Supplementary note for linear programming

1 Farkas' Lemma and Theorems of the Alternative

Lemma 1.1 (Farkas' Lemma)**.** *Let* $\mathbf{A} \in \mathbb{R}^{m\times n}$ *and* $\mathbf{c} \in \mathbb{R}^n$ *be a row vector. Suppose* $\mathbf{x} \in \mathbb{R}^n$ *is a column vector and* $\mathbf{w} \in \mathbb{R}^m$ *is a row vector. Then exactly one of the following systems of inequalities has a solution:*

(1) $\mathbf{Ax} \geq 0$ *and* $\mathbf{cx} < 0$ *or*

(2) $\mathbf{wA} = \mathbf{c}$ *and* $\mathbf{w} \geq \mathbf{0}$

Remark 1.2. *Before proceeding to the proof, it is helpful to restate the lemma in the following way:*

(1) If there is a vector $\mathbf{x} \in \mathbb{R}^n$ *so that* $\mathbf{Ax} \geq \mathbf{0}$ *and* $\mathbf{cx} < 0$*, then there* **is no** *vector* $\mathbf{w} \in \mathbb{R}^m$ *so that* $\mathbf{wA} = \mathbf{c}$ *and* $\mathbf{w} \geq \mathbf{0}$.

(2) Conversely, if there is a vector $\mathbf{w} \in \mathbb{R}^m$ *so that* $\mathbf{wA} = \mathbf{c}$ *and* $\mathbf{w} \geq \mathbf{0}$*, then there* **is no** *vector* $\mathbf{x} \in \mathbb{R}^n$ *so that* $\mathbf{Ax} \geq \mathbf{0}$ *and* $\mathbf{cx} < 0$*.*

Proof. We can prove Farkas' Lemma using the fact that a bounded linear programming problem has an extreme point solution. Suppose that System 1 has a solution $\mathbf{x}$. If System 2 also has a solution $\mathbf{w}$, then

$$\mathbf{wA} = \mathbf{c} \Longrightarrow \mathbf{wAx} = \mathbf{cx}. \tag{1}$$

The fact that System 1 has a solution ensures that $\mathbf{cx} < 0$ and therefore $\mathbf{wAx} < 0$. However, it also ensures that $\mathbf{Ax} \geq \mathbf{0}$. The fact that System 2 has a solution implies that $\mathbf{w} \geq \mathbf{0}$. Therefore we must conclude that:

$$\mathbf{w} \geq \mathbf{0} \text{ and } \mathbf{Ax} \geq \mathbf{0} \Longrightarrow \mathbf{wAx} \geq \mathbf{0}. \tag{2}$$

This contradiction implies that if System 1 has a solution, then System 2 cannot have a solution.

Now, suppose that System 1 has no solution. We will construct a solution for System 2. If System 1 has no solution, then there is no vector $\mathbf{x}$ so that $\mathbf{cx} < 0$ and $\mathbf{Ax} \geq \mathbf{0}$. Consider the linear programming problem:

$$P_F \begin{cases} \min & \mathbf{cx} \\ s.t. & \mathbf{Ax} \geq \mathbf{0} \end{cases} \tag{3}$$

Clearly $\mathbf{x} = \mathbf{0}$ is a feasible solution to this linear programming problem and furthermore is optimal. To see this, note that the fact that there is no $\mathbf{x}$ so that $\mathbf{cx} < 0$ and

$\mathbf{Ax} \geq \mathbf{0}$, it follows that $\mathbf{cx} \geq 0$; i.e., 0 is a lower bound for the linear programming problem P_F. At $\mathbf{x} = \mathbf{0}$, the objective achieves its lower bound and therefore this must be an optimal solution. Therefore P_F is bounded and feasible.

We can convert P_F to standard form through the following steps:

(1) Introduce two new vectors $\mathbf{y}$ and $\mathbf{z}$ with $\mathbf{y}, \mathbf{z} \geq \mathbf{0}$ and write $\mathbf{x} = \mathbf{y} - \mathbf{z}$ (since $\mathbf{x}$ is unrestricted).

(2) Append a vector of surplus variables $\mathbf{s}$ to the constraints.

This yields the new problem:

$$P'_F \begin{cases} \min & \mathbf{cy} - \mathbf{cz} \\ s.t. & \mathbf{Ay} - \mathbf{Az} - \mathbf{I}_m\mathbf{s} = \mathbf{0} \\ & \mathbf{y}, \mathbf{z}, \mathbf{s} \geq \mathbf{0} \end{cases} \tag{4}$$

Problem P'_F in which the reduced costs for the variables are all negative (that is, $z_j - c_j \leq 0$ for $j = 1, \ldots, 2n + m$). Here we have n variables in vector $\mathbf{y}$, n variables in vector $\mathbf{z}$ and m variables in vector $\mathbf{s}$. Let $\mathbf{B} \in \mathbb{R}^{m \times m}$ be the basis matrix at this optimal feasible solution with basic cost vector $\mathbf{c_B}$. Let $\mathbf{w} = \mathbf{c_B}\mathbf{B}^{-1}$ (as it was defined for the revised simplex algorithm).

Consider the columns of the simplex tableau corresponding to a variable x_k (in our original $\mathbf{x}$ vector). The variable $x_k = y_k - z_k$. Thus, these two columns are additive inverses. That is, the column for y_k will be $\mathbf{B}^{-1}\mathbf{A}_k$, while the column for z_k

will be $\mathbf{B}^{-1}(-\mathbf{A}_k) = -\mathbf{B}^{-1}\mathbf{A}_k$. Furthermore, the objective function coefficient will be precisely opposite as well.

Thus the fact that $z_j - c_j \leq 0$ for all variables implies that:

$\mathbf{w}\mathbf{A}_k - c_k \leq 0$ **and**

$-\mathbf{w}\mathbf{A}_k + c_k \leq 0$ **and**

That is, we obtain

$$\mathbf{w}\mathbf{A} = \mathbf{c} \tag{5}$$

since this holds for all columns of $\mathbf{A}$.

Consider the surplus variable s_k. Surplus variables have zero as their coefficient in the objective function. Further, their simplex tableau column is simply $\mathbf{B}^{-1}(-\mathbf{e}_k) = -\mathbf{B}^{-1}\mathbf{e}_k$. The fact that the reduced cost of this variable is non-positive implies that:

$$\mathbf{w}(-\mathbf{e}_k) - \mathbf{0} = -\mathbf{w}\mathbf{e}_k \leq 0 \tag{6}$$

Since this holds for all surplus variable columns, we see that $-\mathbf{w} \leq \mathbf{0}$ which implies $\mathbf{w} \geq \mathbf{0}$. Thus, the optimal basic feasible solution to Problem P_F' must yield a vector $\mathbf{w}$ that solves System 2.

Lastly, the fact that if System 2 does not have a solution, then System 1 does follows from contrapositive on the previous fact we just proved. □

Problem 1. *Suppose we have two statements A and B so that:*

$A \equiv$ *System 1 has a solution.*

$B \equiv$ *System 2 has a solution.*

Our proof showed explicitly that NOT $A \Longrightarrow B$. *Recall that contrapositive is the logical rule that asserts that:*

$$X \Longrightarrow Y \equiv NOT\, Y \Longrightarrow NOT\, X \tag{7}$$

Use contrapositive to prove explicitly that if System 2 has no solution, then System 1 must have a solution. [Hint: NOT NOT $X \equiv X$*.]*

2 Duality

In this section, we show that to each linear programming problem (the primal problem) we may associate another linear programming problem (the dual linear programming problem). These two problems are closely related to each other and an analysis of the dual problem can provide deep insight into the primal problem.

The Dual Problem

Consider the linear programming problem

$$P \begin{cases} \max & \mathbf{c}^T\mathbf{x} \\ s.t. & \mathbf{A}\mathbf{x} \le \mathbf{b} \\ & \mathbf{x} \ge 0 \end{cases} \tag{8}$$

Then the dual problem for Problem P is:

$$D \begin{cases} \min & \mathbf{w}\mathbf{b} \\ s.t. & \mathbf{w}\mathbf{A} \ge \mathbf{c} \\ & \mathbf{w} \ge \mathbf{0} \end{cases} \tag{9}$$

Remark 2.1. *Let* $\mathbf{v}$ *be a vector of surplus variables. Then we can transform Problem* D *into standard form as:*

$$D_S \begin{cases} \min & \mathbf{w}\mathbf{b} \\ s.t. & \mathbf{w}\mathbf{A} - \mathbf{v} = \mathbf{c} \\ & \mathbf{w} \ge \mathbf{0} \\ & \mathbf{v} \ge \mathbf{0} \end{cases} \tag{10}$$

In this formulation, we see that we have assigned a dual variable w_i $(i = 1, \dots, m)$ to each constraint in the system of equations $\mathbf{A}\mathbf{x} \le \mathbf{b}$ of the primal problem. Likewise dual variables $\mathbf{v}$ can be thought of as corresponding to the constraints in $\mathbf{x} \ge \mathbf{0}$.

Lemma 2.2. *The dual of the dual problem is the primal problem.*

Proof. Rewrite Problem D as:

$$\begin{cases} \max & -\mathbf{b}^T\mathbf{w}^T \\ s.t. & -\mathbf{A}^T\mathbf{w}^T \leq -\mathbf{c}^T \\ & \mathbf{w}^T \geq \mathbf{0} \end{cases} \tag{11}$$

Let $\boldsymbol{\beta} = -\mathbf{b}^T$, $\mathbf{G} = -\mathbf{A}^T$, $\mathbf{u} = \mathbf{w}^T$ and $\boldsymbol{\kappa} = -\mathbf{c}^T$. Then this new problem becomes:

$$\begin{cases} \max & \boldsymbol{\beta} \\ s.t. & \mathbf{G}\mathbf{u} \leq \boldsymbol{\kappa} \\ & \mathbf{u} \geq \mathbf{0} \end{cases} \tag{12}$$

Let $\mathbf{x}^T$ be the vector of dual variables (transposed) for this problem. We can formulate the dual problem as:

$$\begin{cases} \min & \mathbf{x}^T\boldsymbol{\kappa} \\ s.t. & \mathbf{x}^T\mathbf{G} \geq \boldsymbol{\beta} \\ & \mathbf{x}^T \geq \mathbf{0} \end{cases} \tag{13}$$

Expanding, this becomes:

$$\begin{cases} \min & -\mathbf{x}^T\mathbf{c}^T \\ s.t. & -\mathbf{x}^T\mathbf{A}^T \geq -\mathbf{b}^T \\ & \mathbf{x}^T \geq \mathbf{0} \end{cases} \tag{14}$$

This can be simplified to:

$$P \begin{cases} \max & \mathbf{c}^T\mathbf{x} \\ s.t. & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{cases} \tag{15}$$

as required. This completes the proof. □

Lemma 2.3. *Shows that the notion of dual and primal can be exchanged and that it is simply a matter of perspective which problem is the dual problem and which is the primal problem. Likewise, by transforming problems into canonical form, we can develop dual problems for any linear programming problem.*

The process of developing these formulations can be exceptionally tedious, as it required enumeration of all the possible combinations of various linear and variable constraints. The following table summarizes the process of converting an arbitrary primal problem into its dual.

Example 2.4. *Consider the following problem:*

$$\begin{aligned} \max \quad & 7x_1 + 6x_2 \\ s.t. \quad & 3x_1 + x_2 + s_1 = 120 \quad && (w_1) \\ & x_1 + 2x_2 + s_2 = 160 \quad && (w_2) \\ & x_1 + s_3 = 35 \quad && (w_3) \\ & x_1, x_2, s_1, s_2, s_3 \geq 0 \end{aligned}$$

	MINIMIZATION PROBLEM		MAXIMIZATION PROBLEM	
VARIABLES	≥ 0		$\leq$	CONSTRAINTS
	≤ 0	$\longleftrightarrow$	$\geq$	
	UNRESTRICTED		$=$	
CONSTRAINTS	$\geq$		≥ 0	VARIABLES
	$\leq$	$\longleftrightarrow$	≤ 0	
	$=$		UNRESTRICTED	

Table 1: Table of Dual Conversions: To create a dual problem, assign a dual variable to each constraint of the form $\mathbf{Ax} \circ \mathbf{b}$, where $\circ$ represents a binary relation. Then use the table to determine the appropriate sign of the inequality in the dual problem as well as the nature of the dual variables.

Here we have placed dual variable named (w_1, w_2 *and* w_3) *next to the constraints to which they correspond.*

The primal problem variables in this case are all positive, so using Table 1 we known that the constraints of the dual problem will be greater-than-or-equal-to constraints. Likewise, we know that the dual variables will be unrestricted in sign since the primal problem constraints are all equality constraints.

The coefficient matrix is:

$$\mathbf{A} = \begin{bmatrix} 3 & 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Clearly we have:

$$\mathbf{c} = \begin{bmatrix} 7 & 6 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix} 120 \\ 160 \\ 35 \end{bmatrix}$$

Since $\mathbf{w} = \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix}$, *we know that* $\mathbf{wA}$ *will be:*

$$\mathbf{wA} = \begin{bmatrix} 3w_1 + w_2 + w_3 & w_1 + 2w_2 & w_1 & w_2 & w_3 \end{bmatrix}$$

This vector will be related to $\mathbf{c}$ *in the constraints of the dual problem.* ***Remember, in this case, all constraints are greater-than-or-equal-to****. Thus we see that the constraints of the dual problem are:*

$$3w_1 + w_2 + w_3 \geq 7$$

$$w_1 + 2w_2 \geq 6$$

$$w_1 \geq 0$$

$$w_2 \geq 0$$

$$w_3 \geq 0$$

We also have the redundant set of constraints that tell us $\mathbf{w}$ *is unrestricted because the primal problem had equality constraints. This will always happen in cases when you've introduced slack variables into a problem to put it in standard form. This should be clear from the definition of the dual problem for a maximization problem in canonical form.*

Thus the whole dual problem becomes:

$$\begin{aligned}
\min \quad & 120w_1 + 160w_2 + 35w_3 \\
s.t. \quad & 3w_1 + w_2 + w_3 \geq 7 \\
& w_1 + 2w_2 \geq 6 \\
& w_1 \geq 0 \\
& w_2 \geq 0 \\
& w_3 \geq 0 \\
& \mathbf{w} \ \textit{unrestricted}
\end{aligned} \tag{16}$$

Again, note that in reality, the constraints we derived from the $\mathbf{wA} \geq \mathbf{c}$ *part of the dual problem make the constraints "*$\mathbf{w}$ *unrestricted" redundant, for in fact* $\mathbf{w} \geq \mathbf{0}$ *just as we would expect it to be if we'd found the dual of the Toy Maker problem given in canonical form.*

Problem 2. *Identify the dual problem for:*

$$\begin{aligned} \max\ & x_1 + x_2 \\ s.t.\ & 2x_1 + x_2 \geq 4 \\ & x_1 + 2x_2 \leq 6 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Problem 3. *Use the table or the definition of duality to determine the dual for the problem:*

$$\begin{cases} \min & \mathbf{cx} \\ s.t. & \mathbf{Ax} \leq \mathbf{b} \\ & \mathbf{x} \geq 0 \end{cases} \tag{17}$$

3 Weak Duality

There is a deep relationship between the objective function value feasibility and boundedness of the problem and the dual problem. We will explore these relationships in the following lemmas.

Lemma 3.1 (Weak Duality)**.** *For the primal problem P and dual problem D let* $\mathbf{x}$ *and* $\mathbf{w}$ *be feasible solutions to Problem P and Problem D respectively, Then:*

$$\mathbf{wb} \geq \mathbf{cx} \tag{18}$$

Proof. Primal feasibility ensures that:

$$\mathbf{Ax} \leq \mathbf{b}.$$

Therefore, we have:

$$\mathbf{wAx} \leq \mathbf{wb}. \tag{19}$$

Dual feasibility ensure that:

$$\mathbf{wA} \geq \mathbf{c}.$$

Therefore, we have:

$$\mathbf{wAx} \geq \mathbf{cx}. \tag{20}$$

Combining Equations 19 and 20 yields Equation 18:

$$\mathbf{wb} \geq \mathbf{cx}.$$

This completes the proof. $\square$

Remark 3.2. *Lemma 3.1 ensures that the optimal solution* $\mathbf{w}^*$ *for Problem D must provide an upper bound to Problem* P*, since for any feasible* $\mathbf{x}$*, we know that:*

$$\mathbf{w}^*\mathbf{b} \geq \mathbf{cx}. \tag{21}$$

Likewise, any optimal solution to Problem P *provides a lower bound on solution* D*.*

Corollary 3.3. *If Problem* P *is unbounded, then Problem* D *is infeasible. Likewise, if Problem* D *is unbounded, then Problem* P *is infeasible.*

Proof. For any $\mathbf{x}$, feasible to Problem P we know that $\mathbf{wb} \geq \mathbf{cx}$ for any feasible $\mathbf{w}$. The fact that Problem P is unbounded implies that for any $V \in \mathbb{R}$ we can find an $\mathbf{x}$ feasible to Problem P that $\mathbf{cx} > V$. If $\mathbf{w}$ were feasible to Problem D, then we would have $\mathbf{wb} > V$ for any arbitrarily chosen V. There can be no finite vector $\mathbf{w}$ with this property and we conclude that Problem D must be infeasible.

The alternative case that when Problem D is unbounded, then Problem P is infeasible follows by reversing the roles of the problem. This completes the proof. □

The Simplex Algorithm

In this section we try to identify some simple way of identifying extreme points. To accomplish this, let us now assume that we write X as:

$$X = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{Ax} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\} \tag{22}$$

We can separate $\mathbf{A}$ into an $m \times m$ matrix B and an $m \times (n-m)$ matrix N and we have the result:

$$\mathbf{x_B} = \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\mathbf{x_N} \tag{23}$$

We know that $\mathbf{B}$ is invertible since we assumed that $\mathbf{A}$ had full row rank. If we assume that $\mathbf{x_N} = \mathbf{0}$, then the solution

$$\mathbf{x_B} = \mathbf{B}^{-1}\mathbf{b} \tag{24}$$

was called a basic solution. Clearly any basic solution satisfies the constrants $\mathbf{Ax} = \mathbf{b}$ but it may not satisfy the constraints $\mathbf{x} \geq \mathbf{0}$.

Definition 3.4 (Basic Feasible Solution)**.** *If $\mathbf{x_B} = \mathbf{B}^{-1}\mathbf{b}$ and $\mathbf{x_N} = \mathbf{0}$ is a basic solution to $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{x_B} \geq 0$, then the solution $(\mathbf{x_B}, \mathbf{x_N})$ is called basic feasible solution.*

Theorem 3.5. *Every basic feasible solution is an extreme point of X. Likewise, every extreme point is characterized by a basic feasible solution of $\mathbf{Ax} = \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$.*

Proof. Since $\mathbf{Ax} = \mathbf{Bx_B} + \mathbf{Nx_N} = \mathbf{b}$ this represents the intersection of m linearly independent hyperplanes (since the rank of $\mathbf{A}$ is m). The fact that $\mathbf{x_N} = \mathbf{0}$ and $\mathbf{x_N}$ contains $n - m$ variables, then we have $n - m$ binding, linearly independent hyperplanes in $\mathbf{x_N} \geq 0$. Thus the point $(\mathbf{x_B}, \mathbf{x_N})$ is the intersection of $m + (n - m) = n$ linearly independent hyperplanes. So $(\mathbf{x_B}, \mathbf{x_N})$ must be an extreme point of X.

Converely, let $\mathbf{x}$ be an extreme point of X. Clearly $\mathbf{x}$ is feasible then it must represent the intersection of n hyperplanes. The fact that $\mathbf{x}$ is feasible implies that $\mathbf{Ax} = \mathbf{b}$. This accounts for m of the intersecting linearly independent hyperplanes. The remaining $n-m$ hyperplanes must come from $\mathbf{x} \geq \mathbf{0}$. That is, $n-m$ variables are zero. Let $\mathbf{x_N} = \mathbf{0}$ be the variables for which $\mathbf{x} \geq \mathbf{0}$ are binding. Denote the remaining variables $\mathbf{x_B}$. We can see that $\mathbf{A} = [\mathbf{B}|\mathbf{N}]$ and that $\mathbf{Ax} = \mathbf{Bx_B} + \mathbf{Nx_N} = \mathbf{b}$. Clearly, $\mathbf{x_B}$ is the unique solution to $\mathbf{Bx_B} = \mathbf{b}$ and thus $(\mathbf{x_B}, \mathbf{x_N})$ is a basic feasible solution. $\square$

Suppose we have a basic feasible solution $\mathbf{x} = (\mathbf{x_B}, \mathbf{x_N})$. We can divide the cost vector $\mathbf{c}$ into its basic and non-basic parts, so we have $\mathbf{c} = [\mathbf{c_B}|\mathbf{c_N}]^T$. Then the objective function becomes:

$$\mathbf{c}^T\mathbf{x} = \mathbf{c}_\mathbf{B}^T\mathbf{x_B} + \mathbf{c}_\mathbf{N}^T\mathbf{x_N} \tag{25}$$

We can substitute Equation (23) into Equation (25) to obtain:

$$\mathbf{c}^T\mathbf{x} = \mathbf{c}_\mathbf{B}^T\left(\mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\mathbf{x_N}\right) + \mathbf{c_N}\mathbf{x_N} = \mathbf{c}_\mathbf{B}^T\mathbf{B}^{-1}\mathbf{b} + \left(\mathbf{c}_\mathbf{N}^T - \mathbf{c}_\mathbf{B}^T\mathbf{B}^{-1}\mathbf{N}\right)\mathbf{x_N} \tag{26}$$

Let $\mathcal{J}$ be the set of indices of non-basic variables. Then we can write Equation (26) as:

$$z(x_1, \dots, x_n) = \mathbf{c}_\mathbf{B}^T\mathbf{B}^{-1}\mathbf{b} + \sum_{j \in \mathcal{J}} \left(\mathbf{c}_j - \mathbf{c}_\mathbf{B}^T\mathbf{B}^{-1}\mathbf{A}_j\right) x_j \tag{27}$$

Consider now the fact $x_j = 0$ for all $j \in \mathcal{J}$. Further, we can see that:

$$\frac{\partial z}{\partial x_j} = \mathbf{c}_j - \mathbf{c}_\mathbf{B}^T\mathbf{B}^{-1}\mathbf{A}_{\cdot j}. \tag{28}$$

This means that if $\mathbf{c}_j - \mathbf{c}_\mathbf{B}^T\mathbf{B}^{-1}\mathbf{A}_{\cdot j} > 0$ and we increase x_j from zero to some new value, then we will increase the value of the objective function. For historic reasons, we actually consider the value $\mathbf{c}_\mathbf{B}^T\mathbf{B}^{-1}\mathbf{A}_{\cdot j} - \mathbf{c}_j$, called the reduced cost and denote it as:

$$-\frac{\partial z}{\partial x_j} = z_j - c_j = \mathbf{c}_\mathbf{B}^T\mathbf{B}^{-1}\mathbf{A}_{\cdot j} - \mathbf{c}_j \tag{29}$$

In a maximization problem, we choose non-basic variables x_j with negative reduced cost to become basic because, in this cases, $\partial z/\partial x_j$ is positive.

Assume we choose x_j, a non-basic variable to become non-zero (because $z_j - c_j < 0$). We wish to know which of the basic variables will become zero as we increase x_j away from zero. We must also be very careful that none of the variables become negative as we do this.

By Equation (23) we know that the only current basic variables will be affected by increasing x_j. Let us focus explicitly on Equation (23) where we include only variable x_j (since all other non-basic variables are kept zero). Then we have:

$$\mathbf{x_B} = \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{A}_{\cdot j}x_j. \tag{30}$$

Let $\overline{\mathbf{b}} = \mathbf{B}^{-1}\mathbf{b}$ be an $m \times 1$ column vector and let that $\overline{\mathbf{a}_j} = \mathbf{B}^{-1}\mathbf{A}_{\cdot j}$ be another $m \times 1$ column. Then we can write:

$$\mathbf{x_B} = \overline{\mathbf{b}} - \overline{\mathbf{a}}_j x_j. \tag{31}$$

Let $\overline{\mathbf{b}} = [\overline{b}_1, \dots, \overline{b}_m]^T$ and $\overline{\mathbf{a}}_j = [\overline{a}_{j_1}, \dots, \overline{a}_{j_m}]$, then we have:

$$\begin{bmatrix} x_{B_1} \\ x_{B_2} \\ \vdots \\ x_{B_m} \end{bmatrix} = \begin{bmatrix} \overline{b}_1 \\ \overline{b}_2 \\ \vdots \\ \overline{b}_m \end{bmatrix} - \begin{bmatrix} \overline{a}_{j_1} \\ \overline{a}_{j_2} \\ \vdots \\ \overline{b}_{j_m} \end{bmatrix} x_j = \begin{bmatrix} \overline{b}_1 - \overline{a}_{j_1}x_j \\ \overline{b}_2 - \overline{a}_{j_2}x_j \\ \vdots \\ \overline{b}_m - \overline{a}_{j_m}x_j \end{bmatrix} \tag{32}$$

We know (a priori) that $\overline{b}_i \geq 0$ for $i = 1, \dots, m$. If $\overline{a}_{j_i} \leq 0$, then as we increase $x_j, \overline{b}_i - \overline{a}_{j_i} \geq 0$ no matter kow large we make x_j. On the other hand, if $\overline{a}_{j_i} > 0$, then as we increase x_j we know that $\overline{b}_i - \overline{a}_{j_i}x_j$ will get smaller and eventually hit zero. In order to ensure that all variables remain non-negative, we cannot increase x_j beyond a certain point.

For each i $(i = 1, \dots, m)$ such that $\overline{a}_{j_i} > 0$, the value of x_j that will make x_{B_i} goto 0 can be found by observing that:

$$x_{B_i} = \overline{b}_i - \overline{a}_{j_i} x_j \tag{33}$$

and if $x_{B_i} = 0$, then we can solve:

$$0 = \overline{b}_i - \overline{a}_{j_i} x_j \implies x_j = \frac{\overline{b}_i}{\overline{a}_{j_i}} \tag{34}$$

Thus, the largest possible value we can assign x_j and ensure that all variables remain positive is:

$$\min \left\{ \frac{\overline{b}_i}{\overline{a}_{j_i}} : i = 1, \dots, m \text{ and } a_{j_i} > 0 \right\}. \tag{35}$$

Expression 35 is called the minimum ratio test. We are interested in which index i is the minimum ratio.

Suppose that in executing the minimum ratio test, we find that $x_j = \overline{b}_k / \overline{a}_{j_k}$. The variable x_j (which was non-basic) becomes basic and the variable $x_{\mathbf{B}_k}$ becomes non-basic. All other basic variables remain basic (and positive). In executing this procedure (of exchanging one basic variable and one non-basic variable) we have moved from one extreme point of X to another.

Theorem 3.6. *If $z_j - c_j \geq 0$ for all $j \in \mathcal{J}$, then the current basic feasible solution is optimal.*

Proof. It is proven that if a linear programming problem has an optimal solution, then it occurs at an extreme point and it is shown that there is a one-to-one correspondence between extreme points and basic feasible solutions. If $z_j - c_j \geq 0$ for all $j \in \mathcal{J}$, then $\partial z / \partial x_j \leq 0$ for all non-basic variables x_j. Thit is, we cannot increase the value of the objective function by increasing the value of any non-basic variable. Thus, since moving to another basic feasible solution (extreme point) will not improve the objective function, it follows we must be at the optimal solution. □

Theorem 3.7. *In a maximization problem, if* $\overline{a}_{j_i} \leq 0$ *for all* $i = 1, \ldots, m$*, and* $z_j - c_j < 0$*, then the linear programming problem is unbounded.*

Proof. The fact that $z_j - c_j < 0$ implies that increasing x_j will improve the value of the objective function. Since $\overline{a}_{j_i} < 0$ for all $i = 1, \ldots, m$, we can increase x_j indefinitely without violating feasibility (no basic variable will ever go to zero). Thus the objective function can be made as large as we like. □

Example 3.8. *Consider the following linear programming problem:*

$$\begin{cases} \max & z(x_1, x_2) = 7x_1 + 6x_2 \\ s.t. & 3x_1 + x_2 \leq 120 \\ & x_1 + 2x_2 \leq 160 \\ & x_1 \leq 35 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{cases}$$

We can convert this problem to standard form by introducing the slack variables s_1*,* s_2 *and* s_3*:*

$$\begin{cases} \max & z(x_1, x_2) = 7x_1 + 6x_2 \\ s.t. & 3x_1 + x_2 + s_1 = 120 \\ & x_1 + 2x_2 + s_2 = 160 \\ & x_1 + s_3 = 35 \\ & x_1, x_2, s_1, s_2, s_3 \geq 0 \end{cases}$$

which yields the matrices

$$\mathbf{c} = \begin{bmatrix} 7 \\ 6 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} \quad \mathbf{A} = \begin{bmatrix} 3 & 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 120 \\ 160 \\ 35 \end{bmatrix}$$

We can begin with the matrices:

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{N} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \\ 1 & 0 \end{bmatrix}$$

In this case we have:

$$\mathbf{x_B} = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} \quad \mathbf{x_N} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \mathbf{c_B} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{c_N} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$$

and

$$\mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} 120 \\ 160 \\ 35 \end{bmatrix} \qquad \mathbf{B}^{-1}\mathbf{N} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \\ 1 & 0 \end{bmatrix}$$

Therefore:

$$\mathbf{c_B^{\mathit{T}}}\mathbf{B}^{-1}\mathbf{b} = 0 \quad \mathbf{c_B^{\mathit{T}}}\mathbf{B}^{-1}\mathbf{N} = [0 \;\; 0] \quad \mathbf{c_B^{\mathit{T}}}\mathbf{B}^{-1}\mathbf{N} - \mathbf{c_N} = [-7 \;\; -6]$$

Using this information, we can compute:

$$\mathbf{c_B^{\mathit{T}}}\mathbf{B}^{-1}\mathbf{A}_{\cdot 1} - \mathbf{c}_1 = -7$$

$$\mathbf{c_B^{\mathit{T}}}\mathbf{B}^{-1}\mathbf{A}_{\cdot 2} - \mathbf{c}_2 = -6$$

and therefore:

$$\frac{\partial z}{\partial x_1} = 7 \;\; \textit{and} \;\; \frac{\partial z}{\partial x_2} = 6$$

Based on this information, we could choose either x_1 or x_2 to enter the basis the value of the objective function would increase. If we choose x_1 to enter the basis, *then we must determine which variable will leave the basis. To do this, we must investigate the elements of* $\mathbf{B}^{-1}\mathbf{A}_{\cdot 1}$ *and the current basic feasible solution* $\mathbf{B}^{-1}\mathbf{b}$*. Since each element of* $\mathbf{B}^{-1}\mathbf{A}_{\cdot 1}$ *is positive, we must perform the minimum ratio test on each element of* $\mathbf{B}^{-1}\mathbf{A}_{\cdot 1}$*. We know that* $\mathbf{B}^{-1}\mathbf{A}_{\cdot 1}$ *is just the first column of* $\mathbf{B}^{-1}\mathbf{N}$ *which is:*

$$\mathbf{B}^{-1}\mathbf{A}_{\cdot 1} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

Performing the minimum ratio test, we see have:

$$\min\left\{\frac{120}{3}, \frac{160}{1}, \frac{35}{1}\right\}$$

In this case, we see that index 3 (35/1) is the minimum ratio. Therefore, variable x_1 *will enter the basis and variable* s_3 *will leave the basis. The new basic and non-basic variables will be:*

$$\mathbf{x_B} = \begin{bmatrix} s_1 \\ s_2 \\ x_1 \end{bmatrix} \quad \mathbf{x_N} = \begin{bmatrix} s_3 \\ x_2 \end{bmatrix} \quad \mathbf{c_B} = \begin{bmatrix} 0 \\ 0 \\ 7 \end{bmatrix} \quad \mathbf{c_N} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$

and the matrices become:

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{N} = \begin{bmatrix} 0 & 1 \\ 0 & 2 \\ 1 & 0 \end{bmatrix}$$

Note we have simply swapped the column corresponding to x_1 *with the column corresponding to* s_3 *in the basis matrix* $\mathbf{B}$ *and the non-basic matrix* $\mathbf{N}$*. We will do this* *repeatedly in the example and we recommend the reader keep track of which variables are being exchanged and why certain columns in* $\mathbf{B}$ *are being swapped with those in* $\mathbf{N}$*.*

Using the new $\mathbf{B}$ *and* $\mathbf{N}$ *matrices, the derived matrices are then:*

$$\mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} 15 \\ 125 \\ 35 \end{bmatrix} \qquad \mathbf{B}^{-1}\mathbf{N} = \begin{bmatrix} -3 & 1 \\ -1 & 2 \\ 1 & 0 \end{bmatrix}$$

The cost information becomes:

$$\mathbf{c}_{\mathbf{B}}^T\mathbf{B}^{-1}\mathbf{b} = 245 \quad \mathbf{c}_{\mathbf{B}}^T\mathbf{B}^{-1}\mathbf{N} = [7 \;\; 0] \quad \mathbf{c}_{\mathbf{B}}^T\mathbf{B}^{-1}\mathbf{N} - \mathbf{c}_{\mathbf{N}} = [7 \;\; -6]$$

using this information, we can compute:

$$\mathbf{c}_{\mathbf{B}}^T\mathbf{B}^{-1}\mathbf{A}_{\cdot 5} - \mathbf{c}_5 = 7$$

$$\mathbf{c}_{\mathbf{B}}^T\mathbf{B}^{-1}\mathbf{A}_{\cdot 2} - \mathbf{c}_2 = -6$$

Based on this information, we can only choose x_2 *to enter the basis to ensure that the value of the objective function increases. We can perform the minimum ration test to figure out which basic variable will leave the basis. We know that* $\mathbf{B}^{-1}\mathbf{A}_{\cdot 2}$ *is just the second column of* $\mathbf{B}^{-1}\mathbf{N}$ *which is:*

$$\mathbf{B}^{-1}\mathbf{A}_{\cdot 2} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

Performing the minimum ratio test, we see have:

$$\min\left\{\frac{15}{1}, \frac{125}{2}\right\}$$

In this case, we see that index 1 (15/1) is the minimum ratio. Therefore, variable x_2 will enter the basis and variable s_1 will leave the basis. The new basic and non-basic variables will be:

$$\mathbf{x_B} = \begin{bmatrix} x_2 \\ s_2 \\ x_1 \end{bmatrix} \quad \mathbf{x_N} = \begin{bmatrix} s_3 \\ s_1 \end{bmatrix} \quad \mathbf{c_B} = \begin{bmatrix} 6 \\ 0 \\ 7 \end{bmatrix} \quad \mathbf{c_N} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and the matrices become:

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{N} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

The derived matrices are then:

$$\mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} 15 \\ 95 \\ 35 \end{bmatrix} \qquad \mathbf{B}^{-1}\mathbf{N} = \begin{bmatrix} -3 & 1 \\ 5 & -2 \\ 1 & 0 \end{bmatrix}$$

The cost information becomes:

$$\mathbf{c_B^T}\mathbf{B}^{-1}\mathbf{b} = 335 \quad \mathbf{c_B^T}\mathbf{B}^{-1}\mathbf{N} = [-11 \;\; 6] \quad \mathbf{c_B^T}\mathbf{B}^{-1}\mathbf{N} - \mathbf{c_N} = [-11 \;\; 6]$$

Based on this information, we can only choose s_3 to (re-enter) the basis to ensure that the value of the objective function increases. We can perform the minimum *ration test to figure out which basic variable will leave the basis. We know that* $\mathbf{B}^{-1}\mathbf{A}_{\cdot 5}$ *is just the fifth column of* $\mathbf{B}^{-1}\mathbf{N}$ *which is:*

$$\mathbf{B}^{-1}\mathbf{A}_{\cdot 5} = \begin{bmatrix} -3 \\ 5 \\ 1 \end{bmatrix}$$

Performing the minimum ratio test, we see have:

$$\min\left\{\frac{95}{5}, \frac{35}{1}\right\}$$

In this case, we see that index 2 (95/5) is the minimum ratio. Therefore, variable s_3 *will enter the basis and variable* s_2 *will leave the basis. The new basic and non-basic variables will be:*

$$\mathbf{x_B} = \begin{bmatrix} x_2 \\ s_3 \\ x_1 \end{bmatrix} \quad \mathbf{x_N} = \begin{bmatrix} s_2 \\ s_1 \end{bmatrix} \quad \mathbf{c_B} = \begin{bmatrix} 6 \\ 0 \\ 7 \end{bmatrix} \quad \mathbf{c_N} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and the matrices become:

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \qquad \mathbf{N} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$$

The derived matrices are then:

$$\mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} 72 \\ 19 \\ 16 \end{bmatrix} \qquad \mathbf{B}^{-1}\mathbf{N} = \begin{bmatrix} 6/10 & -1/5 \\ 1/5 & -2/5 \\ -1/5 & 2/5 \end{bmatrix}$$

The cost information becomes:

$$\mathbf{c}_\mathbf{B}^T\mathbf{B}^{-1}\mathbf{b} = 544 \quad \mathbf{c}_\mathbf{B}^T\mathbf{B}^{-1}\mathbf{N} = [11/5 \;\; 8/5] \quad \mathbf{c}_\mathbf{B}^T\mathbf{B}^{-1}\mathbf{N} - \mathbf{c}_\mathbf{N} = [11/5 \;\; 8/5]$$

Since the reduced costs are now positive, we can conclude that we've obtained an optimal solution because no improvement is possible. The final solution then is:

$$\mathbf{x}_\mathbf{B}^* = \begin{bmatrix} x_2 \\ s_3 \\ x_1 \end{bmatrix} = \mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} 72 \\ 19 \\ 16 \end{bmatrix}$$

Simply, we have $x_1 = 16$ *and* $x_2 = 72$*.*