Perverse Sheaves and Intersection Cohomology

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These are notes I wrote up for my own comprehension while reading the text of Hotta– Takeuchi–Tanisaki.

1 T-Structures

Definition 1. Let \mathcal{D} be a triangulated category, and $\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0}$ its full subcategories. Set $\mathcal{D}^{\leq n} = \mathcal{D}^{\leq 0}[n]$. The pair $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ defines a t-structure on \mathcal{D} if

- $\mathcal{D}^{\leqslant -1} \subseteq \mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 1} \subseteq \mathcal{D}^{\geqslant 0}$
- For all $X \in \mathcal{D}^{\leq 0}$ and $Y \in \mathcal{D}^{\geq 1}$ we have $\operatorname{Hom}_{\mathcal{D}}(X, Y) = 0$.
- For any $X \in \mathcal{D}$ there exists an exact triangle

$$X_0 \to X \to X_1 \stackrel{+1}{\to} \cdots \tag{1}$$

such that $X_0 \in \mathcal{D}^{\leq 0}$ and $X_1 \in \mathcal{D}^{\geq 1}$.

The full subcategory $\mathcal{C} = \mathcal{D}^{\geq 0} \cap \mathcal{D}^{\leq 0}$ is called the heart of the t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$.

The heart of a t-structure allows us to get a good grasp on the ambient derived category.

Theorem 1. $\mathcal{C} = \mathcal{D}^{\geq 0} \cap \mathcal{D}^{\leq 0}$ is an Abelian category, and any exact sequence

$$0 \to X \to Y \to Z \to 0$$

gives rise to a distinguished triangle

$$X \to Y \to Z \stackrel{+1}{\to} \cdots$$

in \mathcal{D} .

This allows us to define the functor H^0 , which sends our derived category to its heart.

Definition 2. The functor

$$H^0: \mathcal{D} \to \mathcal{C} = \mathcal{D}^{\geq 0} \cap \mathcal{D}^{\leq 0}, \qquad X \mapsto \tau^{\leq 0} \tau^{\geq 0} X.$$

Likewise, define $H^n(X) = H^0(X[n])$.

Now let $F : \mathcal{D}_1 \to \mathcal{D}_2$ be a functor of triangulated categories. Denote by $\mathcal{C}_1, \mathcal{C}_2$ the hearts of the respective categories.

Definition 3. The additive functor

$${}^{p}F: \mathcal{C}_{1} \to \mathcal{C}_{2}, \qquad X \mapsto H^{0}(F(X)).$$

We say that ${}^{p}F$ is left t-exact if $F(\mathcal{D}_{1}^{\geq 0}) \subseteq \mathcal{D}_{2}^{\geq 0}$, and analogously for right t-exactness.

2 Perverse Sheaves

Now we define a t-structure on the derived category of constructable sheaves. Recall that we have a Verdier duality functor $\mathbf{D}_X : D^b_{con}(X)^{op} \to D^b_{con}(X)$.

Definition 4. For $F^{\bullet} \in \mathcal{D}^b_{con}(X)$, we have $F^{\bullet} \in \mathcal{D}^{\leq 0}_{con}(X)$ if and only if

 $\dim \operatorname{supp}(H^j(F^{\bullet})) \leqslant -j$

for all $j \in \mathbb{Z}$. Likewise, we have $F^{\bullet} \in \mathcal{D}_{con}^{\geq 0}(X)$ if and only if

$$\dim \operatorname{supp}(H^j(\mathbf{D}_X F^{\bullet})) \leqslant -j$$

for all $j \in \mathbb{Z}$.

It is not obvious that this is indeed a t-structure, so it must be stated seperately.

Theorem 2. The pair $(\mathcal{D}_{con}^{\leq 0}, \mathcal{D}_{con}^{\geq 0})$ forms a t-structure.

Definition 5. We call the heart of this t-structure $\operatorname{Perv}(\mathbb{C}_X)$ the perverse sheaves on X. The functor \mathbf{D}_X induces a duality of $\operatorname{Perv}(\mathbb{C}_X)$. Now, for any functor $F : \mathcal{D}^b_{con}(X) \to \mathcal{D}^b_{con}(Y)$, denote by pF the composition ${}^pH^0 \circ F$, viewed as a functor $\operatorname{Perv}(\mathbb{C}_X) \to \operatorname{Perv}(\mathbb{C}_Y)$. In particular, we get the associated functors

$${}^{p}f^{-1}, {}^{p}f^{!}: \operatorname{Perv}(\mathbb{C}_{Y}) \to \operatorname{Perv}(\mathbb{C}_{X}),$$

and

$${}^{p}Rf_{*}, {}^{p}Rf_{!} : \operatorname{Perv}(\mathbb{C}_{X}) \to \operatorname{Perv}(\mathbb{C}_{Y}).$$

Next, we define minimal extensions of perverse sheaves. Let $X = Z \sqcup U$ with $i : Z \to X$ proper and $j : U \to X$ open. For $F^{\bullet} \in \mathcal{D}^{b}_{con}(X)$, we have natural maps

$$j_!F^\bullet \to Rf_*F^\bullet, \qquad s \mapsto s,$$

mapping a section to itself. If F^{\bullet} is a perverse sheaf on U, we also get maps

$${}^{p}j_{!}F^{\bullet} \to {}^{p}f_{*}F^{\bullet}. \tag{2}$$

Definition 6. We denote by ${}^{p}f_{!*}F^{\bullet}$ the image of (2), and call it the minimal extension of $F^{\bullet} \in \operatorname{Perv}(\mathbb{C}_{U})$.

In other words, we have maps

$${}^{p}j_{!}F^{\bullet} \twoheadrightarrow {}^{p}j_{!*}F^{\bullet} \hookrightarrow {}^{p}j_{*}F^{\bullet}.$$

$$\tag{3}$$

Theorem 3. Let $F^{\bullet} \in Perv(\mathbb{C}_U)$ as above. Then

- (i) ${}^{p}f_{*}F^{\bullet}$ has no non-trivial sub-objects whose support is contained in Z.
- (ii) ${}^{p}j_{!}F^{\bullet}$ has no non-trivial quotient object whose support is contained in Z.
- (iii) ${}^{p}j_{!*}F^{\bullet}$ has neither sub- or quotient objects whose support is contained in Z.

Now suppose that F^{\bullet} is a simple object in $\operatorname{Perv}(\mathbb{C}_U)$. We claim that ${}^p j_{!*}F^{\bullet}$ is also simple in $\operatorname{Perv}(\mathbb{C}_X)$. Indeed, for any subobject $G^{\bullet} \subseteq {}^p j_{!*}F^{\bullet}$ we may consider the exact sequence

$$0 \to G^{\bullet} \to {}^{p}j_{!*}F^{\bullet} \to H^{\bullet} \to 0.$$
(4)

Since $j^! = j^{-1}$ is exact here, we may apply it to obtain

$$0 \to j^{-1}G^{\bullet} \to F^{\bullet} \to j^{-1}H^{\bullet} \to 0.$$
(5)

Since F^{\bullet} is simple, $j^{-1}G^{\bullet}$ or $j^{-1}H^{\bullet}$ is zero, or in other words, one of G^{\bullet} and H^{\bullet} is supported in Z. By the previous theorem, this makes it zero.

Theorem 4. $G = {}^{p}j_{!*}F^{\bullet}$ is the unique perverse sheaf satisfying the conditions

- (i) $G^{\bullet}|_U \simeq F^{\bullet}$,
- (*ii*) $i^{-1}G^{\bullet} \in {}^{p}\mathcal{D}_{con}^{\leqslant -1}(Z),$
- (*iii*) $i^! G^{\bullet} \in {}^p \mathcal{D}_{con}^{\geq 1}(Z).$

3 Intersection Cohomology

Let X be an irreducible complex projective variety (or an irreducible compact analytic space) of dimension d. We may start by defining the intersection cohomology complex in the following way.

Definition 7. Take $U \subseteq X^{\text{reg}}$ to be a Zariski open subset of the smooth locus of X. For $\underline{\mathbb{C}}_{U}[d]$ the constant perverse sheaf on U, define IC_{X}^{\bullet} to be a minimal extension to all of X.

We may explicitly construct IC_X^{\bullet} . Fix a Whitney stratification $X = \bigsqcup_{\alpha} X_{\alpha}$,

$$X_k = \prod_{\dim X_\alpha \leqslant k} X_\alpha, \quad U_k := X \smallsetminus X_{k-1}, \tag{6}$$

with each $X_k \setminus X_{k-1}$ a smooth k-dimensional complex manifold. We have a family of maps $j_k : U_k \hookrightarrow U_{k-1}$, for $k = 1, \ldots, d$. Altogether, this fits into

 $\emptyset \hookrightarrow U_d \hookrightarrow U_{d-1} \hookrightarrow \dots \hookrightarrow U_1 \hookrightarrow X. \tag{7}$

Theorem 5. IC_X^{\bullet} is quasi-isomorphic to the complex

$${}^{p}j_{!*}(\mathbb{C}_{U}[d_{X}]) \simeq \left(\tau^{\leqslant -1}Rj_{1*}\right) \circ \left(\tau^{\leqslant -2}Rj_{2*}\right) \circ \cdots \left(\tau^{\leqslant -d}Rj_{d*}\right) \left(\underline{\mathbb{C}}_{U}[d]\right).$$

$$\tag{8}$$

Proof. Suppose that F^{\bullet} is a perverse sheaf on U_k whose restriction to any strata $X_{\alpha} \subseteq U_k$ has locally constant \mathscr{H}^i . We show that $G^{\bullet} := \tau^{\leq -k} R j_{k*}(F^{\bullet})$ satisfies the unique characterization of a minimal extension from Theorem 4. First, note that U_k consists of strata of dimension $\geq k$, so that $\mathscr{H}^r(F^{\bullet}) = 0$ for r > -k. In particular,

$$\left[\tau^{\leqslant -k} R j_{k*} F^{\bullet}\right]|_{U_k} \simeq F^{\bullet},$$

so that what we have is indeed an extension of F^{\bullet} , satisfying (i). Next, set $Z := U_{k-1} \setminus U_k = \bigsqcup_{\dim X_{\alpha}=k-1} X_{\alpha}$. Denote by $i: Z \to U_k$ the associated closde embedding. Then $i^{-1}G^{\bullet}$ has locally constant cohomology sheaves on each $X_{\alpha} \subseteq Z$, so that $\mathscr{H}^r(i^{-1}G^{\bullet}) = 0$ for r > -k. This implies that $i^{-1}G^{\bullet} \in {}^p\mathcal{D}_{con}^{\leqslant -1}$, thus satisfying condition (ii). Finally, consider the triangle

$$G \to Rj_{k*}F^{\bullet} \to \tau^{\geq -k+1}Rj_{k*}F^{\bullet} \xrightarrow{+1} \cdots$$
 (9)

Applying $i^{!}$, we note that the middle vanishes. This gives us isomorphisms

$$i^!G^\bullet \simeq i^!(\tau^{\ge -k+1}Rj_{k*}F^\bullet)[-1]$$

so that $\mathscr{H}^r(i^!G^{\bullet}) = 0$ for r < -k. But since $i^!G$ has locally constant cohomology on $X_{\alpha} \subseteq Z$, we get $i^!G \in {}^p\mathcal{D}_{con}^{\geq 1}(Z)$, proving (iii). \Box

Importantly, the intersection cohomology complex is self-dual in the way we expect of perverse sheaves.

Theorem 6. $IC_X^{\bullet} \simeq \mathbb{D}_X(IC_X^{\bullet})$. Furthermore, there exist canonical morphisms

$$\underline{\mathbb{C}}_X \to IC_X^{\bullet}[-d] \to \omega_X^{\bullet}[-2d].$$
(10)

Proof. We have an isomorphism in the derived category of constructible sheaves from above

$$\tau^{\leqslant -d} \, {}^{p}j_{!*}\underline{\mathbb{C}}_{X}[d] \simeq Rj_{1*} \circ \cdots \circ Rj_{d*} \left(\underline{\mathbb{C}}_{X}[d]\right) \simeq (j_{*}\mathbb{C})[d]. \tag{11}$$

This is none other than the intersection complex of X, and we see that it admits a canonical map from $\underline{\mathbb{C}}_X$. The second map in the composition comes from taking the Verdier dual, $IC^{\bullet}_X[d] \to \omega^{\bullet}_X$.

With the intersection complex in place, we are ready to define intersection cohomology.

Definition 8. For $i \in \mathbb{Z}$, we define

$$IH^{i}(X) = H^{i}\left(R\Gamma\left(X, IC_{X}^{\bullet}[-d]\right)\right), \qquad (12)$$

$$IH_c^i(X) = H^i\left(R\Gamma_c\left(X, IC_X^{\bullet}[-d]\right)\right).$$
(13)

Importantly, the intersection cohomology of X satisfies Poincare duality.

Theorem 7. Let X be irreducible of dimension d. Then

$$IH^{i}(X) \simeq \left[IH^{2d-i}_{c}(X)\right]^{*}.$$
(14)

Proof. Let $a_X : X \to \{p\}$ be the unique map to a point. We have

$$RHom_{\mathbb{C}}(Ra_{X!}IC^{\bullet},\mathbb{C}) \simeq Ra_{X*}RHom_{\mathbb{C}_X}(IC^{\bullet}_X,\omega^{\bullet}_X).$$
(15)

by Verdier duality. But IC_X^{\bullet} is self-dual, so that

$$R\mathrm{Hom}_{\mathbb{C}_X}(IC_X^{\bullet},\omega_X^{\bullet}) = \mathbf{D}_X(IC_X^{\bullet}) = IC_X^{\bullet}$$

Thus we get an isomorphism

$$[R\Gamma_c(X, IC_X^{\bullet})]^* \simeq R\Gamma(X, IC_X^{\bullet}).$$
(16)

Example 1. Let X be an irreducible complex projective variety with isolated singular points p_1, \ldots, p_k . Then it suffices to consider the stratification

$$X = \{p_1, \ldots, p_k\} \sqcup X^{\operatorname{reg}}.$$

Then we have $X_0 = \cdots = X_{d-1} = \{p_1, \ldots, p_k\}$, and $X_d = X$. Thus, we get the inclusions of complex manifolds

$$\emptyset \to U_d = X^{\operatorname{reg}} = U_1 \xrightarrow{j_1} X_d$$

The only interesting map in this stratification is j_1 , the inclusion of the smooth locus. In particular, by Theorem 5,

$$IC_X^{\bullet} \simeq \tau^{\leqslant -1} \left(Rj_{1*} \underline{\mathbb{C}}_{U_1} \right).$$

Now, associated to truncation is the exact couple

$$IC_X^{\bullet}[-d] \to Rj_{1*}\underline{\mathbb{C}}_{U_1} \to \tau^{\geq d} \left(Rj_{1*}\underline{\mathbb{C}}_{U_1} \right) \stackrel{+1}{\to} \cdots$$
(17)

But now we can apply $R\Gamma$ and take cohomology. For $0 \leq i < d$, we have $IH^i(X) = H^i(X, Rj_{1*}\overline{\mathbb{C}}_{U_1}) = H^i(X^{\operatorname{reg}}, \mathbb{C})$. For i = d, we do not get a clear vanishing on the right, but the map $IH^d(X) \to H^d(X^{\operatorname{reg}}, \mathbb{C})$ is injective. To analyze this case more carefully, consider the canonical morphism $\underline{\mathbb{C}}_X \to IC^{\bullet}_X[-d]$ guarenteed by Theorem 6.

We may associate to this natural map a new exact couple

$$\underline{\mathbb{C}}_X \to IC_X^{\bullet}[-d] \to F^{\bullet} \xrightarrow{+1} \cdots$$
(18)

Here F^{\bullet} is a constructable sheaf supported only on the zero-dimensional closed subset $\{p_1, \ldots, p_k\} = X_0$. In particular, $H^i(F^{\bullet}) = 0$ for all $i \ge d$. Thus, we get $IH^i(X) = H^i(X, \mathbb{C})$ for $d < i \le 2d$. After i = 2d, the cohomology of $\underline{\mathbb{C}}_X$ vanishes, and with it $IH^i(X)$. Summarizing,

$$IH^{i}(X) = \begin{cases} H^{i}(X^{\text{reg}}, \mathbb{C}) & : 0 \leq i < d\\ \inf(H^{i}(X, \mathbb{C}) \to H^{i}(X^{\text{reg}}, \mathbb{C})) & : i = d\\ H^{i}(X, \mathbb{C}) & : d < i \leq 2d\\ 0 & : \text{ otherwise} \end{cases}$$
(19)