Higgs Bundles

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These are notes I wrote up for my own comprehension while reading Simpson's paper. Let X be a compact Kähler manifold with Kähler metric ω. We let E be a vector bundle (either smooth or holomorphic, depending on the context) with associated sheaf of sections $\mathcal{E}$.

Definition 1. A Higgs bundle is a holomorphic vector bundle E on X, equipped with a holomorphic map

$$\theta : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_X, \qquad \theta \wedge \theta = 0 \in \mathrm{End}(E) \otimes \Omega^2_X. \tag{1}$$

We may write θ in local coordinates $z_1, \ldots, z_n$ as $\sum_i \theta_i dz_i$, for some θ_i holomorphic endomorphisms of E. Observe that the condition that $\theta \wedge \theta = 0$ grants us that

$$\theta_i \theta_j dz_i \wedge dz_j - \theta_j \theta_i dz_i \wedge dz_j = 0, \tag{2}$$

so that the matrices θ_i, θ_j commute for every i, j.

Alternatively, we may think of E as a C^∞-bundle equipped with a first-order operator. The holomorphic structure of E comes from an operator $\overline{\partial} : \mathcal{E} \to \mathcal{E} \otimes \mathcal{A}^{0,1}_X$, annihilating precisely the holomorphic sections of E. With this perspective, we may think of θ as a similar operator

$$\theta : \mathcal{E} \to \mathcal{E} \otimes \mathcal{A}^{1,0}_X. \tag{3}$$

we may combine our two operators to form $D'' := \theta + \overline{\partial}$. Conversely, an operator D'' as above defines a Higgs bundle precisely when it satisfies the two conditions

$$D''(fs) = \overline{\partial}(f)s + fD''(s), \qquad (D'')^2 = 0. \tag{4}$$

Indeed, the integrability condition gives us a holomorphic structure via $\overline{\partial}^2 = 0$, forces θ to be holomorphic by $\overline{\partial}(\theta) = 0$, and yields $\theta \wedge \theta = 0$, as needed.

Next, let us establish the correspondance between flat bundles and Higgs bundles. Let V be a flat bundle with sheaf of sections $\mathcal{V}$. A metric K on E or V is a positive definite Hermitian form $(\cdot, \cdot)_K$ on every fiber, varying smoothly over the base. To globalize this construction, we note that a Hermitian form is the same data as an isomorphism

$$K : E \to \overline{E}^\vee, \qquad s \mapsto K(s, -). \tag{5}$$

Theorem 2. *K defines a correspondance between vector bundles with connection (V, D) and vector bundles with an endomorphism-valued one-form (E, θ).*

Proof. Given a Higgs bundle (E, D'') with Hermitian metric K, define the following three analogous operators: ∂_K is the unique operator such that $\overline{\partial} + \partial_K$ preserves K:

$$(\overline{\partial} e, f) + (e, \partial_K f) = \overline{\partial}(e, f), \tag{6}$$

$\bar{\theta}_K$ is the K-adjoint to θ:

$$(\theta e, f) = (e, \bar{\theta}_K f), \tag{7}$$

and $D'_K = \partial_K + \bar{\theta}_K$. Finally, set $D_K = D'_K + D''$; we claim that D_K is a connection. Indeed,

$$\begin{aligned} D_K(ae) &= D'_K(ae) + D''(ae) \\ &= \partial_K(ae) + \bar{\theta}_K(ae) + \overline{\partial}(a)e + aD''(e) \\ &= \partial(a)e + a\partial_K(e) + a\bar{\theta}_K(e) + \overline{\partial}(a)e + aD''(e) \\ &= d(a)e + aD'_K(e) + aD''(e) \\ &= d(a)e + aD_K(e). \end{aligned}$$

In particular, we get a flat structure if and only if the connection is integrable.

Conversely, let (V, D) be a flat bundle with metric K. Decomposing D into $d' + d''$ its $(1, 0)$ and $(0, 1)$ parts respectively, we may define two new operators δ'' and δ' such that $d' + \delta''$ and $\delta' + d''$ preserve K in the sense of (6). Now define our Higgs structure by

$$\partial = (d' + \delta')/2, \quad \overline{\partial} = (d'' + \delta'')/2, \quad \theta = (d' - \delta')/2, \quad \bar{\theta} = (d'' - \delta'')/2. \tag{8}$$

We may describe this structure locally by fixing a flat frame $\{v_i\}$ on $\mathcal{V}$. Define $h_{ij} := (v_i, v_j)_K$, and express θ as

$$\theta = \theta^{ij}_k \otimes v_i \otimes v^*_j \otimes dz_k. \tag{9}$$

Then contraction with v_m allows us to solve for the coefficients of θ:

$$\frac{\partial h_{jm}}{\partial z_k} = \sum_i \theta^{ij}_k h_{im}. \tag{10}$$

Set $D''_K = \overline{\partial} + \theta$. Then we have

$$D''_K(av) = \overline{\partial}(a)v + aD''(v), \tag{11}$$

so D''_K is Leibniz, and defines a Higgs bundle if and only if it is integrable. To see that this is a correspondance, it is helpful to consider the associated operator

$$D^c_K = D''_K - D'_K = \delta'' - \delta'. \tag{12}$$

In particular, one notes that $D''_K = (D + D^c_K)/2$. □

Now suppose that $f : X \to Y$ were a morphism of complex manifolds. Then if E is a Higgs bundle on Y with metric K, f^*E is a Higgs bundle on X, f^*K is a metric, and $f^*D_K = D_{f^*K}$. One can do analogously for flat bundles on Y.

If E and F are Higgs bundles, we may define a Higgs bundle structure on their tensor product by the Leibniz rule:

$$D''(e \otimes f) = D''e \otimes f + e \otimes D''f. \tag{13}$$

Explicitly, the Higgs structure on $E \otimes F$ has associated one-form $\theta_E \otimes 1 + 1 \otimes \theta_F$. Once again, one can do the same procedure to put a flat connection on a tensor product of flat bundles. Moreover, if J and K are metrics on E and F which relate the Higgs operators D'' with the connections D, we claim that the metric on $E \otimes F$ relating the induced operator with the induced flat connection is

$$J \otimes K : (e \otimes f, e' \otimes f') \mapsto (e, e')_J \cdot (f, f')_K. \tag{14}$$

Indeed, write $D'' = \overline{\partial} + \theta$, and define a $D''_C = \overline{\partial} - \theta$. Then $D'_{E\otimes F}$ is defined by the equation

$$(D'_{E\otimes F}(e \otimes f), e' \otimes f') + (e \otimes f, D''_C(e' \otimes f')) = \overline{\partial}(e \otimes f, e' \otimes f'). \tag{15}$$

Now observe that the operator $D'_{J,K} = D'_J \otimes 1 + 1 \otimes D'_K$ also satisfies this equation, and $D = D'' + D'_{J\otimes K}$, as needed.

We may also define a Higgs structure on the dual bundle $E^\vee$. If D'' is a Higgs operator on E, define D'' on $E^\vee$ by

$$D''(\lambda)(e) + \lambda(D''e) = \overline{\partial}(\lambda e). \tag{16}$$

In particular, the associated endomorphism-valued one-form on $E^\vee$ is $-\theta^T$. This insures that $\mathcal{O}_X \to E \otimes E^\vee \to \mathcal{O}_X$ are morphisms of Higgs bundles. We may analogously define a flat connection on the dual of a flat bundle V, and force $\mathbb{C} \to V \otimes V^\vee \to \mathbb{C}$ to be morphisms of flat bundles.

Theorem 3 (Kähler Identities)**.** *If (E, D'', K) is a Higgs bundle, D'_K is the unique operator satisfying*

$$(D'_K)^* = -i[\Lambda, D''], \qquad (D'')^* = -i[\Lambda, D'_K]. \tag{17}$$

Recall from Theorem 2 that while a Higgs bundle always defines a connection and a flat bundle always defines an operator D'', they may fail to satisfy the integrability condition $(D'')^2 = D^2 = 0$. We may measure this failure with and endomorphism valued two-form.

Definition 4. The curvature of (E, D'', K) is the form $F_K = (D_K)^2 = D'_K D'' + D'' D'_K$.

This equality follows from the fact that $(D'_K)^2 = 0$. Moreover, for the same reason, F_K satisfies the Bianchi identities

$$D'' F_K = D'_K F_K = 0. \tag{18}$$

For flat bundles, we have the analogous definition.

Definition 5. The pseudocurvature of (V, D, K) is the form

$$G_K = (D''_K)^2 = (DD^c_K + D^c_K D)/4.$$

As before, $(D^c_K)^2 = 0$, giving us the Bianchi identities

$$DG_K = D^c_K G_K = 0. \tag{19}$$

Thus, we see that our correspondance depends on the choice of a metric K satisfying the equations $F_K = G_K = 0$. As there may be many such choices, we need a canonical one.

Definition 6. For E a Higgs bundle, a metric K is called Hermitian-Yang-Mills if for for some λ depending only on the slope of E we have $\Lambda F_K = \lambda \operatorname{Id}$. For V a flat bundle, a metric K is harmonic if $\Lambda G_K = 0$.

Theorem 7 (Siu, Sampson, Corlette, Deligne)**.** *If K is a harmonic metric, then $G_K = 0$ and V comes from a Higgs bundle.*

With this in mind, we refer to as a harmonic metric a metric K on a Higgs bundle for which $F_K = 0$.

Definition 8. A harmonic bundle (E, K) is a C^∞-bundle equipped with a Higgs structure and a flat structure related by a harmonic metric K.

As before, harmonicity respects pullbacks, tensors, and duals.

Definition 9. We say that a Higgs bundle E is stable if for every subsheaf $M \subseteq E$ preserved by θ with $0 < \operatorname{rk} M < \operatorname{rk} E$,

$$\frac{\deg M}{\operatorname{rk} M} < \frac{\deg E}{\operatorname{rk} E}.$$

When this inequality fails to be sharp, we say that E is semistable. When E is a direct sum of stable Higgs bundles with the same slope, we say that E is polystable.

Theorem 10. *Simple objects in the categories of Higgs bundles and flat bundles are related in the following way:*

(1) A flat bundle V admits a harmonic metric if and only if it is semi-simple.

(2) A Higgs bundle E admits a Hermitian-Yang-Mills metric if and only if it is polystable. Such a metric is harmonic if and only if

$$ch_1(E).[\omega]^{n-1} = ch_2(E).[\omega]^{n-2} = 0. \tag{20}$$

For V a flat bundle, we may consider $H^0_{\mathrm{dR}}(X, V)$ the space of sections v with $Dv = 0$. Analogously, for E a Higgs bundle, we may consider $H^0_{\mathrm{Dol}}(X, E)$ the space of holomorphic sections e with $\theta e = 0$, or equivalently the space of smooth sections v with $D''v = 0$.

Theorem 11. *Suppose that E is a harmonic bundle. Then $H^0_{Dol}(X, E) \simeq H^0_{dR}(X, E)$.*

Proof. Suppose that e is a section with $D''e = 0$. Then by Theorem 3 and the anti-commutativity of D' with D'',

$$(D')^* D'e = i\Lambda D''D'e = -i\Lambda D'D''e = 0.$$

Thus

$$||D'e||^2 = \int_X (D'e, D'e) = \int_X ((D')^* D'e, e) = 0.$$

In particular, $De = D'e + D''e = 0$. The converse follows by making the same argument with $(D^c)^* D^c$. $\square$

Theorem 12 (Non-Abelian Hodge Theorem I)**.** *The categories of semi-simple flat bundles on X, polystable Higgs bundles on X satisfying* (20)*, and harmonic bundles on X are all categorically equivalent.*

Proof. Apply Theorem 11 to get an isomorphism of Hom sets

$$\mathrm{Hom}_{\mathrm{dR}}(E,F) = H^0_{\mathrm{dR}}(X, E^* \otimes F) \simeq H^0_{\mathrm{Dol}}(X, E^* \otimes F) = \mathrm{Hom}_{\mathrm{Dol}}(E,F).$$

$\square$