Some open problems in matchings in graphs

Shmuel Friedland
Univ. Illinois at Chicago

Stability, hyperbolicity, and zero localization of functions,
AIM workshop, Palo Alto, December 9, 2011
Overview

- Matchings in graphs
- Number of $k$-matchings in bipartite graphs and graphs as permanents and haffnians
- Upper bounds on permanents and haffnians: results and conjectures.
- Lower bounds on permanents and haffnians: results and conjectures.
**Matchings**

- **$G = (V, E)$** undirected graph with vertices $V$, edges $E$.
- **matching in $G$:** $M \subseteq E$
  - no two edges in $M$ share a common endpoint.
- $e = (u, v) \in M$ is **dimer**
- $v$ not covered by $M$ is **monomer**.
- $M$ called **monomer-dimer cover** of $G$.
- **$M$ is perfect matching** $\iff$ no monomers.
- **$M$ is $k$-matching** $\iff$ $\#M = k$. 

---

Shmuel Friedland  Univ. Illinois at Chicago  ()  Some open problems in matchings in graphs  Stability, Hyperbolicity, and Zero Localization  / 29
Generating matching polynomial

- \( \phi(k, G) \) number of \( k \)-matchings in \( G \), \( \phi(0, G) := 1 \)
- \( \Phi_G(x) := \sum_k \phi(k, G)x^k \) matching generating polyn.
- roots of \( \Phi_G(x) \) are real nonpositive Heilmann-Lieb 1972. Newton inequalities hold
- \( \Phi_{G_1 \cup G_2}(x) = \Phi_{G_1}(x)\Phi_{G_2}(x) \)

Examples:

\[
\Phi_{K_{2r}}(x) = \sum_{k=0}^{r} \binom{2r}{2k} \frac{\prod_{j=0}^{k-1} \left( \frac{2k-2j}{2} \right)}{k!} x^k = \sum_{k=0}^{r} \frac{(2r)!}{(2r-2k)!2^k k!} x^k
\]

\[
\Phi_{K_{r,r}}(x) = \sum_{k=0}^{r} \binom{r}{k} 2^k k! x^k
\]

\( G(r, 2n) \supset G\mathcal{B}(r, 2n) \) set of \( r \)-regular and regular bipartite graphs on \( 2n \) vertices, respectively

\[ qK_{r,r} \in G\mathcal{B}(r, 2rq) \) a union of \( q \) copies of \( K_{r,r} \).

\[ \Phi_{qK_{r,r}} = \Phi_{K_{r,r}}^q \]
Formulas for \( k \)-matchings in bipartite graphs

\[ G = (V, E) \text{ bipartite } V = V_1 \cup V_2, E \subset V_1 \times V_2, \]
represented by bipartite adjacency matrix
\[ B(G) = B = [b_{ij}]_{i,j=1}^{m \times n} \in \{0, 1\}^{m \times n}, \# V_1 = m, V_2 = n. \]

Example: Any subgraph of \( \mathbb{Z}^d \) is bipartite

CLAIM: \( \phi(k, G) = \text{perm}_k(B(G)) \).

Prf: Suppose \( n = \# V_1 = \# V_2 \).
Then permutation \( \sigma : \langle n \rangle \to \langle n \rangle \) is a perfect match iff \( \prod_{i=1}^{n} b_{i\sigma(i)} = 1 \).
The number of perfect matchings in \( G \) is \( \phi(n, G) = \text{perm} B(G) \).

Computing \( \phi(n, G) \) is \#P-complete problem Valiant 1979

For \( G = \langle 2n \rangle \) bipartite \( G \in \mathcal{GB}(r, 2n) \iff \frac{1}{r} B(G) \in \Omega_n \iff G \) is a disjoint (edge) union of \( r \) perfect matchings
Matching on nonbipartite graphs

\[ G = (V, E), |V| = 2n, \]
\[ A(G) = [a_{ij}] \in S_0(2n, \{0, 1\}) \text{ - adjacency matrix of } G \]

\[ \phi(n, G) = \text{haf}(A(G)) = \sum_{M \in \mathcal{M}(K_{2n})} \prod_{(i,j) \in M} a_{ij} \]
\[ \mathcal{M}(K_{2n}) \text{ the set of perfect matchings in } K_{2n} \]

\[ \phi(k, G) = \text{haf}_k(A(G)) = \sum_{M \in \mathcal{M}_k(K_{2n})} \prod_{(i,j) \in M} a_{ij} \]
\[ \mathcal{M}_k(K_{2n}) \text{ the set of } k \text{ matchings in } K_{2n} \]

Claim \( \text{perm}(A(G)) \geq \text{haf}(A(G))^2 \). Equality holds if \( G \) is bipartite.
Main problems

Find good estimates on

\[ s_n(k, r) := \min_{G \in G(r, 2n)} \phi(k, G) \leq t_n(k, r) := \min_{G \in G_B(r, 2n)} \phi(k, G) \]

\[ S_n(k, r) := \max_{G \in G(r, 2n)} \phi(k, G) \geq T_n(k, r) := \max_{G \in G_B(r, 2n)} \phi(k, G) \]

Completely solved case \( r = 2 \) [8]

\[ S_n(k, 2) = T_n(k, 2) \text{ achieved only for } G = mK_{2,2} \text{ or } G = mK_{2,2} \cup C_6. \]

\( t_n(k, 2) \text{ achieved only for } C_{2n} \)

\( s_n(k, 2) \text{ achieved only for } mC_3, mC_3 \cup C_4 \text{ or } mC_3 \cup C_5. \)
The upper bound conjecture

\[ S_{qr}(k, r) = T_{qr}(k, r) = \phi(k, qK_{r,r}) \]

\( k = qr \) Follows from Bregman’s inequality (see also [3])

\[ \text{perm } A \leq \prod_{i=1}^{n} (r_i!)^{\frac{1}{r_i}} \]

\[ A = [a_{ij}] \in \{0, 1\}^{n \times n} r_i = \sum_{j=1}^{n}, i = 1, \ldots, n \]

Egorichev-Alon-Friedland for \( G = (V, E), |V| = 2n \)

\[ \phi(n, G) \leq \prod_{v \in V} (\text{deg}(v)!)^{\frac{1}{2 \text{deg}(v)}} \]

Equality holds iff \( G \) a union of complete bipartite graphs

\[ S_n(k, r) \leq \binom{2n}{2k} (r!)^k \]

\[ T_n(k, r) \leq \min(\binom{n}{k}^2 (r!)^k, \binom{n}{k} r^k) \]

The lower bounds: Bipartite case

\[ r^k \min_{C \in \Omega_n} \text{perm}_k C \leq \phi(k, G) \] for any \( G \in \mathcal{GB}(r, 2n) \)

\( J_n = B(K_{n,n}) = [1] \) the incidence matrix of the complete bipartite graph \( K_{n,n} \) on \( 2n \) vertices

van der Waerden permanent conjecture 1926:

\[
\min_{C \in \Omega_n} \text{perm} C = \text{perm} \frac{1}{n} J_n \left( = \frac{n!}{n^n} \approx \sqrt{2\pi n} e^{-n} \right)
\]

Tverberg permanent conjecture 1963:

\[
\min_{C \in \Omega_n} \text{perm}_k C = \text{perm}_k \frac{1}{n} J_n \left( = \left( \binom{n}{k} \right)^2 \frac{k!}{n^k} \right)
\]

for all \( k = 1, \ldots, n \).
In 1979 Friedland showed the lower bound $\text{perm } C \geq e^{-n}$ for any $C \in \Omega_n$ following T. Bang’s announcement 1976. This settled the conjecture of Erdös-Rényi on the exponential growth of the number of perfect matchings in $d \geq 3$-regular bipartite graphs 1968, Voorhoeve 1979.

van der Waerden permanent conjecture was proved by Egorichev and Falikman 1981.

Tverberg conjecture was proved by Friedland 1982

79 proof is tour de force according to Bang

81 proofs involve directly (Egorichev) and indirectly (Falikman) use of Alexandroff mixed volume inequalities with the conditions for the extremal matrix

82 proof uses methods of 81 proofs with extra ingredients

There are new simple proofs using nonnegative hyperbolic polynomials e.g. Gurvits, Friedland-Gurvits
Lower matching bounds for bipartite graphs

Voorhoeve-1979 ($r = 3$) Schrijver-1998

$$
\phi(n, G) \geq \left( \frac{(r - 1)^{r-1}}{r^{r-2}} \right)^n \quad \text{for} \quad G \in \mathcal{GB}(r, 2n)
$$

Gurvits 2006: $A \in \Omega_n$, each column has at most $r$ nonzero entries:

$$
\text{perm} A \geq \frac{r!}{r^r} \left( \frac{r}{r - 1} \right) r^{(r-1)} \left( \frac{r - 1}{r} \right)^{(r-1)n}.
$$

**Cor:**

$$
\phi(n, G) \geq \frac{r!}{r^r} \left( \frac{r}{r - 1} \right) r^{(r-1)} \left( \frac{r - 1}{r^{r-2}} \right)^n
$$

**Con FKM 2006:**

$$
\phi(k, G) \geq \binom{n}{k}^2 (\frac{nr - k}{nr})^{nr-k} (\frac{kr}{n})^k, \quad G \in \mathcal{GB}(r, 2n)
$$

F-G 2008 showed weaker inequalities
Positive hyperbolic polynomials

A polynomial \( p = p(x) = p(x_1, \ldots, x_n) : \mathbb{R}^n \rightarrow \mathbb{R} \) is called \textit{positive hyperbolic} if

\( \phi(t) := p(x + tu), \) for \( t \in \mathbb{R} \), has \( m \)-real \( t \)-roots for each \( u > 0 \) and each \( x \).

Ex. 1: \( A = (a_{ij})_{i,j=1}^{m,n} \in \mathbb{R}^{m \times n} \)

\[ \rho_{k,A}(x) := \sum_{1 \leq i_1 < \ldots < i_k \leq m} \prod_{j=1}^{k} (Ax)_{i_j}, \] \( x \in \mathbb{R}^n \)

Ex. 2: \( A_1, \ldots, A_n \in \mathbb{C}^{m \times m} \) hermitian, nonnegative definite matrices such that \( A_1 + \ldots + A_n \) is a positive definite matrix. Let \( p(x) = \det \sum_{i=1}^{n} x_i A_i \). Then \( p(x) \) is positive hyperbolic.

Ex. 3: \( B \in \mathbb{R}^{m \times m}_+ \) symmetric. Then \( x^\top Bx \) positive hyperbolic iff \( B \) has exactly one positive eigenvalue.
$\rho(\mathbf{x}) : \mathbb{R}^n \rightarrow \mathbb{R}$ positive hyperbolic polynomial of degree $m \geq 1$.

**Gurvits** $\operatorname{Cap} \rho := \inf_{x > 0, x_1 \ldots x_n = 1} \rho(\mathbf{x})$

$A \in \mathbb{R}_{+}^{n \times n}$ doubly stochastic. Then $\operatorname{Cap} \rho_{k,A} = \binom{n}{k}$.

Let $B = D_1 A D_2$, $D_1, D_2$ positive diagonal, $A$ doubly stochastic matrix. Let $\rho_{n,B}$ be defined as above. Then $\operatorname{Cap} \rho_{n,B} = \frac{1}{\det D_1 D_2}$.

**Lemma:** $\rho : \mathbb{R}^n \rightarrow \mathbb{R}$ positive hyperbolic of degree $m \geq 1$. Assume that $\operatorname{Cap} \rho > 0$. Then $\deg_i \rho \geq 1$ for $i = 1, \ldots, n$. For $m = n \geq 2$

$\operatorname{Cap} \frac{\partial \rho}{\partial x_i}(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) \geq \left(\frac{\deg_i \rho - 1}{\deg_i \rho}\right)^{\deg_i \rho - 1} \operatorname{Cap} \rho$ for $i = 1, \ldots, n$, where $0^0 = 1$. 

Friedland-Gurvits inequality

Let \( p : \mathbb{R}^n \to \mathbb{R} \) be positive hyperbolic of degree \( m \in [1, n] \). Assume that \( \deg_i p \leq r_i \in [1, m] \) for \( i = 1, \ldots, n \). Rearrange the sequence \( r_1, \ldots, r_n \) in an increasing order \( 1 \leq r_1^* \leq r_2^* \leq \ldots \leq r_n^* \). Let \( k \in [1, n] \) be the smallest integer such that \( r_k^* > m - k \). Then

\[
\sum_{1 \leq i_1 < \ldots < i_m \leq n} \frac{\partial^m p}{\partial x_{i_1} \cdots \partial x_{i_m}}(0) \geq \frac{n^{n-m}}{(n-m)!} \frac{(n-k+1)!}{(n-k+1)^{n-k+1}} \prod_{j=1}^{k-1} \left( \frac{r_j^* + n - m - 1}{r_j^* + n - m} \right)^{r_j^* + n - m - 1} \text{Cap } p. \quad (0.1)
\]

(Here \( 0^0 = 1 \), and the empty product for \( k = 1 \) is assumed to be 1.) If \( \text{Cap} > 0 \) and \( r_i = m \) for \( i = 1, \ldots, m \) equality holds if and only if \( p = C \left( \frac{x_1 + \ldots + x_n}{n} \right)^m \) for each \( C > 0 \).
$p$-matching and total matching entropies

$G = (V, E)$ infinite, degree of each vertex bounded by $N$,

$p \in [0, 1]$-matching entropy, $(p$-dimer entropy) of $G$

$$h_G(p) = \sup_{\text{on all sequences}} \limsup_{l \to \infty} \frac{\log \phi(k, G_l)}{\# V_l}$$

and total matching entropy, (monomer-dimer entropy)

$$h_G = \sup_{\text{on all sequences}} \limsup_{l \to \infty} \frac{\log \sum_{k=0}^{0.5(\# V_l)} \phi(k, G_l)}{\# V_l},$$

$G_l = (E_l, V_l), l \in \mathbb{N}$ a sequence of finite graphs converging to $G$, and

$$\lim_{l \to \infty} \frac{2k_l}{\# V_l} = p$$

$$h_G = \max_{p \in [0, 1]} h_G(p)$$
Asymptotic versions

\[ Sa(p, r) = \limsup_{n_j \to \infty, \frac{k_j}{n_j} \to p \in [0,1]} \frac{\log S_{n_j}(k_j, r)}{2n_j} \]

\[ Ta(p, r) = \limsup_{n_j \to \infty, \frac{k_j}{n_j} \to p \in [0,1]} \frac{\log T_{n_j}(k_j, r)}{2n_j} \]

\[ sa(p, r) = \liminf_{n_j \to \infty, \frac{k_j}{n_j} \to p \in [0,1]} \frac{\log s_{n_j}(k_j, r)}{2n_j} \]

\[ ta(p, r) = \liminf_{n_j \to \infty, \frac{k_j}{n_j} \to p \in [0,1]} \frac{\log t_{n_j}(k_j, r)}{2n_j} \]

Next slide gives the graphs of AUMC and the upper bounds for \( Ta(p, 4) \).
Expected values of $k$-matchings for bipartite graphs

- **Permutation** $\sigma : \langle nr \rangle \to \langle nr \rangle$ induces $G(\sigma) \in GB_{\text{mult}}(r, 2n)$ and vice versa.

  $G(\sigma) = \{(i, \lceil \frac{\sigma((i-1)r+j)}{r} \rceil), j = 1, \ldots, r, i = 1, \ldots, n\} \subset \langle n \rangle \times \langle n \rangle$

- The number of different $\sigma$ inducing the same simple $G$ is $(r!)^n$.

- **$\mu$ probability measure on $GB_{\text{mult}}(r, 2n)$:**

  $\mu(G(\sigma)) = ((nr)!)^{-1}$

- **FKM 06:**

  $E(k, n, r) := E(\phi(k, G)) = \left(\frac{n}{k}\right) r^{2k} k! (nr - k)! (nr)!^{-1}$, $k = 1, \ldots, n$

- For $1 \leq k_l \leq n_l, l = 1, \ldots, n$, increasing sequences of integers such that $\lim_{l \to \infty} \frac{k_l}{n_l} = p \in [0, 1]$. Then

  $$\lim_{l \to \infty} \log \frac{E(k_l, n_l, r)}{2n_k} = f(p, r)$$

  $$f(p, r) := \frac{1}{2} (p \log r - p \log p - 2(1-p) \log(1-p) + (r-p) \log(1 - \frac{p}{r}))$$
Asymptotic Lower and Upper Matching conjectures

FKLM JOSS 08:

\[ G_l = (E_l, V_l) \in \mathcal{G}(r, \# V_l), l = 1, 2, \ldots, \] and \( \lim_{l \to \infty} \frac{2k_l}{\# V_l} = p. \)

\[ \text{low}_r(p) := \inf \text{lim inf}_{l \to \infty} \frac{\log \phi(k_l, G_l)}{\# V_l} \]

\textbf{ALMC:} \( \text{low}_r(p) = f(p, r) \) (For most of the sequences \( \lim \inf = f(p, r) \))

\[ \text{upp}_r(p) := \sup \text{lim sup}_{l \to \infty} \frac{\log \phi(k_l, G_l)}{\# V_l} \]

\textbf{AUMC:} \( \text{upp}_r(p) = h_{K(r)}(p), K(r) \) countable union of \( K_{r,r} \)

\[ P_r(t) := \frac{\log \sum_{k=0}^{r} \binom{r}{k}^2 k! e^{2kt}}{2r}, \quad t \in \mathbb{R}, \]

\[ p(t) := P'_r(t) \in (0, 1), \quad h_{K(r)}(p(t)) := P_r(t) - tp(t) \]
Some open problems in matchings in graphs
Some open problems in matchings in graphs

Stability, hyperbolicity, and zero localization.
$r = 4$ upper bounds

**Figure:** $h_{K(4)}$-green, $\text{upp}_{4,1}$-blue, $\text{upp}_{4,2}$-orange
Lower asymptotic bounds Friedland-Gurvits 2008

Thm: $r \geq 3, s \geq 1$ integers,

$B_n \in \Omega_n, n = 1, 2, \ldots$ each column of $B_n$ has at most $r$-nonzero entries.

$k_n \in [0, n] \cap \mathbb{N}, n = 1, 2, \ldots$, $\lim_{n \to \infty} \frac{k_n}{n} = p \in (0, 1]$ then

$$\liminf_{n \to \infty} \frac{\log \text{perm}_{k_n} B_n}{2n} \geq \frac{1}{2} (-p \log p - 2(1 - p) \log(1 - p)) +$$

$$\frac{1}{2} (r + s - 1) \log (1 - \frac{1}{r + s}) - \frac{1}{2} (s - 1 + p) \log (1 - \frac{1 - p}{s})$$

Prf combines properties positive hyperbolic polynomials, capacity and the measure on $G(r, 2n)$

- Cor: $r$-ALMC holds for $p_s = \frac{r}{r+s}, s = 0, 1, \ldots$
- Con: under Thm assumptions

$$\liminf_{n \to \infty} \frac{\log \text{perm}_{k_n} B_n}{2n} \geq f(r, p) - \frac{p}{2} \log r$$

- For $p_s = \frac{r}{r+s}, s = 0, 1, \ldots$, conjecture holds
Lower bounds for matchings in regular non-bipartite graphs

Petersen’s THM: A bridgeless cubic graph has a perfect match

Problem: Find the minimum of the biggest match in \( G(r, 2n) \) for \( r > 2 \).

Does every \( G \in G(r, 2n) \) has a match of size \( \lfloor \frac{2n}{3} \rfloor \)? (True for \( r = 2 \).)

Esperet-Kardos-King-Král-Norine:
Every cubic bridgeless graph has at least \( 2^{3656} \) perfect matchings

Cygan-Pilipczuk-Skrekovski:
\( \exists \) inf-family of cubic 3-colored connected graphs \( G = (V, E) \) s.t.
\( \text{haf}(A(G)) \approx c_F |V|(\frac{1+\sqrt{5}}{2}) \frac{|V|}{12} \), \( |V| = 12k + 4, k = 1, 2, \ldots \)
THM Edmonds 1965: A symmetric doubly stochastic matrix with zero diagonal of even order $A = [a_{ij}]_{i,j=1}^{2n}$ is a convex combination of symmetric permutation matrices with zero diagonal if and only if
\[ \sum_{i,j \in S} a_{ij} \leq |S| - 1 \]
for any odd subset $S \subset \{1, \ldots, 2n\}$ (*)

Denote by $\psi_{2n}$ the subset of all symmetric doubly stochastic matrices of the above form.

Problem: Find $\mu_{n,n} := \min \text{haf}(A), A \in \psi_{2n}$

FALSE CONJECTURE: The minimum is achieved only for the matrix $\frac{1}{2n-1} A(K_{2n})$

\[ \text{haf}(\frac{1}{2n-1} A(K_{2n})) \approx e^{-n\sqrt{2e}} < \text{haf}(\frac{1}{n} A(K_{n,n})) \approx e^{-n\sqrt{2\pi n}} \]

CONJECTURE: $\mu := \lim_{n \to \infty} \frac{\log \mu_{n,n}}{n} > -\infty$

C-P-S $\mu \leq \frac{\log \frac{1+\sqrt{5}}{2}}{6} - \log 3$
Hyperbolic polynomials

**THM:** Good lower bounds hold for $\text{haf}_k(A)$ if $A \in \Psi_{2n} n-1 n-1$ eigenvalues of $A$ are nonpositive

**Outline of proof:** Fact $\mathbf{x}^\top A \mathbf{x}$ is a hyperbolic polynomial for a nonnegative symmetric matrix iff $A$ has all but one nonpositive eigenvalues [5]

$$\text{haf}_k A = (2^k k!)^{-1} \sum_{1 \leq i_1 < \ldots < i_{2k} \leq 2n} \frac{\partial^{2k}}{\partial x_{i_1} \ldots \partial x_{i_{2k}}} \left( \mathbf{x}^\top A \mathbf{x} \right)^k$$

Use the arguments of [2] to show

$$\text{haf}_n(B) \geq \left( \frac{n-1}{n} \right)^{(n-1)n} \approx e^{-n} \sqrt{e}$$

$$\text{haf}_k(B) \geq \frac{(2n)^{2n-2k} (2n-k)! (2n)^k}{(2n-2k)! (2n-k)^{2n-k} 2^k k!} \left( \frac{(2n-k-1)}{2n-k} \right)^{(2n-k-1)k}$$
References


References


S. Friedland, Analogs of the van der Waerden and Tverberg conjectures for haffnians, arXiv:1102.2542.


References


