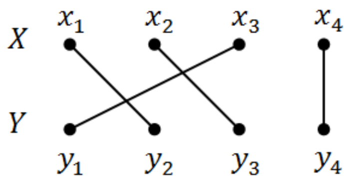
Summary of Lectures

Definition 1. A **matching** in a graph G is a set of non-loop edges with no shared endpoints. The vertices incident to the edges of a matching M are **saturated** by M; the others are **unsaturated** (we say M-saturated and M-unsaturated). A **perfect matching** in a graph is a matching that saturates every vertex.

Example 2 (Perfect matchings in $K_{n,n}$)**.** Consider $K_{n,n}$ with partite sets $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$. A perfect matching defines a bijection from X to Y. Successively finding mates for $x_1, x_2, \ldots$ yields $n!$ perfect matchings.

Each matching is represented by a permutation of $[n]$, mapping i to j when x_i is matched to y_j. We can express the matchings as matrices. With X and Y indexing the rows and columns, we let position i, j be 1 for each edge $x_i y_j$ in a matching M to obtain the corresponding matrix. There is one 1 in each row and each column.



$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Definition 3. A **maximal matching** in a graph is a matching that cannot be enlarged by adding an edge. A **maximum matching** is a matching of maximum size among all matchings in the graph.

A matching M is maximal if every edge not in M is incident to an edge already in M. Every maximum matching is a maximal matching, but the converse need not hold.

Example 4 (Maximal $\neq$ maximum)**.** The smallest graph having a maximal matching that is not a maximum matching is P_4. If we take the middle edge, then we can all no other, but the two end edges form a larger matching. Below we show this phenomenon in P_4 and in P_6.

In Example 4, replacing the bold edges by the solid edges yields a larger matching. This gives us a way to look for larger matchings.

Definition 5. Given a matching M, an M-**alternating path** is a path that alternates between edges in M and edges not in M. An M-alternating path whose enpoints are unsaturated by M is an M-**augmenting path**.

Definition 6. If G and H are graphs with vertex set V, then the **symmetric defference** $G \triangle H$ is the graph with vertex set V whose edges are all those edges appearing in exactly one of G and H. We also use this notation for sets of edges; in particular, if M and M' are matchings, then $M \triangle M' = (M - M') \cup (M' - M)$.

Lemma 7. Every component of the symmetric difference of two matchings is a path or an even cycle.

Proof. Let M and M' be matchings, and let $F = M \triangle M'$. Since M and M' are matchings, every vertex has at most one incident edge from each of them. Thus F has at most two edges at each vertex. Since $\Delta(F) \leq 2$, every component of F is a path or a cycle. Furthermore, every path or cycle in F alternates between edges of $M - M'$ and edges of $M' - M$. Thus each cycle has even length, with an equal number of edges from M and from M'. □

Theorem 8 (Berge [1957])**.** A matching M in a graph G is a maximum matching in G if and only if G has no M-augmenting path.

Proof. We prove the contrapositive of each direction; G has a matching larger than M if and only if G has an M-augmenting path. We have observed that an M-augmenting path can be used to produce a matching larger than M.

For the converse, let M' be a matching in G larger than M; we costruct an M-augmenting path. Let $F = M \triangle M'$. By Lemma 7, F consists of paths and even cycles; the cycles have the same number of edges from M and M'. Since $|M'| > |M|$, F must have a component with more edges of M' than of M. Such a component can only be a path that starts and ends with an edge of M'; thus it is an M-augmenting path in G. □

Hall's matching condition:

Consider an X, Y-bigraph (bipartite graph with bipartition X, Y), we seek a matching that satures X.

If a matching M satures X, then for every $S \subseteq X$, there must be at least $|S|$ vertices that have neighbors in S, because the vertices matched to S must be chosen from that set. We use $N_G(S)$ or simply $N(S)$ to denote the set of vertices having neighbors in S. Thus $|N(S)| \geq |S|$ is a necessary condition. The condition "For all $S \subseteq X$, $|N(S)| \geq |S|$" is Hall's Condition. Hall proved that this obvious necessary condition is also sufficient.

Theorem 9 (Hall's Theorem)**.** An X, Y bigraph G has a matching that satures X if and only if $|N(S)| \geq |S|$ for all $S \subseteq X$.

Proof. Necessity: The $|S|$ vertices matched to S must lie in $N(S)$.

Sufficiency: Assume to the cotrary, there is no matching that satures X. If M

is a maximum matching in G, then it does not sature X. Let $u \in X$ be a vertex unsaturated by M. Define S the set of all vertices in X reachable from u by M-alternating paths in G. Note that $u \in S$. Also define T the set of all vertices in Y reachable from u by M-alternating paths in G. We claim that M matches T with $S - \{u\}$. The M-alternating paths from u reach Y along edges not in M and return to X along edges in M. Hence every vertex of $S - \{u\}$ is reached by an edge in M from a vertex in T. Since there is no M-augmenting path, every vertex of T is saturated. (Note that the reason that there is no M-augmenting path is immediate by Berge's theorem, also the reason that every vertex of T is saturated is that otherwise we get M-augmenting path). Thus an M-alternating path reaching $y \in T$ extends via M to a vertex of S. Hence these edges of M yield a bijection from T to $S - \{u\}$, and we have $|T| = |S - \{u\}|$.

This implies $|T| = |S - \{u\}|$. The matching between T and $S - \{u\}$ yields $T \subseteq N(S)$. In fact, $T = N(S)$. Suppose that $y \in Y - T$ has a neighbor $v \in S$. The edge vy cannot be in M, since u is unsaturated and the rest of S is matched to T by M. Thus adding vy to an M-alternating path reaching v yields an M-alternating path to y. This contradicts $y \notin T$, and hence vy cannot exist.

With $T = N(S)$, we have proved $|N(S)| = |T| = |S| - 1 < |S|$, for this choice of S. This completes the proof of the contrapositive. □

When the sets of the bipartition have the same size, Hall's Theorem is the Marriage Theorem, proved originally by Frobenius [1917]. The name arises from the setting of the compatibility relation between a set of n men and a set of n women. If every man is compatible with k women and every woman is compatible with k men, then a perfect matching must exist. Again multiple edges are allowed, which enlarge the scope of applications.

Theorem 10 (Marriage Theorem)**.** Consider an X, Y-bigraph G with $|X| = |Y|$. Then G has a perfect matching if and only if $|S| \leq |N(S)|$, for any $S \subseteq X$.

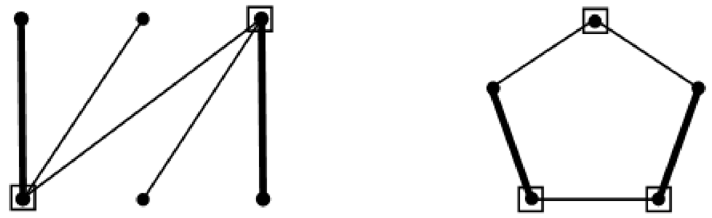
Corollary 11. For $k > 0$, every k-regular bipartite graph has a perfect matching.

Proof. Let G be a k-regular X, Y-bigraph. Counting the edges by endpoints in X and by endpoints in Y shows that $k|X| = k|Y|$, so $|X| = |Y|$. Hence it suffices to verify Hall's Condition; a matching that saturates X will also saturate Y and be a perfect matching.

Consider $S \subseteq X$. Let m be the number of edges from S to $N(S)$. Since G is k-regular, $m = k|S|$. These m edges are incident to $N(S)$, so $m \leq k|N(S)|$. Hence $k|S| \leq k|N(S)|$, which yields $|N(S)| \geq |S|$, when $k > 0$. Having chosen $S \subseteq X$ arbitrarily, we have established Hall's Condition. □

Definition 12. A **vertex cover** of a graph G is a set $Q \subseteq V(G)$ that contains at least one endpoint of every edge. The vertices in Q *cover* $E(G)$.

Example 13 (Matchings and vertex covers)**.** In the graph on the left below we mark a vertex cover of size 2 and show a matching of size 2 in bold. The vertex cover of size 2 prohibits matchings with more than 2 edges, and illustrated on the right, the optimal values differ by 1 for an odd cycle. The difference can be arbitrarily large.



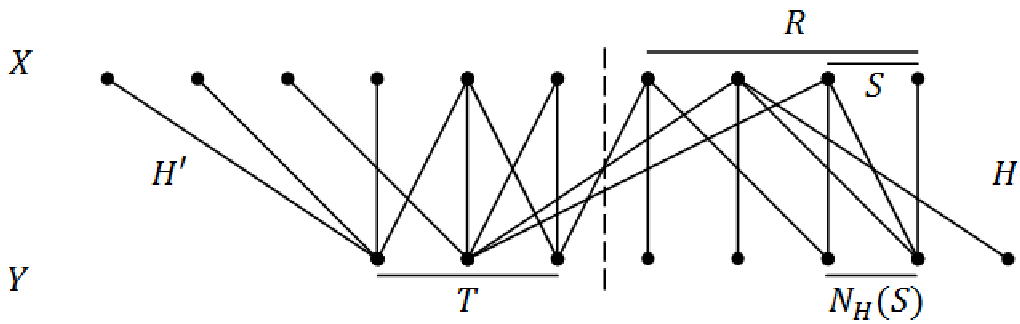
Theorem 14 (König [1931], Egerváy [1931])**.** If G is a bipartite graph, then the maximum size of a matching in G equals the minimum size of a vertex cover of G.

Proof. Let G be an X, Y-bigraph. Since distinct vertices must be used to cover the edges of a matching, $|Q| \geq |M|$ whenever Q is a vertex cover and M is a matching in G. Given a smallest vertex cover Q of G, we construct a matching of size $|Q|$ to prove that equality can always be achieved.

Partition Q by letting $R = Q \cap X$ and $T = Q \cap Y$. Let H and H' be the subgraphs of G induced by $R \cup (Y - T)$ and $T \cup (X - R)$. We use Hall's Theorem to show that H has a matching that saturates R into $Y - T$ and H' has a matching that saturates T. Since H and H' are disjoint, the two matchings together form a matching of size $|Q|$ in G.

Since $R \cup T$ is a vertex cover, G has no edge from $Y - T$ to $X - R$. For each $S \subseteq R$, we consider $N_H(S)$, which is contained in $Y - T$. If $|N_H(S)| < |S|$, then we can substitute $N_H(S)$ for S in Q to obtain a smaller vertex cover, since $N_H(S)$ covers all edges incident to S that are not covered by T.

The minimality of Q thus yields Hall's Condition in H, and hence H has a matching that saturates R. Applying the same argument to H' yields the matching that saturates T. □



An application of Hall Theorem:

Recall that a permutation matrix is a square matrix that has exactly one entry of 1 in each row and each column and zero elsewhere. Now, we define a more general family of matrices called doubly stochastic as mentioned in Section **??**.

Definition 15. A matrix with no negative entries whose column (rows) sums are 1 is called a column stochastic (row stochastic) matrix. In some references column stochastic (row stochastic) matrix is called a stochastic matrix. Both types of these matrices are also called Markov matrices.

Definition 16. A doubly stochastic matrix is a square matrix $A = [a_{ij}]$ of non-negative real entries, each of whose rows and columns sum 1, i.e.

$$\sum_i a_{ij} = \sum_j a_{ij} = 1.$$

The set of all $n \times n$ doubly stochastic matrices is denoted by Ω_n. If we denote all $n \times n$ permutation matrices by $\mathcal{P}_n$, then clearly $\mathcal{P}_n \subset \Omega_n$.

Definition 17. A subset A of a real finite-dimensional vector space is said to be convex if $\lambda \mathbf{x} + (1 - \lambda)\mathbf{y} \in A$, for all vectors $\mathbf{x}, \mathbf{y} \in A$ and all scalars $\lambda \in [0, 1]$. Via induction, this can be seen to be equivalent to the requirement that $\sum_{i=1}^{n} \lambda_i \mathbf{x}_i \in A$, for all vectors $\mathbf{x}_1, \dots, \mathbf{x}_n \in A$ and all scalars $\lambda_1, \dots, \lambda_n \geqslant 0$ with $\sum_{i=1}^{n} \lambda_i = 1$. A point $\mathbf{x} \in A$ is called an extreme point of A if $\mathbf{y}, \mathbf{z} \in A$, $0 < t < 1$, and $\mathbf{x} = t\mathbf{y} + (1-t)\mathbf{z}$ imply $\mathbf{x} = \mathbf{y} = \mathbf{z}$. We denote by ext A the set of all extreme points of A. With these restrictions on λ_i's, an expression of the form $\sum_{i=1}^{n} \lambda_i \mathbf{x}_i$ is said to be a convex combination of $\mathbf{x}_1, \dots, \mathbf{x}_n$. The convex hull of a set $B \subset \mathbf{V}$ is defined as $\{\sum \lambda_i \mathbf{x}_i : \mathbf{x}_i \in B, \lambda_i \geq 0 \text{ and } \sum \lambda_i = 1\}$. The convex hull of B can also be defined as the smallest convex set containing B. (Why?) It is denoted by conv B.

Theorem 18 (Krein-Milman)**.** Let $A \subset \mathbb{R}^n$ be a nonempty compact convex set. Then

1. The set of all extreme points of A is non-empty.

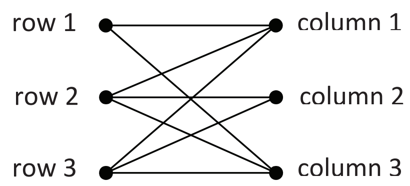
2. The convex hull of the set of all extreme points of A is A itself.

The following theorem is a direct application of matching theory to express the relation between two sets of matrices $\mathcal{P}_n$ and Ω_n.

Theorem 19 (Birkhoff)**.** Every doubly stochastic matrix can be written as a convex combination of permutation matrices.

Proof. We use Philip Hall Theorem to prove this theorem. We associate to our doubly stochastic matrix $A = [a_{ij}]$ a bipartite graph as follows. We represent each row and each column with a vertex and we connect the vertex representing row i with the vertex representing row j if the entry a_{ij} is non-zero.

For example if $A = \begin{bmatrix} \frac{7}{12} & 0 & \frac{5}{12} \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{bmatrix}$, the graph associated to A is given in the picture below.



We claim that the associated graph of any doubly stochastic matrix has a perfect matching. Assume to the contrary, A has no perfect matching. Then, by Philip Hall Theorem there is a subset E of the vertices in one part such that the set $R(E)$ of all vertices connected to some vertex in E has strictly less than $\#E$ elements. Without loss of generality, we may assume that A is a set of vertices representing rows, the set $R(A)$ consists then of vertices representing columns. Consider now the sum $\sum_{i\in E, j\in R(E)} a_{ij} = \#E$, the sum of all entries located in columns belonging to $R(E)$. (by the definition of the associated graph). Thus

$$\sum_{i\in E, j\in R(E)} a_{ij} = \#E.$$

Since the graph is doubly stochastic and the sum of elements located in any of given $\#E$ rows is $\#E$. On the other hand, the sum of all elements located in all columns belonging to $R(E)$ is at least $\sum_{i\in E, j\in R(E)} a_{ij}$, since the entries not belonging to a row in E are non-negative. Since the matrix is doubly stochastic, the sum of all elements located in all columns belonging to $R(E)$ is also exactly $\#R(E)$. Thus, we obtain

$$\sum_{i\in E, j\in R(E)} a_{ij} \leq \#R(E) < \#E = \sum_{i\in E, j\in R(E)} a_{ij},$$

a contradiction. Then, A has a perfect matching.

Now, we are ready to prove the theorem. We proceed by induction on the number of non-zero entries in the matrix. As we proved, associated graph of A has a perfect matching. Underline the entries associated to the edges in the matching. For example in the associated graph above, $\{(1,3),(2,1),(3,2)\}$ is a perfect matching so we underline a_{13}, a_{23} and a_{32}. Thus, we underline exactly one element in each row and each column. Let α_0 be the minimum of the underlined entries. Let P_0 be the permutation matrix that has a 1 exactly at the position of the underlined elements. If $\alpha_0 = 1$, then all underlined entries are 1, and $A = P_0$ is a permutation matrix. If $\alpha_0 < 1$, then the matrix $A - \alpha_0 P_0$ has non-negative entries, and the sum of the entries in any row or any column is $1 - \alpha_0$. Dividing each entry by $(1-\alpha_0)$ in $A - \alpha_0 P_0$

gives a doubly stochastic matrix A_1. Thus, we may write $A = \alpha_0 P_0 + (1-\alpha_0)A_1$, where A_1 is not only doubly stochastic but has less non-zero entries than A. By our induction hypothesis, A_1 may be written as $A_1 = \alpha_1 P_1 + \cdots + \alpha_n P_n$, where $P_1, \ldots, P_n$ are permutation matrices, and $\alpha_1 P_1 + \cdots + \alpha_n P_n$ is a convex combination. But then we have

$$A = \alpha_0 P_0 + (1-\alpha_0)\alpha_1 P_1 + \cdots + (1-\alpha_0)\alpha_n P_n,$$

where $P_0, P_1, \ldots, P_n$ are permutation matrices and we have a convex combination. Since $\alpha_0 \geq 0$, each $(1-\alpha_0)\alpha_i$ is non-negative and we have

$$\alpha_0 + (1-\alpha_0)\alpha_1 + \cdots + (1-\alpha_0)\alpha_n = \alpha_0 + (1-\alpha_0)(\alpha_1 + \ldots + \alpha_n) = \alpha_0 + (1-\alpha_0) = 1.$$

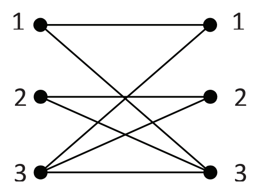
In our example

$$P_0 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

and $\alpha_0 = \frac{1}{6}$. Thus, we get

$$A_1 = \frac{1}{1-\frac{1}{6}}\left(A - \frac{1}{6}P_0\right) = \frac{6}{5}\begin{bmatrix} \frac{7}{12} & 0 & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} \frac{7}{10} & 0 & \frac{3}{10} \\ 0 & \frac{3}{5} & \frac{2}{5} \\ \frac{3}{10} & \frac{2}{5} & \frac{3}{10} \end{bmatrix}.$$

The graph associated to A_1 is the following:



A perfect matching is $\{(1,1),(2,2),(3,3)\}$, the associated permutation matrix is

$$P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$