

# THE MONODROMY OF COMPACT LAGRANGIAN FIBRATIONS

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**ABSTRACT.** We study the monodromy representations underlying compact Lagrangian fibrations. In the case where the associated period map is generically immersive, we prove that the monodromy representation is irreducible over  $\mathbb{C}$ . In the alternative case where the fibration is isotrivial, we recover a result of [KLM23], proving that its fibers are isogeneous to a power of an elliptic curve. We show that over  $\mathbb{C}$ , the monodromy representation underlying an isotrivial Lagrangian fibration is a direct sum of two irreducible  $\mathbb{C}$ -local systems.

## 1. INTRODUCTION

The Beauville–Bogomolov decomposition theorem states that any compact Kähler manifold  $X$  with  $K_X$  trivial is up to isogeny a product of complex tori, strict Calabi–Yau manifolds, and hyperkähler manifolds. In recent years, the latter of the three have become increasingly popular. For a survey, see for instance [Deb22].

**Definition 1.1.** Let  $X$  be a simply-connected complex manifold. We say that  $X$  is hyperkähler if  $H^0(X, \Omega_X^2) = \mathbb{C}\sigma$  for some nowhere degenerate holomorphic 2-form  $\sigma$ .

Since  $\sigma$  is symplectic,  $X$  must have even complex dimension  $2n$ . We may hope to understand the geometry of a hyperkähler by expressing it as the total space of a fibration. To that end, consider the following definition.

**Definition 1.2.** Let  $f : X \rightarrow B$  be a holomorphic map from a hyperkähler to a normal analytic variety of dimension  $0 < \dim B < 2n$ . We say that  $f$  is a fibration if it is proper and surjective with connected fibers. If for some fiber  $X_b \subseteq X$  we have  $\sigma|_{X_b} = 0$ , we say that  $X_b$  is isotropic. If every fiber of  $f$  is Lagrangian (isotropic and pure  $n$ -dimensional) we say that  $f$  is a Lagrangian fibration.

While the condition of being Lagrangian may seem quite strict, it turns out to be forced, by [Mat99]. In fact, if  $f : X \rightarrow B$  is a fibration, then  $f$  is Lagrangian and  $B$  is projective; see, for instance, [GL14, Proposition 2.2]. Of special note are the smooth fibers of  $f$ ; by [Mat99], they are all Abelian varieties. Additionally, Matsushita establishes many constraints on the base of a Lagrangian fibration. In particular,  $B$  is  $\mathbb{Q}$ -Fano,  $\mathbb{Q}$ -factorial, and log terminal [Mat15].  $B$  has Picard rank 1, and has the same  $\mathbb{Q}$ -intersection cohomology as  $\mathbb{P}^n$ . This fact holds even on the level of Hodge decompositions, in that every non-zero cohomology class on  $B$  is Hodge. A large conjecture of Matsushita predicts that  $B \simeq \mathbb{P}^n$  for every Lagrangian fibration  $f : X \rightarrow B$ . This result is known to be true in several cases, including when  $B$  is a curve or a surface [HX22]. In fact, Hwang proves in [Hwa08] that  $B \simeq \mathbb{P}^n$  whenever  $B$  is smooth.

Denote by  $f : X^\circ \rightarrow B^\circ$  the smooth fibers of  $f$ . We have an associated period map  $B^\circ \rightarrow \Gamma \backslash \mathcal{D}$ , mapping each Abelian variety to its associated weight one Hodge structure. By Torelli, the smooth fibers of  $f$  are completely determined by the associated polarized variation of Hodge structures  $V_{\mathbb{Q}} = R^1 f_* \mathbb{Q}_{X^\circ}$ . Thus, a common approach for understanding

the geometry of  $f$  is understanding the structure of local systems underlying variations of Hodge structures on  $B$ . [Bak22] affirms a conjecture of Matsushita, proving that the period map associated to  $f$  is either constant or generically immersive. In the first case, we say that  $f$  is **isotrivial**; in the second,  $f$  is **of maximal variation**. Voisin proves that  $V_{\mathbb{R}} = V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R}$  is irreducible as a variation of Hodge structures [Voi18]. To further strengthen this result, we consider the isotrivial and maximal variation cases separately.

**Theorem 1.3** (=Corollary 4.5). *Suppose that  $V_{\mathbb{Q}}$  is the local system associated to the variation of Hodge structures arising from a maximal variation Lagrangian fibration. Then  $V_{\mathbb{C}} = V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}$  is irreducible as a representation of  $\pi_1(B^{\circ})$ .*

As will be evident from the proof of the previous theorem, the assumption that  $f$  is of maximal variation is critical. Not only is there no analogous result in the isotrivial case, we prove that the  $\mathbb{C}$ -local system underlying an isotrivial Lagrangian fibration is never irreducible! The isotrivial case was studied in depth by Kim, Laza, and Martin [KLM23]; we recover their structure theorem [KLM23, Theorem 1.3].

**Theorem 1.4** (=Theorem 5.2). *Suppose that  $f : X \rightarrow B$  is isotrivial. Then there is an elliptic curve  $E$  such that  $E^n$  is isogenous to the smooth fiber  $X_b$  of  $f$ .*

This allows us to understand the splitting behaviour of  $V_{\mathbb{C}}$  into irreducible  $\mathbb{C}$ -local systems. It turns out that any splitting of  $V_{\mathbb{C}}$  occurs over a finite extension field  $K$  of  $\mathbb{Q}$ , which we identify with the CM field of the associated elliptic curve.

**Theorem 1.5** (=Proposition 5.4). *Let  $f : X \rightarrow B$  be an isotrivial Lagrangian fibration, and let  $V_{\mathbb{Q}} \simeq E^n$  as in Theorem 5.2. Then if  $K = \text{End}_{\text{QHS}}(E)$ , there is a splitting of  $V_K$  into  $K$ -local systems of rank  $n$ ,  $V_K = U_1 \oplus U_2$ , such that  $U_1$  and  $U_2$  are irreducible over  $\mathbb{C}$ . Furthermore,  $U_1 \simeq U_2$  if and only if they are defined over  $\mathbb{Q}$ .*

Finally, we relate our results to an alternative classification of isotrivial Lagrangian fibrations found in [KLM23] to produce a more geometric description of the irreducible isotrivial  $V_{\mathbb{C}}$ . This extends the discussion [KLM23, Remark 1.8] expressing the CM structure of the elliptic curve  $E$  to a trivializing cover of the local system  $V_{\mathbb{C}}$ .

**Proposition 1.6** (=Proposition 5.11). *Suppose that  $V_{\mathbb{C}}$  is an irreducible isotrivial complex variation of polarized Hodge structures associated to  $f : X \rightarrow B$ . Then a minimal trivializing cover  $C^{\circ} \rightarrow B^{\circ}$  of  $V_{\mathbb{C}}$  compactifies to an Abelian torsor  $A \supseteq C^{\circ}$ .*

*Remark 1.7.* Suppose further that  $f$  admits a rational section, and that for every Lagrangian fibration  $X' \dashrightarrow B'$  birational to  $f$  the conjecture of Matsushita holds (i.e.  $B' \cong \mathbb{P}^n$ ). Then as an immediate consequence of [KLM23, Theorem 1.8],  $f$  is birational to a  $\text{K3}^{[n]}$ -fibration or a  $\text{Kum}_n$ -fibration.

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**Notation and conventions.** Throughout this paper, we mainly stick to the following notations:

- $X$  a (compact) hyperkähler manifold of dimension  $2n$  with symplectic form  $\sigma$
- $f : X \rightarrow B$  a Lagrangian fibration
- $B^{sm}$  the smooth locus of  $B$ ,  $B^{\circ} \subseteq B^{sm}$  the regular values of  $f$

- $j : B^{sm} \rightarrow B$  the natural inclusion,  $\Omega_B^{[k]} = j_* \Omega_{B^{sm}}^k$  the sheaf of reflexive differentials
- $f : X^\circ \rightarrow B^\circ$  the restriction of  $f$  to the smooth fibers
- $V_{\mathbb{Q}}$  the local system  $R^1 f_* \mathbb{Q}_{X^\circ}$  on  $B^\circ$
- $(V_{\mathbb{Q}}, (\mathcal{V}, \nabla), F^\bullet \mathcal{V}, q)$  the associated variation of Hodge structures, where
- $\mathcal{V} = V \otimes_{\mathbb{Q}} \mathcal{O}_{B^\circ}$  is a bundle equipped with flat connection  $\nabla$
- $F^\bullet$  a decreasing filtration of  $\mathcal{O}_{B^\circ}$ -modules on  $\mathcal{V}$
- $q : V_{\mathbb{Q}} \otimes V_{\mathbb{Q}} \rightarrow \mathbb{Q}_{B^\circ}(-1)$  a polarization form
- $\mathcal{V}^{>-1}$  Deligne's canonical extension to  $\mathcal{V}$  to  $B^{sm}$ , with residue eigenvalues in  $(-1, 0]$
- $V_K = V_{\mathbb{Q}} \otimes_{\mathbb{Q}} K$  for  $\mathbb{Q} \subseteq K \subseteq \mathbb{C}$  a field extension
- $U \subseteq V_{\mathbb{C}}$  an irreducible  $\mathbb{C}$ -local system
- $W \subseteq V_{\mathbb{R}}$  an  $\mathbb{R}$ -local system

## 2. STRUCTURE THEORY OF COMPLEX REPRESENTATIONS

Before we begin our discussion of complex representation arising from real representations, we review complex variations of Hodge structures. As with rational or real variations, we have two descending filtrations  $F^\bullet$  and  $\overline{F}^\bullet$  on  $V$  a  $\mathbb{C}$ -local system, holomorphic and anti-holomorphic respectively. For a clean summary, see [Bak22]. The key difference between the more conventional notions of variations of Hodge structure comes from the lack of bigrading; since there is no longer a clear notion of complex conjugation,  $V$  only admits a (mono)grading  $V^p := F^p V \cap \overline{F}^{-p} V$ .

The key invariant for complex variations is the level; this is the difference between the maximum and minimum  $p$  for which  $V^p \neq 0$ . For example,  $\mathbb{C}(-p)$  is level zero, while the variation associated to a family of elliptic curves is level one. Notice that if  $V$  is level  $l$  and  $V'$  is level  $l'$ , the tensor product  $V \otimes V'$  is level  $l + l'$ .

Drawing upon the theorem of the fixed part [Sch73], Deligne proves the following crucial structure theorem for complex variations of Hodge structure.

**Proposition 2.1** ([Del87], Proposition 1.13). *Suppose  $V$  is a  $\mathbb{C}$ -local system underlying a polarized variation of Hodge structures. We have a splitting of  $\mathbb{C}$ -local systems*

$$(2.1) \quad V = \bigoplus_i B_i \otimes U_i$$

where the  $U_i$  are irreducible and pairwise non-isomorphic  $\mathbb{C}$ -local systems, and the  $B_i$  are  $\mathbb{C}$ -vector spaces. Moreover,

- (i) Each  $U_i$  underlies a polarized complex variation of Hodge structures, unique up to shifting the Hodge grading, and
- (ii) Each polarizable complex variation of Hodge structure arises from (i) by equipping each  $U_i$  with its unique (up to shifts) polarizable complex variation of Hodge structures and each  $B_i$  with a polarizable complex Hodge structure coming from [Sch73], namely  $B_i = \text{Hom}(V, U_i)$ .

The subsequent observation made in [Bak22] is that if  $V$  is a  $\mathbb{C}$ -local system equipped with a polarized complex variation of Hodge structures, then  $V = W_{\mathbb{C}}$  for some polarized real variation of Hodge structures  $W$  if and only if complex conjugation of  $V$  flips the Hodge grading. Thus, every local system arising from a complex polarizable variation of Hodge structure is semi-simple.

This allows us to draw on the following classification theorem, which is a classical result of representation theory. Despite the fact that this theorem is well known, we could not find a satisfactory reference for representations of (non-finite) groups, so we have included a proof below.

**Proposition 2.2.** *If  $V_{\mathbb{R}}$  is an irreducible real representation underlying a polarizable variation of Hodge structures, then the isotypic factors of  $V_{\mathbb{C}}$  coming from Proposition 2.1 have the following distinct forms:*

- (R)  $B \otimes W_{\mathbb{C}}$ , for  $W$  an irreducible  $\mathbb{R}$ -local system,
- (C)  $(B \otimes U) \oplus (\overline{B} \otimes \overline{U})$  for  $U$  an irreducible  $\mathbb{C}$ -local system,
- (Q)  $B \otimes U$ , where  $U^{\oplus 2} = W_{\mathbb{C}}$  for  $W$  an irreducible  $\mathbb{R}$ -local system.

In the first case, we say that  $V_{\mathbb{C}}$  is of **real type**; in the second,  $V_{\mathbb{C}}$  is of **complex type**; and in the third case,  $V_{\mathbb{C}}$  is of **quaternionic type**.

*Proof.* Suppose that  $U$  is a complex irreducible factor as above. If  $U \neq \overline{U}$ , then somewhere in the factorization of  $V_{\mathbb{C}}$  there is a factor  $\overline{U}$ , putting us in the second case. Thus, we may suppose that  $U = \overline{U}$ . The restriction of the polarization form to  $U$  gives a non-degenerate pairing  $q : U \times U \rightarrow \mathbb{C}(-d)$ , inducing an isomorphism of  $\mathbb{C}$ -vector spaces

$$(2.2) \quad \varphi : U \rightarrow U^{\vee}, \quad z \mapsto q(-, z).$$

We may associate to  $\varphi$  the maps  $\varphi^{\text{sym}} = \varphi + \varphi^T$  and  $\varphi^{\text{alt}} = \varphi - \varphi^T$ , for  $\varphi^T$  the transpose. If both maps are non-zero, observe that they are linearly independent. But notice that the Hermitian form  $h : U \times \overline{U} \rightarrow \mathbb{C}(-d)$  gives a similar isomorphism

$$(2.3) \quad \psi : \overline{U} \rightarrow U^{\vee}, \quad \overline{z} \mapsto h(-, \overline{z}).$$

Since  $U = \overline{U}$ , we see that  $\varphi$ ,  $\varphi^{\text{sym}}$ , and  $\varphi^{\text{alt}}$  live in  $\text{End}(U)$ . By Schur's lemma, they are therefore given by multiplication by a complex number. In particular, the latter two are linearly dependent, a contradiction. Thus, one of them is zero. This shows that  $\varphi$  is either symmetric or alternating.

Next, consider the following diagram:

$$\begin{array}{ccc} U & \xrightarrow{\varphi} & U^{\vee} \\ \sigma \downarrow & & \parallel \\ \overline{U} & \xrightarrow{\psi} & U^{\vee} \end{array}$$

We may uniquely define  $\sigma = \psi^{-1} \circ \varphi : U \rightarrow \overline{U}$  an anti-linear map relating the polarization form  $q$  with the Hermitian form  $h$ :

$$(2.4) \quad q(x, \sigma(y)) = h(x, y).$$

Notice that  $\sigma^2 : U \rightarrow U$  is  $\mathbb{C}$ -linear, so again by Schur's lemma we may express  $\sigma^2 = \lambda \cdot \text{id}$  for some non-zero  $\lambda \in \mathbb{C}$ . Applying this in conjunction with our observation that  $\varphi$  is either

symmetric or alternating, we get

$$\begin{aligned}
h(\sigma(x), \sigma(x)) &= q(\sigma(x), \sigma^2(x)) \\
&= q(\sigma(x), \lambda x) \\
&= \lambda q(\sigma(x), x) \\
&= \pm \lambda q(x, \sigma(x)) \\
&= \pm \lambda h(x, x).
\end{aligned}$$

But because  $h$  is positive definite, we see that  $\lambda$  is a positive real precisely when  $\varphi$  is symmetric, and a negative real when  $\varphi$  is alternating.

Now suppose that  $\lambda > 0$ . Rescaling  $\sigma$  by  $\lambda^{-\frac{1}{2}}$ , we get  $\sigma^2 = \text{id}$ . Let  $\text{Res}_{\mathbb{C}/\mathbb{R}}(U)$  be the underlying  $\mathbb{R}$ -vector space, and consider the induced map  $(\sigma - \text{id}) : \text{Res}_{\mathbb{C}/\mathbb{R}}(U) \rightarrow \text{Res}_{\mathbb{C}/\mathbb{R}}(U)$ . This map has a half-dimensional kernel, namely the 1-eigenspace, which we denote  $W$ . Noting that  $W_{\mathbb{C}} = U$  shows that  $W$  must be irreducible, placing us in the first case. Alternatively, suppose that  $\lambda < 0$ . Rescaling  $h$  by  $\lambda^{-\frac{1}{2}}$ , we get  $\sigma^2 = -\text{id}$ . We can see that  $U$  has no underlying real structure, as such a structure comes with a symmetric form, and as we have shown above, all forms on  $U$  are alternating. However, we may define

$$(2.5) \quad \Sigma = \begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix} : U^{\oplus 2} \rightarrow U^{\oplus 2}.$$

Then we once again have  $\Sigma^2 = \text{id}$ , so as above, we take  $W$  to be the 1-eigenspace of  $\text{Res}_{\mathbb{C}/\mathbb{R}}(U^{\oplus 2})$ . We must now argue that  $W$  is irreducible as an  $\mathbb{R}$ -representation. Suppose not; let  $W = W' \oplus W''$ , and notice that  $U^{\oplus 2} = (W')_{\mathbb{C}} \oplus (W'')_{\mathbb{C}}$ . Now,  $U$  is irreducible, so either  $(W')_{\mathbb{C}} = (W'')_{\mathbb{C}} = U$ , or one of the factors is zero. The first case is impossible, as that would imply that  $U$  has an underlying real structure  $W'$ . This proves that  $W$  is irreducible, putting us in the third case.  $\square$

Notice from the proof of Proposition 2.2 that the crucial object to consider is  $\bigwedge^2 V_{\mathbb{C}}$ . As a local system, its global sections probe for irreducible complex subrepresentations of every type, as we will see later. In particular, we have the following corollary.

**Corollary 2.3.** *Let  $U$  be an irreducible complex variation of Hodge structures, with  $U \simeq \overline{U}$ . Then  $h^0(\text{Sym}^2 U) = \mathbb{C}\varphi \neq 0$  if and only if  $U$  is of real type, and  $h^0(\bigwedge^2 U) = \mathbb{C}\varphi \neq 0$  if and only if  $U$  is of quaternionic type.*

Voisin proves another very useful theorem regarding the irreducibility of  $V_{\mathbb{R}}$  by studying the global sections of  $\bigwedge^2 V_{\mathbb{R}}$ .

**Lemma 2.4** ([Voi18], Lemma 4.5). *Let  $f : X \rightarrow B$  be a Lagrangian fibration, and  $V_{\mathbb{R}} = R^1 f_* \mathbb{R}_{X^\circ}$ .  $V_{\mathbb{R}}$  is irreducible as a polarizable variation of real Hodge structures.*

*Proof.* By [Mat16], the restriction  $H^2(X, \mathbb{R}) \rightarrow H^2(X_b, \mathbb{R})$  is rank 1 for  $X_b$  a generic fiber. But by Deligne's global invariant cycles theorem [Del71], the map

$$(2.6) \quad H^2(X, \mathbb{R}) \rightarrow H^0(B^\circ, R^2 \pi_* \mathbb{R}_{X^\circ})$$

is surjective. In particular, the dimension of the codomain of (2.6) is at most 1. Note that  $R^2 \pi_* \mathbb{R}_{X^\circ} = \bigwedge^2 V_{\mathbb{R}}$ , so if  $V_{\mathbb{R}}$  split as a variation, the space of sections of  $\bigwedge^2 V_{\mathbb{R}}$  would be greater than one-dimensional, a contradiction.  $\square$

*Remark 2.5.* Throughout this paper,  $X$  and  $B$  are assumed to be compact. This is not strictly necessary of an assumption for many of our tools (i.e. Proposition 2.1, Corollary 2.3, Lemma 4.3), but Lemma 2.4 is not guaranteed when  $B$  is non-compact.

### 3. REVIEW OF ALGEBRAIC FOLIATIONS

We closely follow the survey of Araujo [Ara18], taking their conventions.

**Definition 3.1.** A foliation  $\mathcal{E}$  on a normal variety  $B$  is a saturated coherent subsheaf of the tangent sheaf which is closed under the ambient Lie bracket. The canonical class  $K_{\mathcal{E}}$  is any Weil divisor for which  $\det(\mathcal{E}) \simeq \mathcal{O}_B(-K_{\mathcal{E}})$ . We say that  $\mathcal{E}$  is  $(\mathbb{Q})$ -Gorenstein if  $K_{\mathcal{E}}$  is  $(\mathbb{Q})$ -Cartier. If  $-K_{\mathcal{E}}$  is ample, we say that  $\mathcal{E}$  is  $(\mathbb{Q})$ -Fano.

If we denote the (generic) rank of  $\mathcal{E}$  by  $r$ , we may consider the so-called Pfaff field ([Ara18, Definition 2.2])

$$(3.1) \quad \Omega_B^r \rightarrow \Omega_B^{[r]} \rightarrow \det(\mathcal{E})^\vee \simeq \mathcal{O}_B(K_{\mathcal{E}}).$$

Here and throughout this paper the notation  $\Omega_B^{[r]}$  denotes the reflexivization  $(\bigwedge^r T_B)^\vee$ . We define the singular locus of  $\mathcal{E}$  to be the closed subscheme of  $B$  whose ideal sheaf is the image of the associated map  $(\Omega_B^r \otimes \det(\mathcal{E}))^{\vee\vee} \rightarrow \mathcal{O}_B$ . On the smooth locus of  $B$ , this is simply the subvariety where  $\mathcal{E}$  is not a bundle.

**Definition 3.2.** An analytic subvariety  $Z \subseteq B$  not contained in the singular locus of  $\mathcal{E}$  is called invariant if on the smooth locus of  $Z$ , (3.1) factors through the induced map on tangent bundles as

$$\begin{array}{ccc} \Omega_B^r|_{Z^{sm}} & \longrightarrow & \Omega_Z^r|_{Z^{sm}} \\ & \searrow & \downarrow \\ & & \mathcal{O}_B(K_{\mathcal{E}})|_{Z^{sm}} \end{array}$$

A maximal invariant subvariety of dimension  $r$  is called a leaf of  $\mathcal{E}$ .

On the regular locus of the foliation, this definition, coupled with a generalization of the classical Frobenius theorem, ensures the existence of smooth submanifolds whose tangent bundles recover  $\mathcal{E}$ . For details, see for instance [BM16, Proposition 1.3.3]. However, despite the fact that  $\mathcal{E}$  is an algebraic sheaf, it is generally false that the leaves of  $\mathcal{E}$  are themselves algebraic. To this end, Bogomolov and McQuillan give a sufficient condition to ensure the existence of rationally-connected algebraic leaves.

**Proposition 3.3** ([BM16], Main Theorem). *Let  $\mathcal{E}$  be a foliation on a normal projective variety  $B$ , and let  $C \subseteq B$  be a curve on which  $\mathcal{E}|_C$  is ample. Then for any  $p \in C$ , the leaf  $Z \ni p$  is an algebraic variety. The leaf through a general point of  $C$  is moreover rationally connected.*

The difficult part of the theorem is proving that the leaf  $Z$  is indeed rationally connected. We do not need this part of the theorem, so we omit its proof. Instead, we will sketch their argument to show that the leaf  $Z$  is algebraic.

*Proof.* To begin, notice that over the curve  $C$ , the foliation has locally the structure of a fibration  $Z_\Delta \rightarrow \Delta \subseteq C$ . It is possible that when we try to extend this structure globally, the leaves  $Z_\alpha$  may intersect  $C$  at more than one point. Thus, we instead consider the graph  $\Gamma$  of



the inclusion  $C \rightarrow B$ . This sits as a curve inside the product  $C \times B$ , and  $\mathcal{E}$  naturally pulls back to a foliation on this product, with leaves through  $\Gamma$  lying over the leaves of  $\mathcal{E}$  through  $C$ . Notice further that this construction has the effect of separating the leaves: two leaves through  $\Gamma$  map to the same leaf  $Z \subseteq B$  if and only if  $Z$  meets  $C$  twice. Thus we produce a fibration  $W \rightarrow \Gamma$ .

We aim to show that the total space  $W$  is algebraic. This will show that each of the individual fibers is algebraic, thereby making its image in  $B$  algebraic. Let  $r$  be the rank of  $\mathcal{E}$ ; we need to show that the Zariski closure  $\overline{W}$  of  $W$  has the expected dimension  $r + 1$ . We proceed asymptotically: if  $\mathcal{L}$  is any line bundle on  $\overline{W}$ , it suffices to show that

$$(3.2) \quad h^0(\overline{W}, \mathcal{L}^{\otimes n}) \leq C(\mathcal{L}) \cdot n^{r+1},$$

for  $C(\mathcal{L})$  a constant depending only on  $\mathcal{L}$ .

Take  $\widehat{W}$  to be the completion of  $W$  along  $\Gamma$ . The inclusion  $H^0(\overline{W}, \mathcal{L}) \hookrightarrow H^0(\widehat{W}, \mathcal{L})$  tells us that it is sufficient to show (3.2) for  $\widehat{W}$ . If  $W_m$  denotes the  $m$ -th infinitesimal thickening of  $\overline{W}$ , we have the exact sequence

$$(3.3) \quad 0 \rightarrow H^0(\Gamma, \text{Sym}^m N_{\Gamma/W}^\vee \otimes \mathcal{L}^{\otimes n}) \rightarrow H^0(W_{m+1}, \mathcal{L}^{\otimes n}) \rightarrow H^0(W_m, \mathcal{L}^{\otimes n}).$$

Here  $N_{\Gamma/W}$  is the normal bundle of  $\Gamma$ . But notice that by construction, it admits a natural map  $\mathcal{F} \rightarrow N_{\Gamma/W}$  from the pullback of  $\mathcal{E}$ , which is generically an isomorphism. By assumption,  $\mathcal{F}$  is positive, there is some  $m \leq C(\mathcal{L}) \cdot n$  above which the left term of (3.3) vanishes. Thus

$$\begin{aligned} h^0(\widehat{W}, \mathcal{L}^{\otimes n}) &= h^0(W_m, \mathcal{L}^{\otimes n}) \\ &\leq \sum_{k=0}^m h^0(\Gamma, \text{Sym}^k N_{\Gamma/W}^\vee \otimes \mathcal{L}^{\otimes n}) \\ &\leq \sum_{k=0}^m C'(\mathcal{L}) \cdot n^r \\ &= C'(\mathcal{L}) \cdot m \cdot n^{r+1} \end{aligned}$$

Here  $C'(\mathcal{L})$  and  $m$  both only depend on  $\mathcal{L}$ , as required.  $\square$

#### 4. FIBRATIONS OF MAXIMAL VARIATION

We first begin by refining Proposition 2.1 under the assumption that  $V_{\mathbb{Q}}$  is of maximal variation. Write  $V_{\mathbb{C}} = \bigoplus_i B_i \otimes U_i$ . We claim that in fact, there is more than one isotypic component in this sum if and only if it is of the complex form  $V_{\mathbb{C}} = (B \otimes U) \oplus (\overline{B} \otimes \overline{U})$ . Indeed,  $V_{\mathbb{C}}$  comes from a real representation, so any isotypic component of  $V_{\mathbb{C}}$  has a complex conjugate (possibly itself) also contained in  $V_{\mathbb{C}}$ . We may add any isotypic component to its complex conjugate to produce  $W_{\mathbb{C}} \subseteq V_{\mathbb{C}}$ ; this must arise from a real variation of Hodge structure  $W \subseteq V_{\mathbb{R}}$ . But  $V_{\mathbb{R}}$  is irreducible as a variation by Lemma 2.4, so we get  $W_{\mathbb{C}} = V_{\mathbb{C}}$  as needed.

In fact, we can say much more about the decomposition of  $V_{\mathbb{C}}$  into irreducibles.

**Lemma 4.1.** *Let  $V_{\mathbb{C}}$  be a variation of Hodge structures arising from a maximal variation Lagrangian fibration. Then for some irreducible complex variation of Hodge structures  $U$ ,*

$$V_{\mathbb{C}} = \begin{cases} U \\ U \oplus U \\ U \oplus \bar{U} \end{cases}.$$

*Proof.* Recall that  $V_{\mathbb{Q}}$  is a variation of weight one Hodge structures. Since it is of maximal variation, the irreducible factor  $U \subseteq V_{\mathbb{C}}$  has level one by [Bak22, Lemma 4]. Observe additionally that  $V_{\mathbb{C}}$  has only a  $V^{1,0}$  piece and a  $V^{0,1}$  piece, so it too is of level one. But now notice that this severely limits the level of  $B$ . Indeed, if  $U$  and  $V_{\mathbb{C}} = B \otimes U$  are both level one, then  $B$  is of level zero. This means that any vector subspace  $C \subseteq B$  is a Hodge subvariation. Concretely, if  $V_{\mathbb{C}}$  has real type,  $V_{\mathbb{C}} = B \otimes W_{\mathbb{C}}$ , then any one-dimensional subspace  $\mathbb{C}(0) \subseteq W_{\mathbb{C}}$  yields a Hodge subvariation  $\mathbb{C}(0) \otimes W_{\mathbb{C}} \subseteq V_{\mathbb{C}}$  coming from a real variation  $\mathbb{R}(0) \otimes W \subseteq V_{\mathbb{R}}$ . But by Lemma 2.4,  $W = V_{\mathbb{R}}$ , so that  $B$  is one-dimensional.

The complex case is completely analogous; if  $V_{\mathbb{C}} = (B \otimes U) \oplus (\bar{B} \otimes \bar{U})$ , then both  $U$  and  $\bar{U}$  are level one, forcing  $B$  and  $\bar{B}$  to be level zero. But now  $\mathbb{C}(0) \subseteq B$  and  $\bar{\mathbb{C}}(0) \subseteq \bar{B}$  are Hodge substructures. This yields  $U \oplus \bar{U} \subseteq V_{\mathbb{C}}$ , which is underlied by a real variation. But again by Lemma 2.4, the underlying real variation is  $V_{\mathbb{R}}$ , forcing  $\dim B = 1$ .

Finally, when  $V_{\mathbb{C}} = B \otimes U$  is quaternionic, we know that  $V_{\mathbb{C}}$  is itself real, so that  $\dim B \geq 2$ . Again, by the same argument,  $B$  has level zero, so any subspace of  $B$  yields a subvariation. This time, consider  $\mathbb{C}(0)^{\oplus 2} \subseteq B$ . As before, this is a subvariation. Moreover,  $\mathbb{C}(0)^{\oplus 2} \otimes U \subseteq V_{\mathbb{C}}$  is a subvariation underlied by a real variation of Hodge structures. Again by Lemma 2.4, we get equality  $B = \mathbb{C}(0)^{\oplus 2}$ , so that  $V_{\mathbb{C}} \simeq U^{\oplus 2}$ .  $\square$

With this description, we can compute some explicit examples. In particular, we see that starting from an elliptic K3 surface, we can get a concrete understanding of the associated  $K3^{[n]}$ -type Lagrangian fibration.

*Example 4.2.* Let  $X \rightarrow \mathbb{P}^1$  be an elliptic K3 surface which is of maximal variation. By Lemma 4.1, the associated complex variation of Hodge structures  $V_{\mathbb{C}} = U$  is irreducible; otherwise, each of the irreducible factors would have to be one-dimensional, thereby failing to be of level one. Consider the associated  $K3^{[n]}$ -type Lagrangian fibration; it comes with the induced maps  $(X^{[n]})^{\circ} \rightarrow (\mathbb{P}^n)^{\circ}$  the locus of smooth fibers of  $f^{[n]}$ , and  $X^{\circ} \rightarrow U$  the smooth fibers of  $f$ . We have a rational map  $U^n \rightarrow U^{(n)} \dashrightarrow \mathbb{P}^n$  to the  $n$ -th symmetric power of  $U$ ; let  $(U^n)^{\circ} \subseteq (\mathbb{P}^1)^n$  be the locus on which it is regular. This gives us the diagram

$$\begin{array}{ccccc} (X^n)^{\circ} & \longrightarrow & (X^{[n]})^{\circ} & \hookrightarrow & X^{[n]} \\ \downarrow & & \downarrow & & \downarrow \\ (U^n)^{\circ} & \longrightarrow & (\mathbb{P}^n)^{\circ} & \hookrightarrow & \mathbb{P}^n \end{array}$$

Notice that this yields a right-exact sequence of groups

$$(4.1) \quad \pi_1((U^n)^{\circ}) \rightarrow \pi_1((\mathbb{P}^n)^{\circ}) \rightarrow \mathfrak{S}_n \rightarrow 1.$$

Take  $H = \pi_1((U^n)^{\circ})$  and  $G = \pi_1((\mathbb{P}^n)^{\circ})$ , and consider the induction  $\text{Ind}_{G/H}(V_{\mathbb{C}})$ . Notice that by construction, this representation is expressible as  $B \otimes W$ , where  $W$  is the representation associated to  $R^1 f_* \mathbb{C}_{X^{\circ}}$ . Now notice that  $\sum_{p \in \mathfrak{S}_n} pW$  carries the structure of a  $G$ -subrepresentation of  $V_{\mathbb{C}}$ . Moreover,  $\mathfrak{S}_n$  acts transitively on the irreducible factors of



$B \otimes W$ , so if  $\sum_{p \in \mathfrak{S}_n} pW$  split as a sum of distinct irreducible  $G$ -representations, there would have to be a non-zero map between them.

Thus  $V_{\mathbb{C}}$  has a single isotypic component. For each irreducible factor  $W'$ , we may consider its restriction to  $H$ . Since  $\mathfrak{S}_n$  acts transitively on the irreducible factors of  $B \otimes W$ , there is some element of  $\mathfrak{S}_n$  which acts by permuting the irreducible factors. But each  $W'$  was a  $G$ -representation, so it must have been closed under the action of  $\mathfrak{S}_n$ . Therefore  $W' = V_{\mathbb{C}}$  is a single irreducible factor. By Lemma 4.1,  $V_{\mathbb{C}}$  has real type. In particular, we see that real type Lagrangian fibrations occur over projective space of every dimension.

We will soon see every Lagrangian fibration of maximal variation is of real type. Consider the map  $f : X \rightarrow B$  on three levels:

$$\begin{array}{ccccc} X^{\circ} & \hookrightarrow & X_{B^{sm}} & \hookrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ B^{\circ} & \xrightarrow{i} & B^{sm} & \xrightarrow{j} & B \end{array}$$

Here  $X_{B^{sm}} \rightarrow B^{sm}$  is the restriction of  $f$  to the smooth part of  $B$ , and  $X^{\circ} \rightarrow B^{\circ}$  is the further restriction to the smooth fibers.

Let  $(\mathcal{V}, \nabla, F^{\bullet})$  be the flat bundle on  $B^{\circ}$  associated to  $V$ , equipped with sub-bundle  $F^1\mathcal{V}$ . By a formulation of Matsushita's theorem due to Schnell [Sch23, §4],  $R^1f_*\mathcal{O}_{X_{B^{sm}}} \simeq \Omega_{B^{sm}}^1$ . If we further restrict to  $B^{\circ}$  and take duals, we get  $T_{B^{\circ}} \simeq f_*\Omega_{X^{\circ}/B^{\circ}}^1 = F^1\mathcal{V}$ . In fact, Bakker and Schnell use Saito's theory of Hodge modules to extend this isomorphism over  $B^{sm}$ .

**Lemma 4.3** ([BS23], §2, (7)). *Deligne's canonical extension lifts the isomorphism on  $B^{\circ}$  to*

$$T_{B^{sm}} \simeq \mathcal{V}^{>-1} \cap i_*(F^1\mathcal{V}).$$

Now we are finally prepared to refine Lemma 4.1. While dimension-counting has narrowed down the types of isotypic decomposition which may occur to only three cases, it has failed to rule out the possibility that  $V_{\mathbb{C}}$  may be of complex type or quaternionic.

**Theorem 4.4.** *The monodromy representation associated to any maximal variation Lagrangian fibration is of real type.*

*Proof.* Suppose that  $V_{\mathbb{C}}$  were quaternionic, i.e.  $V_{\mathbb{C}} = U^{\oplus 2}$ , for  $U$  complex irreducible. Take  $\mathcal{U}$  to be flat bundle on  $B^{\circ}$  associated to  $U$ ; we naturally have  $\mathcal{V} = \mathcal{U}^{\oplus 2}$ . Since this is a decomposition on the level of variations of Hodge structure,  $T_{B^{\circ}} \simeq F^1\mathcal{V} = (F^1\mathcal{U})^{\oplus 2}$ . But now we may apply Lemma 4.3 to get

$$(4.2) \quad T_{B^{sm}} \simeq (\mathcal{U}^{>-1} \cap i_*(F^1\mathcal{U}))^{\oplus 2}.$$

In particular, the sheaf of reflexive 1-forms  $j_*(T_{B^{sm}}^{\vee}) = \Omega_B^{[1]}$  on  $B$  is a square. Thus the space  $H^1(B, \Omega_B^{[1]})$  is even-dimensional. But by [BL20, Corollary 2.3],  $H^{p,q}(B) \simeq H^q(X, \Omega_B^{[p]})$  for  $p + q \leq 2$ . Since  $H^0(B, \Omega_B^{[2]})$  contains the image of  $\bar{\sigma}$  under the identification given by Matsushita's theorem [Sch23], we know that  $H^2(B, \mathbb{C})$  is non-empty. Thus it has dimension at least 2. But  $B$  is known to have the same rational cohomology as  $\mathbb{P}^n$  (see, for instance, [HM22, Theorem 2.1]). As  $H^2(\mathbb{P}^n, \mathbb{Q}) \simeq \mathbb{Q}$ , this is a contradiction.

Now we show that  $V_{\mathbb{C}}$  is not complex. Once again, suppose for contradiction that it were, let  $V_{\mathbb{C}} = U \oplus \bar{U}$ , for  $U$  an irreducible  $\mathbb{C}$ -local system. Then on the level of flat bundles,  $\mathcal{V}$

splits as  $\mathcal{U}_1 \oplus \mathcal{U}_2$ . Each  $\mathcal{U}_i$  comes equipped with a sub-bundle  $F^1 \mathcal{V}_i$ , which respects the Hodge filtration on  $\mathcal{V}$ , i.e.

$$(4.3) \quad T_{B^\circ} \simeq F^1 \mathcal{V} = F^1 \mathcal{U}_1 \oplus F^1 \mathcal{U}_2.$$

Now apply Lemma 4.3 to get a splitting of  $T_{B^{sm}}$  as

$$(4.4) \quad T_{B^{sm}} \simeq (\mathcal{U}_1^{>-1} \cap i_*(F^1 \mathcal{U}_1)) \oplus (\mathcal{U}_2^{>-1} \cap i_*(F^1 \mathcal{U}_2)) =: \mathcal{E}_1 \oplus \mathcal{E}_2.$$

Note that both  $\mathcal{E}_1$  and  $\mathcal{E}_2$  are foliations. Their integrability follows from [BS23, Lemma 3.3]. We will now show that at any general point of  $B^{sm}$ , at least one of the two foliations admits an algebraically integrable subfoliation. This will imply that a general point of  $B^{sm}$  is contained in a closed subscheme of  $B^{sm}$  along which the period map drops rank, contradicting our assumption that  $f$  is of maximal variation.

Note that  $\det(j_* \mathcal{E}_1) \otimes \det(j_* \mathcal{E}_2) = \omega_B^\vee$  is  $\mathbb{Q}$ -ample. By [Mat15, Theorem 1.1],  $B$  has Picard rank 1, so one of the two summands must therefore have ample determinant. In other words, we may assume without loss of generality that  $j_* \mathcal{E}_1$  is a  $\mathbb{Q}$ -Fano foliation. We will use the argument of [Ara18, Corollary 2.12] to show that there exists  $\mathcal{F} \subseteq j_* \mathcal{E}_1$  an algebraically integrable subfoliation.

Fix an ample class  $H$  on  $B$ , and consider the Harder-Narasimhan filtration of  $j_* \mathcal{E}_1$  induced by the corresponding slope  $\mu_H$ :

$$(4.5) \quad 0 \subsetneq \mathcal{F} = \mathcal{F}_1 \subsetneq \cdots \subsetneq \mathcal{F}_k = j_* \mathcal{E}_1, \quad \mu(\mathcal{F}) > \mu(\mathcal{F}_2/\mathcal{F}_1) > \cdots > \mu(\mathcal{F}_k/\mathcal{F}_{k-1}).$$

We claim that  $\mathcal{F}$  is a subfoliation of  $j_* \mathcal{E}_1$ . Indeed, given a local section  $v \in \mathcal{F}(U)$ , we may consider the image of the Lie bracket  $[v, -] : \mathcal{F}(U) \rightarrow \mathcal{E}_1(U)$ . This is not a morphism of  $\mathcal{O}_B(U)$ -modules, since for any  $f \in \mathcal{O}_B(U)$  and  $w \in \mathcal{F}(U)$  we have

$$(4.6) \quad [v, fw] = \nabla_v(f) \cdot w + f[v, w].$$

However, we may compose this map with the projection  $\mathcal{E}_1 \rightarrow \mathcal{E}_1/\mathcal{F}$ , in which case the first term of (4.6) dies. This composition is now  $\mathcal{O}_B$ -linear. Moreover, since the choice of local section  $v$  was arbitrary, we in reality have a morphism of  $\mathcal{O}_B$ -modules  $[-, -] : \mathcal{F}^{\otimes 2} \rightarrow \mathcal{E}_1/\mathcal{F}$ . Suppose this morphism was anywhere non-zero; then there would need to be some graded piece of the Harder-Narasimhan filtration on which the Lie bracket had non-zero image, say

$$(4.7) \quad \mathcal{F}^{\otimes 2} \rightarrow \mathcal{E}_1/\mathcal{F} \rightarrow \mathcal{F}_i/\mathcal{F}_{i-1}.$$

By assumption, both  $\mathcal{F}$  and  $\mathcal{F}_i/\mathcal{F}_{i-1}$  are semistable with the slope of the former strictly greater than the slope of the latter. But then there are no non-zero maps  $\mathcal{F} \rightarrow \mathcal{F}_i/\mathcal{F}_{i-1}$ . But in characteristic zero, the tensor product of stable sheaves is semistable; see, for instance, [Lan05, §2.1.2]. Thus  $\mathcal{F}^{\otimes 2}$  is also semi-stable of slope  $2\mu_H(\mathcal{F})$ . In particular, it too admits no non-zero maps to  $\mathcal{F}_i/\mathcal{F}_{i-1}$ , proving  $[\mathcal{F}, \mathcal{F}] \subseteq \mathcal{F}$ .

Now construct a general complete intersection curve  $C \subseteq B$  avoiding the singular loci of  $B$  and  $\mathcal{F}$  as follows: for  $m_1, \dots, m_{n-1}$  sufficiently large integers, let  $H_i$  be a general member of the complete linear series  $|m_i H|$ , and set  $C := H_1 \cap \cdots \cap H_{n-1}$ . By the Mehta-Ramanathan Theorem [MR84], the restriction of (4.5) to  $C$  is still the Harder-Narasimhan filtration of  $j_* \mathcal{E}_1|_C$ . In particular, we see that every filtered piece  $\mathcal{F}_i|_C$  is locally free. Since  $j_* \mathcal{E}_1$  was  $\mathbb{Q}$ -Fano, we have  $\mu_H(j_* \mathcal{E}_1) > 0$ , so it has positive degree on  $C$ . Consequently,  $\mathcal{F}|_C$  also has positive degree. By a theorem of Hartshorne [Har71, Theorem 2.4],  $\mathcal{F}|_C$  is ample if and only if every quotient bundle of  $\mathcal{F}|_C$  has positive degree. But  $\mathcal{F}|_C$  is the maximal destabilizing subsheaf, so it admits no nontrivial quotients. Thus, by Theorem 3.3, every leaf of  $\mathcal{F}$  through  $C$  is algebraic.

Now take  $Z \subsetneq B^{sm}$  of  $\mathcal{F}$  to be any such leaf. Consider the period map  $\mathcal{P} : B^\circ \rightarrow \Gamma \backslash \mathcal{D}$ ; we have constructed  $\mathcal{E}_2$  to be perpendicular to  $T_Z = \mathcal{F} \subseteq \mathcal{E}_1$ . Thus  $d\mathcal{P}|_Z$  kills all of  $\mathcal{E}_2|_Z$ , so that  $d\mathcal{P}$  drops rank along  $C$ . This means that  $\mathcal{P}$  is not immersive on any sufficiently general complete intersection curve in  $B$ , and consequently at any general point of  $B^\circ$ . Therefore  $V_{\mathbb{C}}$  must not have been of maximal variation, and we have reached our desired contradiction.  $\square$

**Corollary 4.5.**  $V_{\mathbb{C}}$  is irreducible as a  $\mathbb{C}$ -local system.

*Proof.* This follows immediately from Theorem 4.4 and Lemma 4.1.  $\square$

## 5. ISOTRIVIAL FIBRATIONS

To more cleanly deal with the case where  $f : X \rightarrow B$  is isotrivial, we introduce a definition.

**Definition 5.1.** Let  $\Gamma$  be a finitely generated group,  $K$  a field. A  $K$ -Hodge representation of  $\Gamma$  is the data of a polarized  $K$ -Hodge structure  $(V, \bigoplus_{p+q=w} V_{\mathbb{C}}^{p,q}, q)$  and a representation  $\rho : \Gamma \rightarrow \mathrm{GL}(V, q)$  acting by Hodge automorphisms.

The notion of a Hodge representation is useful precisely because it captures the data of isotrivial variations of Hodge structure. In particular, since  $f$  is isotrivial, we may fix a basepoint  $b \in B^\circ$  and note that  $V_{\mathbb{Q}}$  has the structure of a weight 1 Hodge representation  $((V_{\mathbb{Q}})_b, (V_{\mathbb{C}})_b^{1,0} \oplus (V_{\mathbb{C}})_b^{0,1}, q, \rho)$  with respect to the monodromy representation

$$\rho : \Gamma = \pi_1(B^\circ, b) \rightarrow \mathrm{GL}(V_{\mathbb{Q}}, q).$$

The choice of the basepoint does not matter; such a Hodge representation completely captures the data of  $V_{\mathbb{Q}}$ . Hence, we will abuse notation by dropping the subscript, identifying  $V_{\mathbb{Q}}$  as a  $\mathbb{Q}$ -Hodge representation of  $\pi_1(B^\circ)$ .

Now we recover the structure theorem of Kim, Laza, and Martin.

**Theorem 5.2** ([KLM23], Theorem 1.3). *There exists an elliptic curve  $E$  along with a morphism of vector spaces  $H^1(E, \mathbb{Q})^{\oplus n} \rightarrow V_{\mathbb{Q}}$  which is an isomorphism of  $\mathbb{Q}$ -Hodge structures.*

*Remark 5.3.* Note also that we do not claim this map to be an isomorphism of Hodge representations; indeed, we will see that  $H^1(E, \mathbb{Q})$  only admits the structure of a Hodge representation locally. Phrased in terms of isotrivial variations, this is an identification of Hodge structures between a general fiber of  $V_{\mathbb{Q}}$  and  $H^1(E, \mathbb{Q})^{\oplus n}$ . Equivalently, the smooth fibers of  $f : X \rightarrow B$  are isogeneous to  $E^n$ .

*Proof.* Let  $D = B^{sm} \setminus B^\circ$ . We consider the local monodromy of  $V_{\mathbb{Q}}$  in a neighborhood  $U \subseteq B^{sm}$  where  $D \cap U$  is smooth. Set  $U^\circ = U \cap B^\circ$ , and choose a basepoint  $b \in U^\circ$ . Let  $T_b$  denote the matrix representing the local monodromy operator around  $b$ . By [BS23, Lemma 3.6],  $T_b$  has only two non-trivial conjugate eigenvalues  $\lambda, \bar{\lambda}$  with

$$(5.1) \quad (\lambda, \bar{\lambda}) \in \{(\zeta_6, \bar{\zeta}_6), (\zeta_4, \bar{\zeta}_4), (\zeta_3, \bar{\zeta}_3), (-1, -1)\}.$$

We have a natural Hodge substructure  $(V_{\mathbb{Q}})^{T_b} \subseteq V_{\mathbb{Q}}$  given by the 1-eigenspace of  $T_b$ ; denote by  $H$  its orthogonal complement with respect to the Hermitian form on  $V_{\mathbb{Q}}$ .  $H$  too is a Hodge substructure, and because  $H$  is two-dimensional and weight one, it is irreducible. Notice that there exists an isotrivial family of elliptic curves  $\mathcal{E} \rightarrow U^\circ$  with fiber  $E$ , such that we have an isomorphism of  $\pi_1(U^\circ)$ -Hodge representations  $H^1(E, \mathbb{Q}) \simeq H$ .

There are no Hodge automorphisms between isotypic components of a Hodge structure, so  $V_{\mathbb{Q}}$  has multiple isotypic components as a Hodge structure if and only if it has multiple

isotypic components as a Hodge representation. But by Lemma 2.4,  $V_{\mathbb{Q}}$  is irreducible as a Hodge representation, so it has only a single isotypic component as a Hodge structure. Since  $H \subseteq V_{\mathbb{Q}}$  is irreducible, we must get an isomorphism of  $\mathbb{Q}$ -Hodge structures  $V_{\mathbb{Q}} \simeq H^{\oplus n}$ .  $\square$

Next we seek to understand the structure of  $V_{\mathbb{C}}$  by applying Proposition 2.2. As in Lemma 4.1, we will see that in each of the three cases, the dimensions of  $B$  and  $U$  are determined. Moreover, due to the previous theorem, we will see that the isotypic decomposition of  $V_{\mathbb{C}}$  actually occurs over a degree two extension of  $\mathbb{Q}$ .

**Proposition 5.4.** *Let  $E$  be the elliptic curve from above,  $H = H^1(E, \mathbb{Q})$ .*

- (a)  $V_{\mathbb{C}}$  is of real type if and only if we have the isomorphism of  $\mathbb{Q}$ -Hodge representations  $V_{\mathbb{Q}} = H \otimes W$  for  $W$  a  $\mathbb{Q}$ -local system irreducible over  $\mathbb{C}$ .
- (b)  $V_{\mathbb{C}}$  is of complex type if and only if  $E$  has CM and we have an isomorphism of  $K$ -Hodge representations  $V_K \simeq V_K^{1,0} \oplus V_K^{0,1}$  for  $K = \text{End}_{\mathbb{Q}HS}(H)$  the CM field of  $E$ , where  $V_K^{1,0}$  and  $V_K^{0,1}$  are non-isomorphic  $K$ -local systems irreducible over  $\mathbb{C}$ .
- (c)  $V_{\mathbb{C}}$  is not quaternionic.

*Proof.* If  $V_{\mathbb{C}}$  is real or quaternionic, it must be of the form  $V_{\mathbb{C}} = B \otimes U$ , where  $U$  is an irreducible  $\mathbb{C}$ -local system. However, since  $V_{\mathbb{C}}$  is isotrivial,  $U$  must have level zero. Since  $V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$ ,  $B$  must have level one. In particular, it is at least two-dimensional. We first claim that  $V_{\mathbb{C}}$  again cannot be quaternionic. Indeed, if it were, then by Corollary 2.3,  $h^0(\bigwedge^2 U)$  would contain an alternating form. Consider the containment  $\bigwedge^2 U \oplus \bigwedge^2 U \subseteq \bigwedge^2 V_{\mathbb{C}}$ . We know by Deligne's global invariant cycles theorem that  $h^0(V_{\mathbb{C}}) = 1$ , so this containment is impossible, a contradiction.

Next, suppose that  $V_{\mathbb{C}} = B \otimes W_{\mathbb{C}}$  has real type. Furthermore, we claim that the dimension of  $B$  is exactly two. Consider once again the containments

$$(5.2) \quad \bigwedge^2 V_{\mathbb{C}} \supseteq (W_{\mathbb{C}}^{\otimes 2})^{\oplus \binom{\dim B}{2}} \supseteq (\text{Sym}^2 W_{\mathbb{C}})^{\oplus \binom{\dim B}{2}}$$

Again by global invariant cycles, we know that the  $h^0(\bigwedge^2 V_{\mathbb{C}}) = 1$ . But by Corollary 2.3,  $h^0(\text{Sym}^2 W_{\mathbb{C}}) = 1$ . This forces  $\dim B = 2$ .

Otherwise, suppose that  $V_{\mathbb{C}}$  is of complex type,  $V_{\mathbb{C}} = (B \otimes U) \oplus (\overline{B} \otimes \overline{U})$ . Then  $B$  and  $\overline{B}$  both have level zero. But now they must both be one-dimensional. If not, any subspace of  $C \subseteq B$  is a Hodge subvariation, with  $(C \otimes U) \oplus (\overline{C} \otimes \overline{U}) \subsetneq V_{\mathbb{C}}$  defined over  $\mathbb{R}$ . This contradicts the irreducibility of  $V_{\mathbb{R}}$  as a Hodge structure. Therefore  $V_{\mathbb{C}} \simeq U \oplus \overline{U}$ .

So far, we have not used the Hodge substructure  $H \subseteq V_{\mathbb{Q}}$ . Consider the natural evaluation map

$$(5.3) \quad H \otimes_{\mathbb{Q}} \text{Hom}_{\mathbb{Q}HS}(H, V_{\mathbb{Q}}) \rightarrow V_{\mathbb{Q}}.$$

This map is always surjective. Suppose that  $E$  is not CM, i.e.  $\text{End}_{\mathbb{Q}HS}(H) = \mathbb{Q}$ . Then by the previous theorem,  $\dim H \otimes_{\mathbb{Q}} \text{Hom}_{\mathbb{Q}HS}(H, H^{\oplus n}) = 2n$ , so that this map is an isomorphism of  $\mathbb{Q}$ -Hodge structures. Now notice that we may equip the domain with a Hodge representation, where  $\pi_1(B^{\circ})$  acts trivially on  $H$  and  $\gamma \cdot \varphi : h \rightarrow \varphi(\gamma h)$  for  $h \in H$  and  $\varphi \in \text{Hom}_{\mathbb{Q}HS}(H, V_{\mathbb{Q}})$ . This extends the evaluation map to an isomorphism of Hodge representations. In particular,  $V_{\mathbb{C}}$  has real type.

Now suppose that  $E$  does have CM,  $\text{End}_{\mathbb{Q}HS}(H) = K$  is an imaginary quadratic extension of  $\mathbb{Q}$ . Since  $H_K = H_K^{1,0} \oplus H_K^{0,1}$  as a Hodge structure, the previous theorem gives us an analogous splitting of Hodge structures  $V_K = V_K^{1,0} \oplus V_K^{0,1}$ . Moreover, Hodge automorphisms

preserve the Hodge grading, so this gives a splitting of the Hodge representation  $V_K = V_K^{1,0} \oplus V_K^{0,1}$  into  $K$ -local systems. If they are non-isomorphic, then  $V_{\mathbb{C}} = V_{\mathbb{C}}^{1,0} \oplus V_{\mathbb{C}}^{0,1}$  is a splitting of  $V_{\mathbb{C}}$  into non-isomorphic  $\mathbb{C}$ -local systems. As we showed above, this is the complex type case, so that  $V_{\mathbb{C}}^{1,0}$  and  $V_{\mathbb{C}}^{0,1}$  are irreducible  $\mathbb{C}$ -local systems.

Alternatively, suppose that  $V_K^{1,0} \simeq V_K^{0,1}$ , so that  $V_K \simeq B \otimes U$  for  $U$  an irreducible  $K$ -local system. If  $U = W_K$  for  $W$  a  $\mathbb{Q}$ -local system, then we are again in the real type case. Thus  $B$  is two-dimensional and level one, so it is again identified with the Hodge substructure  $H \subseteq V_{\mathbb{Q}}$ , and  $V_{\mathbb{Q}} = H \otimes W$ . Finally, suppose  $U$  is an irreducible  $K$ -local system which is not defined over  $\mathbb{Q}$ . Then  $U_{\mathbb{C}}$  is a  $\mathbb{C}$ -local system which is not defined over  $\mathbb{R}$ . As we have seen, it cannot be quaternionic, so it must be of complex type. But if  $U_{\mathbb{C}} \neq \overline{U}_{\mathbb{C}}$ , then  $U \neq \overline{U}$ . This contradicts our assumption.  $\square$

*Remark 5.5.* One may have expected that  $V_{\mathbb{C}}$  has complex type if and only if  $E$  is CM. The reason why it is not sufficient for  $E$  to be CM is because the monodromy may never actually make use of the CM structure. Indeed,  $V_{\mathbb{C}}$  has complex type precisely when there is a local generator  $\gamma \in \pi_1(B^{\circ})$  which acts on  $H$  by a CM automorphism.

There is an alternative perspective one may take in classifying isotrivial Lagrangian fibrations. Given  $f : X \rightarrow B$ , one may ask for the intermediate trivialization  $\pi : C^{\circ} \rightarrow B^{\circ}$ , the smallest cover of  $B$  trivializing  $R^1 f_* \mathbb{Q}_{X^{\circ}}$  over the pullback of  $B^{\circ}$ . We may then form the pullback diagram

$$\begin{array}{ccc} Z^{\circ} & \longrightarrow & X^{\circ} \\ \downarrow & & \downarrow f \\ C^{\circ} & \xrightarrow{\pi} & B^{\circ} \end{array}$$

By [KLM23],  $C^{\circ}$  must compactify to a variety which is either of general type or an Abelian torsor. In fact, if we assume that the singular fibers of  $f$  are not multiple, [KLM23] show that the singular fibers of any Lagrangian fibration follow the Kodaira classification of elliptic surface singularities.

**Proposition 5.6** ([KLM23], Proposition 4.20). *Suppose that  $f : X \rightarrow B$  is an isotrivial Lagrangian fibration whose singular fibers are non-multiple. A general singular fiber of  $f$  can be of Kodaira type II, III, IV,  $I_0^*$ ,  $II^*$ ,  $III^*$ , or  $IV^*$ .*

We exhibit some examples of isotrivial Lagrangian fibrations over  $\mathbb{P}^1$  to show how their classification relates to our own. We owe these examples in part to [Saw14].

*Example 5.7.* Let  $f : X \rightarrow \mathbb{P}^1$  be an elliptic K3 surface, and suppose that  $X$  has singular fibers of Kodaira type II (cuspidal cubic curve). Each one has Euler characteristic 2, so there are twelve of them; around each of the twelve punctures  $b_i \in \mathbb{P}^1$ , the monodromy matrix can be locally expressed as

$$(5.4) \quad T_{b_i} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}.$$

Since the matrix is of order six, we may construct a degree six cover  $\pi : C^{\circ} \rightarrow B^{\circ}$  trivializing  $R^1 f_* \mathbb{Z}_{X^{\circ}}$ . But  $C^{\circ}$  then extends to a ramified cover  $\pi : C \rightarrow \mathbb{P}^1$ . By Riemann-Hurwitz,

$$(5.5) \quad \chi(C) = 6 \cdot 2 - (6 - 1) \cdot 12 = -48$$

so  $C$  has genus twenty-five. In particular, it is of general type. The monodromy matrix acts on a general fiber  $E$  by complex multiplication, so the associated monodromy representation is of complex type.

*Example 5.8.* Take  $E \times E \rightarrow E$  for  $E$  an elliptic curve, and quotient by the involution. After blowing up the sixteen fixed points, we get a Lagrangian fibration  $f : X \rightarrow \mathbb{P}^1$ . Since the map  $E \rightarrow \mathbb{P}^1$  has four branch points, we see that  $f$  has four singular fibers, each of which is of Kodaira type  $I_0^*$ . The associated Dynkin diagram is  $\widetilde{D}_4$ , with fibers consisting of four  $\mathbb{P}^1$  meeting a doubled  $E/\pm 1 = \mathbb{P}^1$  at the four branch points. The local monodromy matrix over each fiber is given by

$$(5.6) \quad T_b = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

This lattice automorphism is not CM, so the local system is of real type. The associated covering map is  $E \rightarrow \mathbb{P}^1$  of degree 2, where  $E$  is Abelian.

*Example 5.9.* Take  $E \times E \rightarrow E$  as in Example 5.8, but this time we instead fix  $E = \mathbb{C}/\langle 1, \zeta \rangle$  for  $\zeta$  a primitive sixth root of unity. Quotient by the diagonal action of the group

$$(5.7) \quad G = \left\langle \begin{pmatrix} \zeta^2 & 0 \\ 0 & \zeta^{-2} \end{pmatrix} \right\rangle$$

and blowing up the nine fixed points gives us a Lagrangian fibration  $f : X \rightarrow \mathbb{P}^1$ . The three singular fibers above each of the fixed points of  $\langle \zeta^2 \rangle \backslash E$  consist of  $E_6$  singularities of Kodaira type  $IV^*$ . In particular, the local monodromy matrices have the form

$$T_b = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}.$$

It has order three, so in particular, it is of complex type.  $B^\circ$  is covered by a degree three map from  $E$ , so it is of Abelian type.

*Example 5.10.* In Example 5.9, replace  $E$  by  $\mathbb{C}/\langle 1, i \rangle$  and  $G$  by  $\left\langle \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \right\rangle$ . We once again produce a Lagrangian fibration

$$(5.8) \quad f : X = G \backslash E \times E \longrightarrow \langle i \rangle \backslash E \simeq \mathbb{P}^1.$$

The action of  $G$  on  $E$  has three fixed points, giving three singular fibers. Two are of Kodaira class  $III^*$  with Dynkin diagram  $\widetilde{E}_7$ , while the last is forced to be  $I_0^*$  as in Example 5.8. The new fibers have local monodromy

$$(5.9) \quad T_b = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

of order four, so it is CM. Once again,  $(\mathbb{P}^1)^\circ$  is covered and compactified to  $E$ , so it has Abelian type.

Notice that while we managed to produce examples of Abelian type fibrations which were of either representation type, we do not have an example of a general type fibration which is of real type. In fact, we claim that this is the case in higher dimensions as well.



**Proposition 5.11.**  $V_{\mathbb{C}}$  is of complex type if and only if it splits over  $\mathbb{Q}(i)$  or  $\mathbb{Q}(\sqrt{-3})$ . If  $V_{\mathbb{C}}$  has real type, then the intermediate trivialization  $C^{\circ}$  must compactify to an Abelian torsor.

*Proof.* The local monodromy group around any singular fiber is  $\mu_n$  for  $n = 2, 3, 4, 6$ . If  $V_{\mathbb{C}}$  is to be of real type, we must have  $n = 2$ , which is attained only when the singular fiber is of Kodaira type  $I_0^*$ . But now the first assertion follows directly from Proposition 5.4 and Proposition 5.6. The second comes from [KLM23, Proposition 2.24], which states that  $C^{\circ}$  compactifies to an Abelian torsor if and only if the general singular fibers of  $f$  are of Kodaira type  $I_0^*$ ,  $II^*$ ,  $III^*$ , or  $IV^*$ , so that all real-type  $V_{\mathbb{C}}$  are trivialized by an Abelian torsor.  $\square$

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