HW3, Math 506, Fall 2016, Due 9/14

September 7, 2016

1. (1 point) Suppose that $\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2$ are $\mathcal{L}$-structures with elementary embeddings $f_i : \mathcal{M}_0 \to \mathcal{M}_i$ for $i = 1, 2$. Show that there is an $\mathcal{L}$-structure with $\mathcal{N}$ with elementary embeddings $g_i : \mathcal{M}_i \to \mathcal{N}$ for $i = 1, 2$ such that $g_1 \circ f_1 = g_2 \circ f_2$.

2. (2 points) A theory T has a $\forall\exists$-axiomatization if it can be axiomatized by sentences of the form $\forall \bar{v} \exists \bar{w} \varphi(\bar{v}, \bar{w})$ where φ is quantifier free. Suppose you are given a chain of models of a $\forall\exists$-theory T. Show that the union of the chain also satisfies T. Now we will show the converse - theories which are preserved by unions of chains have $\forall\exists$-axiomatizations. Let $S = \{\varphi \,|\, \varphi \text{ is } \forall\exists,\, T \models \varphi\}$. Let $\mathcal{M} \models S$. We wish to show that $\mathcal{M} \models T$.

   - Show that there is a $\mathcal{N} \models T$ such that for any $\exists\forall$sentence ψ if $\mathcal{M} \models \psi$ then $\mathcal{N} \models \psi$.
   - Show that there is $\mathcal{N}'$ with $\mathcal{M} \subseteq \mathcal{N}'$ and $\mathcal{N} \equiv \mathcal{N}$
   - Show that there is $\mathcal{M}' \subseteq \mathcal{N}'$ with $\mathcal{M}$ an elementary substructure of $\mathcal{M}'$.
   - Iterate the construction and complete the proof.

3. (1 point) We say that $\mathcal{M} \models T$ is existentially closed if whenever $\mathcal{N} \models T$, $\mathcal{M} \subset \mathcal{N}$ and $\mathcal{N} \models \exists \bar{v} \varphi(\bar{v}, \bar{a})$ with $\bar{a} \in \mathcal{M}$ and φ quantifier-free then $\mathcal{M} \models \exists \bar{v} \varphi(\bar{v}, \bar{a})$.

   Now fix some model $\mathcal{M}_0 \models T$. Prove that if T is $\forall\exists$-axiomatizable, then T has an existentially closed model $\mathcal{N}_0$ containing $\mathcal{M}_0$ such that $|\mathcal{N}_0| = |\mathcal{M}| + |\mathcal{L}| + \aleph_0$.

4. (1 point) Suppose that T has built-in Skolem functions. Show that T has a universal axiomatization.

5. (1 point) Give an example of an $\mathscr{L}_{\omega_1\omega}$ sentence Φ such that every finite subsentence og Φ is satisfiable, but Φ is not. (So compactness fails).

6. (2 points) Axiomatize the following classes of structures with some single sentence in some language using $\mathscr{L}_{\omega_1\omega}$:

   - Torsion-free abelian groups.
   - Finitely generated fields.
   - Linear orders isomorphic to $(\mathbb{Z}, <)$.
   - Connected graphs.
   - Finite valence graphs.

- Cycle-free graphs.

7. (1 point) Give an example of a countable language $\mathscr{L}$ and an $\mathscr{L}_{\omega_1\omega}$ sentence Φ such that every model of Φ has cardinality at least $2^{\aleph_0}$. (So Downward Löwenheim-Skolem fails).

1 A primer on infinitary logic

Given a signature τ we now define the *infinitary* language $\mathscr{L}_{\infty\omega}$ associated to τ. Roughly speaking the two subscripts describe how many conjunction/disjuntions we are allowed to use and how many quantifications we are allow. The first subscript '∞' indicates that we will allow infinitely many conjunctions and disjunctions. The second subscript 'ω' indicates that we will allow only finitely many quantifiers in a row.

The symbols of $\mathscr{L}_{\infty\omega}$ are all symbols from the signature τ together with the usual logical symbols:

$$=, \neg, \bigwedge, \bigvee, \forall, \exists.$$

The terms, atomic formulae, and literals are defined in the same way as before (i.e. for first-order logic).

Definition: $\mathscr{L}_{\infty\omega}$ is the smallest class such that

- all atomic formulae are in $\mathscr{L}_{\infty\omega}$
- if $\varphi \in \mathscr{L}_{\infty\omega}$ then $\neg\varphi \in \mathscr{L}_{\infty\omega}$
- if $\Phi \subseteq \mathscr{L}_{\infty\omega}$ then $\bigvee \Phi$ and $\bigwedge \Phi$ are in $\mathscr{L}_{\infty\omega}$
- if $\varphi \in \mathscr{L}_{\infty\omega}$ then $\forall x\varphi$ and $\exists x\varphi$ are in $\mathscr{L}_{\infty\omega}$

Remark: We are allowing $\Phi \subseteq \mathscr{L}_{\infty\omega}$ to be an *arbitrary* subset, so we are allowing arbitrary conjunctions and disjunctions, contrary to the case for the usual first-order logic.

Given an $\mathscr{L}$-structure $\mathfrak{A}$ (with domain A) we can now extend the notion of satisfaction "$\models$" to arbitrary formulae of $\mathscr{L}_{\infty\omega}$;

- For atomic formulae the $\models$ relation is the same as before.
- Given $\varphi(\bar{x}) \in \mathscr{L}_{\infty\omega}$ then $\mathfrak{A} \models \neg\varphi(\bar{a})$ if and only if it is not the case that $\mathfrak{A} \models \varphi(\bar{a})$.
- Given $\Phi(\bar{x}) \subseteq \mathscr{L}_{\infty\omega}$ then $\mathfrak{A} \models \bigwedge \Phi(\bar{a})$ if and only if, for all $\varphi(\bar{x}) \in \Phi(\bar{x})$ $\mathfrak{A} \models \varphi(\bar{a})$.
- Given $\Phi(\bar{x}) \subseteq \mathscr{L}_{\infty\omega}$ then $\mathfrak{A} \models \bigvee \Phi(\bar{a})$ if and only if, for at least one of $\varphi(\bar{x}) \in \Phi(\bar{x})$ we have $\mathfrak{A} \models \varphi(\bar{a})$.
- Given $\varphi(y, \bar{x}) \in \mathscr{L}_{\infty\omega}$, then $\mathfrak{A} \models \forall y\varphi(y, \bar{a})$ if and only if for all $b \in A$ we have $\mathfrak{A} \models \varphi(b, \bar{a})$.
- Given $\varphi(y, \bar{x}) \in \mathscr{L}_{\infty\omega}$, then $\mathfrak{A} \models \exists y\varphi(y, \bar{a})$ if and only if for at least one $b \in A$ we have $\mathfrak{A} \models \varphi(b, \bar{a})$.

Now we say that **first-order logic** is the language $\mathscr{L}_{\omega\omega}$ where we allow only finite subsets Φ (in other words we have only finite conjunctions and disjunctions), and only finitely many quantifiers. In general for some cardinal κ we get a language $\mathscr{L}_{\kappa\omega}$ where we allow the subsets $\Phi \subseteq \mathscr{L}_{\kappa\omega}$ to have size $< \kappa$.

In model theory we most often either work within $\mathscr{L}_{\omega\omega}$, and occasionally in $\mathscr{L}_{\omega_1\omega}$. The latter language allows *countably* many conjunctions and disjunctions. There are several properties of first-order logic that the infinitary logics fail to have (see above exercises).