\mathcal{O}(3)|$ which remain tangent to L_X. Now, consider the unique conics through any of the five nodes of X, say $n_1, \dots, n_5$. The type 1 theta lines through n_i all intersect this conic in two distinct points. But under the map above, $|\mathcal{O}(2) - E_1 - \cdots - \widehat{E_i} - \cdots - E_5|$ is contracted to a point in $\mathbb{P}^2$, so the image of the theta lines through n_i intersect a point of $\mathbb{P}^2$ with multiplicity 2. Therefore the image of any type 1 theta line is a unique nodal cubic which is tangent to L_X. This gives us 24 nodal cubics tangent to the same line L_X in $\mathbb{P}^2$.

In the dual projective space $\check{\mathbb{P}}^2$ we have, by the degree formula (2), 24 plane quartics intersecting at a common point, namely the image of L_X. Now suppose that Y is another 6-nodal quintic in $|\mathcal{O}(5) - 2E_1 - \cdots - 2E_6|$ with $\theta(Y) = \theta(X)$. Denoting by L_Y its image under the rational map, we see that the type 1 theta lines of Y, previously sent to nodal cubics tangent to L_X, are also tangent to L_Y. In the dual space, we get a second point at which the 24 quartics also all meet. However, since two quartics intersect in at most 16 distinct points, there can only be at most 16 distinct 6-nodal quintics Y in $\mathbb{P}^2$ with $\theta(Y) = \theta(X)$. ☐

We now know that a general 6-nodal quintic shares its theta lines with finitely many other curves. The next step is to show that the only way you can have n curves with the same theta lines is if $n = 1$. To do this, we once again employ Zariski Connectedness.

Theorem 4.2. *The general 6-nodal quintic is determined by its bitangents.*

Proof. Consider the incidence correspondence J, defined in Lemma 3.1. In the proof of this lemma, we argued by the Zariski Connectedness Principle that the fibers of $\pi_1 : V \times \mathrm{Sym}^n(\check{\mathbb{P}}^2) \longrightarrow V$ are connected because J itself is connected and irreducible. The same rhetoric applies to $\pi_1 : V \times \mathrm{Sym}^n(\check{\mathbb{P}}^2) \longrightarrow \mathrm{Sym}^n(\check{\mathbb{P}}^2)$, so in particular, we can be sure that for any curve X in V there is a connected fiber of curves X_t in V with $\theta(X) = \theta(X_t)$. But by the previous lemma, the fiber of π_2 corresponding to a 6-nodal quintic is finite. Because the only connected finite sets are those with cardinality zero or one, we can be sure that X is the only curve with the theta curve $\theta(X)$. ☐

From here on out, we will proceed by deforming smooth quintics to the 6-nodal case. We will show that if there exists a deformation of a general δ-nodal quintic to a 6-nodal quintic, then the general δ-nodal quintic must have the theta property.

Theorem 4.3. *The general 5-nodal plane quintic has the theta property.*

Proof. Suppose the contradiction that the statement is false. Let $\mathcal{X} \longrightarrow T$ be a family over a smooth curve T with a distinguished point $t_0 \in T$ such that away from t_0, the fiber $X_t \in V$ has five nodes at fixed points $e_1, \ldots, e_5 \in \mathbb{P}^2$. The special fiber X_0 has, in addition to the five aforementioned nodes, a sixth node $e_0 \in \mathbb{P}^2$. By our assumption, we may suppose that X_t does not have the theta property; thus, we construct a second family $\mathcal{Y} \longrightarrow T$ with $Y_t \in V$ such that $\theta(X_t) = \Theta_t = \theta(Y_t)$.

First, notice that Y_t is also a five-nodal quintic with nodes at $e_1, \ldots, e_5$, with a sixth node e_0 at t_0. Notice also that the special fibers X_0 and Y_0 are equal, since they have the same theta curves and since they have the theta property. Denote by W the linear subspace of $\mathbb{P}^{20}$ consisting of quintics with singularities at $e_1, \ldots, e_5$. Then we may consider the map

$$\theta|_W : W \cap V \longrightarrow \mathrm{Sym}^{120}(\check{\mathbb{P}}^2). \tag{21}$$

We claim that $\theta|_W$ is an immersion at X_0. This implies that $X_t = Y_t$, a contradiction.

We start by considering the normalization of any point X, given by $\nu : X^\nu \longrightarrow X$. By Zariski [2], we have that

$$T_X W \cong H^0\left(X^\nu, \nu^*(\mathcal{O}_X(5) \otimes \mathcal{O}_X(-2e_1 - 2e_2 - 2e_3 - 2e_4 - 2e_5))\right). \tag{22}$$

If $\mathcal{L} = \nu^*(\mathcal{O}_X(5) \otimes \mathcal{O}_X(-2e_1 - 2e_2 - 2e_3 - 2e_4 - 2e_5))$, then $\deg \mathcal{L} = 25 - 4 \cdot 5 = 5$. Let l_1, l_2, l_3 be three theta lines of type 0 in Θ_0, and let W_{l_1,l_2,l_3} be the subset of curves in W bitangent to l_1, l_2, and l_3. If $p_1, p_2, \ldots, p_6 \in \mathbb{P}^2$ are the points of intersection of l_1, l_2, l_3 with X_0, then we have once again by Zariski's theory

$$T_{X_0} W_{l_1,l_2,l_3} = H^0(X^\nu, \mathcal{L} \otimes \mathcal{O}(-p_1 - p_2 - p_3 - p_4 - p_5 - p_6)). \tag{23}$$

In particular, since $\deg \mathcal{L} \otimes \mathcal{O}(-p_1 - \cdots - p_6) < 0$, we have that the tangent space $T_{X_0} W_{l_1,l_2,l_3} = 0$. Finally, observe that

$$T_{X_0} \theta|_W^{-1}(\Theta_0) \subset T_{X_0} W_{l_1,l_2,l_3} = 0, \tag{24}$$

so that $T_{X_0} \theta|_W^{-1}(\Theta_0) = 0$, and therefore $\theta|_W$ is an immersion at X_0, as required. □

We prove the 4-nodal case analogously.

Theorem 4.4. *A general 4-nodal quintic has the theta property.*

Proof. Define $\mathcal{X}, \mathcal{Y} \longrightarrow T$ as above, except that the common nodes are now $e_2, e_3, e_4, e_5 \in \mathbb{P}^2$. At the distinguished t_0, X_0 (and Y_0) are six-nodal quintics with two new nodes e_0, e_1. By the same argument as above, $X_0 = \Theta_0 = Y_0$. Let W be the linear subspace of $\mathbb{P}^{20}$ consisting of curves with singularities at e_1, e_2, e_3, e_4. We will show that $\theta|_W$ is an immersion at X_0.

As before, for any X and its normalization X^ν,

$$T_X W \cong H^0\left(X^\nu, \mathcal{L}\right) \tag{25}$$

where $\mathcal{L} = \nu^*(\mathcal{O}_X(5) \otimes \mathcal{O}_X(-2e_1 - 2e_2 - 2e_3 - 2e_4))$ has degree 9. Choose $l_1, \ldots, l_5$ to be theta lines of type 0 in Θ_0, tangent to X_0 at points $p_1, \ldots, p_{10}$. Then if $W_{l_1,\ldots,l_5}$ is the subset of curves in W bitangent to $l_1, \ldots, l_5$, then $\deg \mathcal{L} \otimes \mathcal{O}(-p_1 \cdots - p_{10}) < 0$, so

$$T_{X_0} W_{l_1,\ldots,l_5} = H^0(X^\nu, \mathcal{L} \otimes \mathcal{O}(-p_1 \cdots - p_{10})) = 0 \tag{26}$$

and therefore $\theta|_W$ is an immersion at X_0. □

Even the 3-nodal case follows essentially the same argument.

Theorem 4.5. *A general 3-nodal quintic has the theta property.*

Proof. We need a bit more preparation for this case. Suppose that the general 3-nodal quintic does not have the theta property. Fix two lines $L, K \in \check{\mathbb{P}}^2$, and consider a family of smooth quintics $\mathcal{X} \longrightarrow T$ such that away from a distinguished $t_0 \in T$, X_t has fixed nodes $e_3, e_4, e_5 \in \mathbb{P}^2$, and such that at t_0, X_0 has three additional nodes:

$e_1 \in L,\ e_2 \in K,\ e_0 = L \cap K$. For if $V_{L,K}$ is the subset of V whose quintics have L and K in their theta curves. We may consider the following (dominant) rational map:

$$V_{L,K} \dashrightarrow \mathrm{Sym}^2(L) \times \mathrm{Sym}^2(K) \tag{27}$$

which sends a quintic in $V_{L,K}$ to its two tangent points with L and K respectively. The general fiber of this map is a $\mathbb{P}^{12}$ of quintics in V, determined by fixing four tangent points. This shows that $V_{L,K}$ is a 16 dimensional irreducible subvariety of V, and our deformation is well-defined.

Thus suppose that X_t does not have the theta property away from t_0. Let $\mathcal{Y} \longrightarrow T$ be a second family with $\theta(X_t) = \theta(Y_t)$. Then as above, $X_0 = Y_0$, and Y_t is 3-nodal away from Y_0, with nodes at e_3, e_4, e_5. Let

$$\theta_{L,K} = \theta|_{V_{L,K}} : V_{L,K} \longrightarrow \mathrm{Sym}^{120}(\check{\mathbb{P}}^2). \tag{28}$$

We claim that $\theta_{L,K}$ is an immersion at X_0. Indeed, if X is tangent to L and K at $p_1, \ldots, p_4$, then

$$T_{X_0} V_{L,K} = H^0(X_0^\nu, \nu^* \mathcal{O}_{X_0}(5) \otimes \mathcal{O}_{X_0}(-e_0 - \cdots - e_5)) \tag{29}$$

where the relevant line bundle is denoted by $\mathcal{L}$, and has degree 13. Recall that Θ_0 has 12 theta lines of type 0. Let $l_1, \ldots, l_7$ be seven of them, tangent to X_0 at $p_1, \ldots, p_{14}$. Then if $V_{L,K,l_1,\ldots,l_7}$ is the subset of $V_{L,K}$ bitangent to $l_1, \ldots, l_7$,

$$T_{X_0} V_{L,K,l_1,\ldots,l_7} \subset H^0(X^\nu, \mathcal{L} \otimes \nu^* \mathcal{O}_{X_0}(-p_1 \cdots - p_{14})) = 0 \tag{30}$$

the second equation being an evident consequence of the negative degree of the line bundle. Finally,

$$T_{X_0} \theta_{L,K}^{-1}(\Theta_0) \subset T_{X_0} V_{L,K,l_1,\ldots,l_7} = 0 \tag{31}$$

implies that $\theta_{L,K}$ is an immersion at X_0, as required. □

5 Nodal Sextics

The final cases to consider are the nodal sextics.

A smooth sextic $S \in V^0$ has only type 0 theta lines, all 324 of which are distinct. As we showed before, S has the theta property. Thus, we are most interested in δ-nodal sextics, $\delta = 1, \ldots, 10$. As in the previous section, we resolve most cases with a structure lemma. This follows from the Plücker formulas, with a proof identical to that of Lemma 4.1.

Lemma 5.1. *A plane δ-nodal sextic has $324 - 50\delta + 2\delta^2$ theta lines of type 0.*

We can now use Bézout's theorem on the dual curve to great effect.

Theorem 5.1. *All δ_0-nodal cubics, $\delta_0 \leqslant 8$, have the theta property.*

Proof. Recall that a dual curve is determined by Bézout's theorem if $4b_0 > (d^*)^2$. In our case, $b_0 = 324 - 50\delta_0 + 2\delta_0^2$. For all values of $\delta_0 \leqslant 8$,

$$4(324 - 50\delta_0 + 2\delta_0^2) > (30 - 2\delta_0)^2. \tag{32}$$

□

As in the previous section, we know the number of type 2 theta lines to be $\binom{\delta}{2}$. Similarly, we can use Riemann-Hurwitz to to verify that the number of type 1 theta lines is $2\delta(13 - \delta)$. Table 3 summarizes the counts of the theta lines of nodal sextics.

Once again, theta lines of type 0, 1, and 2 appear with multiplicities 1, 2, and 4 respectively. We can therefore determine all the nodes of the relevant curves by their theta curves. We can now set up analogous arguments for the two cases which are not covered by Bézout's theorem. We start with the 10-nodal sextic.

Table 3: Theta Lines Counts of Nodal Sextics

Nodes δ	Type 0 lines b_0	Type 1 lines b_1	Type 2 lines b_2
0	324	0	0
1	276	24	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$
8	52	80	28
9	36	72	36
10	24	60	45

Theorem 5.2. *An irreducible 10-nodal sextic has the theta property.*

Proof. Two distinct 10-nodal quintics sharing the same theta curves also share the same nodes. Thus, they intersect in 40 points, counting by multiplicity. But by Bézout's theorem, they may intersect at only 36 points, a contradiction. □

Now all that is left is the 9-nodal sextic. By now, the proof scheme should be quite clear. We will outline it one last time.

Theorem 5.3. *A general 9-nodal plane sextic has the theta property.*

Proof. Assume the contradiction that the statement above is false. Construct two families $\mathcal{X}, \mathcal{Y} \longrightarrow T$ of sextics which share the same theta curves and which share the same nine nodes $e_1, \ldots, e_9$, such that they converge to 10-nodal sextics X_0, Y_0. By the theorem above, $X_0 = Y_0$. Denote the tenth node e_{10}. Let W be the subspace of $\mathbb{P}^{27}$ consisting of sextics with nodes at $e_1, \ldots, e_9$. The map $\theta|_W$ to $\mathrm{Sym}^{324}(\check{\mathbb{P}}^2)$ will be shown to be an immersion, thereby proving the theorem.

To any X in $\mathcal{X}$ assign its normalization X^ν; we have that

$$T_X W \cong H^0(X^\nu, \nu^*(\mathcal{O}_X(5) \otimes \mathcal{O}_X(-2e_1 - \cdots - 2e_9))). \tag{33}$$

If $\mathcal{L} = \nu^*(\mathcal{O}_X(5) \otimes \mathcal{O}_X(-2e_1 - \cdots - 2e_9))$, then $\deg \mathcal{L} = 36 - 36 = 0$. Let l be any theta line of type 0 in the theta curve of X_0; let W_l be the subset of W bitangent to l. If $p_1, p_2 \in \mathbb{P}^2$ are the points of tangency of l with X_0, we have

$$T_{X_0} W_l = H^0(X^\nu, \mathcal{L} \otimes \mathcal{O}(-p_1 - p_2)) = 0. \tag{34}$$

Thus $T_{X_0}\theta|_W^{-1}(\theta(X_0)) \subset T_{X_0} W_l = 0$, so $\theta|_W$ is an immersion at X_0. □

6 Questions

We have proven that a general irreducible curve of arbitrary degree greater than 3 and with arbitrarily many nodes has the theta property. This is, however, far weaker than the desired result for irreducible curves. Thus, we pose the following questions.

- *Given an irreducible plane curve with a fixed number of nodes, do we know that it has the theta property?* Theorem 2.2, Theorem 4.1, and Theorem 5.2 do not make any assumption of starting with a general nodal curve. They are based on Bézout's theorem, and hold regardless of your choice of curve. By the same argument, we know that an irreducible nodal plane curve of degree greater than 6 must have the theta property. However, our proofs for some of the special quintic and sextics, which we in part adopted from Caporaso and Sernesi's proofs for quartics, assume that you start with a general δ-nodal curve of degree d. It would be best if the proofs were modified in such a way that we did not need the given nodal curve to be general. Alternatively, if it turns out that there are irreducible curves which do not have the theta property, the degree and number of nodes may only belong to a finite list. (See Table 4.)

- *Do all of these reducible curves have the theta property?* By restricting ourselves to the case of irreducible curves, we have missed many interesting cases. For example, Caporaso and Sernesi prove that the quartic consisting of two conics has the theta property when the two conics are distinct. Similarly, one can consider the union of a conic with a cubic, the union of two cubics, and so on.

- *Are all irreducible curves with fixed singularity types determined by their bitangents?* There are many types of singularities for curves. In this paper, we have considered nodal curves, but we can consider worse singularities, such as cusps and tacnodes. For example, a general plane sextic of genus three has the theta property, but Bézout's theorem does not guarantee this for a sextic with four nodes and three cusps.

- *Which curves are determined by their type 0 theta lines?* We used the term 'theta line' to generalize a bitangent for a singular curve. However, one can make the case that a theta line of type greater than zero is not a true bitangent. This problem can be examined by fixing points on the dual curve as we did in Theorem 2.2 and the like. Are there two curves which share the same type zero theta lines but are not equal?

- *Given two curves of degree d, how many bitangents can they share before they are guaranteed to be equal?* If we can restrict our focus to just bitangents, can we perhaps restrict our focus to a subset of bitangents? In particular, this generalizes to the following question:

- *Is there ever a finite number of curves of degree d sharing the same k bitangents, or does the number jump from an infinite family to one unique curve?*

  When we chose which divisors of the dual curve to look at, we made the debatable decision to examine the nodes; this gave us a collection of lines in $\mathbb{P}^2$ which we initially called the bitangents. However, we can just as easily look at another class of divisors. For example, let C be a smooth cubic in $\mathbb{P}^2$. $\check{C}$ is determined by the Plücker formulas (2) and (9) to be a sextic with nine cusps and zero nodes. Hence, C has nine flex lines. This gives the following natural question:

- *Do the flex lines of a cubic curve determine it? If not, are there two cubics with the same configuration of flex lines?*

  A nodal cubic C' can be examined in the same way. Its dual curve $\widecheck{C'}$ is a three-cuspidal quartic; thus, C' has three distinct flex lines.

- *Do the three flex lines of C' determine it, or is there another nodal cubic which has the same flex lines? By the same reasoning, as cuspidal cubic has a single unique flex line; how many other cuspidal cubics share this flex line?*

- *Can two curves of degree d share the same bitangent conics? How many bitangent conics does it take to determine a unique curve of degree d?* After all, we do not even need to restrict ourselves to tangent lines. Steiner's conic problem asks for the number of smooth conics tangent to five general conics in $\mathbb{P}^2$; the answer comes out to 3264, a large but finite number. Perhaps some part of this problem can be adopted to show that the tangent (or bitangent) conics to a curve determine it.

  In full generality, this problem can be extended to surfaces in $\mathbb{P}^n$. We know by duality that a surface is determined by its tangent hyperplanes; any such hyperplane intersects the surface in a singular curve. As such, we may consider the Severi variety of nodal curves on the surface.

- *To what extent does this variety determine the surface?*

Degree d	Nodes δ
4	0
4	1
4	2
5	3
5	4
5	5
6	9

Table 4: Curves which may fail to have the theta property

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