Topics in Tensors I
Ranks of 3-tensors

A Summer School by Shmuel Friedland[1]
July 6-8, 2011
given in Department of Mathematics
University of Coimbra, Portugal

[1] Department of Mathematics, Statistics and Computer Science, University of Illinois at Chicago, Chicago, Illinois 60607-7045, USA, e-mail:friedlan@uic.edu

Basic notions

scalar $a \in \mathbb{F}$, vector $\mathbf{x} = (x_1, \ldots, x_n)^\top \in \mathbb{F}^n$, matrix $A = [a_{ij}] \in \mathbb{F}^{m\times n}$,
3-tensor $\mathcal{T} = [t_{i,j,k}] \in \mathbb{F}^{m\times n\times l}$, p-tensor $\mathcal{T} = [t_{i_1,\ldots,i_p}] \in \mathbb{F}^{n_1\times\ldots\times n_p}$

Basic notions

scalar $a \in \mathbb{F}$, vector $\mathbf{x} = (x_1, \ldots, x_n)^\top \in \mathbb{F}^n$, matrix $A = [a_{ij}] \in \mathbb{F}^{m \times n}$,
3-tensor $\mathcal{T} = [t_{i,j,k}] \in \mathbb{F}^{m \times n \times l}$, p-tensor $\mathcal{T} = [t_{i_1,\ldots,i_p}] \in \mathbb{F}^{n_1 \times \ldots \times n_p}$

Abstractly $\mathbb{U} := \mathbb{U}_1 \otimes \mathbb{U}_2 \otimes \mathbb{U}_3$ $\dim \mathbb{U}_i = m_i, i = 1, 2, 3$, $\dim \mathbb{U} = m_1 m_2 m_3$

Basic notions

scalar $a \in \mathbb{F}$, vector $\mathbf{x} = (x_1, \ldots, x_n)^\top \in \mathbb{F}^n$, matrix $A = [a_{ij}] \in \mathbb{F}^{m \times n}$,
3-tensor $\mathcal{T} = [t_{i,j,k}] \in \mathbb{F}^{m \times n \times l}$, p-tensor $\mathcal{T} = [t_{i_1,\ldots,i_p}] \in \mathbb{F}^{n_1 \times \ldots \times n_p}$

Abstractly $\mathbb{U} := \mathbb{U}_1 \otimes \mathbb{U}_2 \otimes \mathbb{U}_3$ $\dim \mathbb{U}_i = m_i, i = 1, 2, 3$, $\dim \mathbb{U} = m_1 m_2 m_3$
Tensor $\tau \in \mathbb{U}_1 \otimes \mathbb{U}_2 \otimes \mathbb{U}_3$

Basic notions

scalar $a \in \mathbb{F}$, vector $\mathbf{x} = (x_1, \ldots, x_n)^\top \in \mathbb{F}^n$, matrix $A = [a_{ij}] \in \mathbb{F}^{m \times n}$,
3-tensor $\mathcal{T} = [t_{i,j,k}] \in \mathbb{F}^{m \times n \times l}$, p-tensor $\mathcal{T} = [t_{i_1,\ldots,i_p}] \in \mathbb{F}^{n_1 \times \ldots \times n_p}$

Abstractly $\mathbb{U} := \mathbb{U}_1 \otimes \mathbb{U}_2 \otimes \mathbb{U}_3$ $\dim \mathbb{U}_i = m_i, i = 1, 2, 3$, $\dim \mathbb{U} = m_1 m_2 m_3$
Tensor $\tau \in \mathbb{U}_1 \otimes \mathbb{U}_2 \otimes \mathbb{U}_3$

HISTORY: Tensors-as now W. Voigt 1898
Tensor calculus 1890 G. Ricci-Curbastro: absolute differential calculus,
T. Levi-Civita: 1900, A. Einstein: General relativity 1915

Basic notions

scalar $a \in \mathbb{F}$, vector $\mathbf{x} = (x_1, \ldots, x_n)^\top \in \mathbb{F}^n$, matrix $A = [a_{ij}] \in \mathbb{F}^{m \times n}$,
3-tensor $\mathcal{T} = [t_{i,j,k}] \in \mathbb{F}^{m \times n \times l}$, p-tensor $\mathcal{T} = [t_{i_1,\ldots,i_p}] \in \mathbb{F}^{n_1 \times \ldots \times n_p}$

Abstractly $\mathbb{U} := \mathbb{U}_1 \otimes \mathbb{U}_2 \otimes \mathbb{U}_3$ $\dim \mathbb{U}_i = m_i, i = 1, 2, 3$, $\dim \mathbb{U} = m_1 m_2 m_3$
Tensor $\tau \in \mathbb{U}_1 \otimes \mathbb{U}_2 \otimes \mathbb{U}_3$

HISTORY: Tensors-as now W. Voigt 1898
Tensor calculus 1890 G. Ricci-Curbastro: absolute differential calculus,
T. Levi-Civita: 1900, A. Einstein: General relativity 1915

Rank one tensor $t_{i,j,k} = x_i y_j z_k$, $(i, j, k) = (1, 1, 1), \ldots, (m_1, m_2, m_3)$
or decomposable tensor $\mathbf{x} \otimes \mathbf{y} \otimes \mathbf{z}$

Basic notions

scalar $a \in \mathbb{F}$, vector $\mathbf{x} = (x_1, \ldots, x_n)^\top \in \mathbb{F}^n$, matrix $A = [a_{ij}] \in \mathbb{F}^{m \times n}$,
3-tensor $\mathcal{T} = [t_{i,j,k}] \in \mathbb{F}^{m \times n \times l}$, p-tensor $\mathcal{T} = [t_{i_1,\ldots,i_p}] \in \mathbb{F}^{n_1 \times \ldots \times n_p}$

Abstractly $\mathbb{U} := \mathbb{U}_1 \otimes \mathbb{U}_2 \otimes \mathbb{U}_3$ $\dim \mathbb{U}_i = m_i, i = 1, 2, 3$, $\dim \mathbb{U} = m_1 m_2 m_3$
Tensor $\tau \in \mathbb{U}_1 \otimes \mathbb{U}_2 \otimes \mathbb{U}_3$

HISTORY: Tensors-as now W. Voigt 1898
Tensor calculus 1890 G. Ricci-Curbastro: absolute differential calculus,
T. Levi-Civita: 1900, A. Einstein: General relativity 1915

Rank one tensor $t_{i,j,k} = x_i y_j z_k$, $(i, j, k) = (1, 1, 1), \ldots, (m_1, m_2, m_3)$
or decomposable tensor $\mathbf{x} \otimes \mathbf{y} \otimes \mathbf{z}$

basis of $\mathbb{U}_j$: $[\mathbf{u}_{1,j}, \ldots, \mathbf{u}_{m_j,j}]$ $j = 1, 2, 3$
basis of $\mathbb{U}$: $\mathbf{u}_{i_1,1} \otimes \mathbf{u}_{i_2,2} \otimes \mathbf{u}_{i_3,3}$, $i_j = 1, \ldots, m_j, j = 1, 2, 3,$

Basic notions

scalar $a \in \mathbb{F}$, vector $\mathbf{x} = (x_1, \ldots, x_n)^\top \in \mathbb{F}^n$, matrix $A = [a_{ij}] \in \mathbb{F}^{m \times n}$,
3-tensor $\mathcal{T} = [t_{i,j,k}] \in \mathbb{F}^{m \times n \times l}$, p-tensor $\mathcal{T} = [t_{i_1,\ldots,i_p}] \in \mathbb{F}^{n_1 \times \ldots \times n_p}$

Abstractly $\mathbb{U} := \mathbb{U}_1 \otimes \mathbb{U}_2 \otimes \mathbb{U}_3$ $\dim \mathbb{U}_i = m_i, i = 1,2,3$, $\dim \mathbb{U} = m_1 m_2 m_3$
Tensor $\tau \in \mathbb{U}_1 \otimes \mathbb{U}_2 \otimes \mathbb{U}_3$

HISTORY: Tensors-as now W. Voigt 1898
Tensor calculus 1890 G. Ricci-Curbastro: absolute differential calculus,
T. Levi-Civita: 1900, A. Einstein: General relativity 1915

Rank one tensor $t_{i,j,k} = x_i y_j z_k$, $(i,j,k) = (1,1,1), \ldots, (m_1, m_2, m_3)$
or decomposable tensor $\mathbf{x} \otimes \mathbf{y} \otimes \mathbf{z}$

basis of $\mathbb{U}_j$: $[\mathbf{u}_{1,j}, \ldots, \mathbf{u}_{m_j,j}]$ $j = 1,2,3$
basis of $\mathbb{U}$: $\mathbf{u}_{i_1,1} \otimes \mathbf{u}_{i_2,2} \otimes \mathbf{u}_{i_3,3}$, $i_j = 1, \ldots, m_j, j = 1,2,3,$
$\tau = \sum_{i_1=i_2=i_3=1}^{m_1,m_2,m_3} t_{i_1,i_2,i_2} \mathbf{u}_{i_1,1} \otimes \mathbf{u}_{i_2,2} \otimes \mathbf{u}_{i_3,3}$

Ranks of tensors

Ranks of tensors

Unfolding tensor: in direction 1:
$\mathcal{T} = [t_{i,j,k}]$ view as a matrix $A_1 = [t_{i,(j,k)}] \in \mathbb{F}^{m_1 \times (m_2 \cdot m_3)}$

Ranks of tensors

Unfolding tensor: in direction 1:
$\mathcal{T} = [t_{i,j,k}]$ view as a matrix $A_1 = [t_{i,(j,k)}] \in \mathbb{F}^{m_1 \times (m_2 \cdot m_3)}$

$R_1 := \operatorname{rank} A_1$:
dimension of row or column subspace spanned in direction 1

Ranks of tensors

Unfolding tensor: in direction 1:
$\mathcal{T} = [t_{i,j,k}]$ view as a matrix $A_1 = [t_{i,(j,k)}] \in \mathbb{F}^{m_1 \times (m_2 \cdot m_3)}$

$R_1 := \mathrm{rank}\, A_1$:
dimension of row or column subspace spanned in direction 1

$T_{i,1} := [t_{i,j,k}]_{j,k=1}^{m_2,m_3} \in \mathbb{F}^{m_2 \times m_3}, i = 1, \ldots, m_1$

Ranks of tensors

Unfolding tensor: in direction 1:
$\mathcal{T} = [t_{i,j,k}]$ view as a matrix $A_1 = [t_{i,(j,k)}] \in \mathbb{F}^{m_1 \times (m_2 \cdot m_3)}$

$R_1 := \mathrm{rank}\, A_1$:
dimension of row or column subspace spanned in direction 1

$T_{i,1} := [t_{i,j,k}]_{j,k=1}^{m_2,m_3} \in \mathbb{F}^{m_2 \times m_3}, i = 1, \ldots, m_1$
$\mathcal{T} = \sum_{i=1}^{m_1} T_{i,1}\mathbf{e}_{i,1}$ (convenient notation)

Ranks of tensors

Unfolding tensor: in direction 1:
$\mathcal{T} = [t_{i,j,k}]$ view as a matrix $A_1 = [t_{i,(j,k)}] \in \mathbb{F}^{m_1 \times (m_2 \cdot m_3)}$

$R_1 := \mathrm{rank}\, A_1$:
dimension of row or column subspace spanned in direction 1

$T_{i,1} := [t_{i,j,k}]_{j,k=1}^{m_2,m_3} \in \mathbb{F}^{m_2 \times m_3}, i = 1, \ldots, m_1$
$\mathcal{T} = \sum_{i=1}^{m_1} T_{i,1} \mathbf{e}_{i,1}$ (convenient notation)
$R_1 := \dim \mathrm{span}(T_{1,1}, \ldots, T_{m_1,1})$.

Ranks of tensors

Unfolding tensor: in direction 1:
$\mathcal{T} = [t_{i,j,k}]$ view as a matrix $A_1 = [t_{i,(j,k)}] \in \mathbb{F}^{m_1 \times (m_2 \cdot m_3)}$

$R_1 := \mathrm{rank}\, A_1$:
dimension of row or column subspace spanned in direction 1

$T_{i,1} := [t_{i,j,k}]_{j,k=1}^{m_2,m_3} \in \mathbb{F}^{m_2 \times m_3}, i = 1, \ldots, m_1$
$\mathcal{T} = \sum_{i=1}^{m_1} T_{i,1} \mathbf{e}_{i,1}$ (convenient notation)
$R_1 := \dim \mathrm{span}(T_{1,1}, \ldots, T_{m_1,1})$.

Similarly, unfolding in directions 2, 3

Ranks of tensors

Unfolding tensor: in direction 1:
$\mathcal{T} = [t_{i,j,k}]$ view as a matrix $A_1 = [t_{i,(j,k)}] \in \mathbb{F}^{m_1 \times (m_2 \cdot m_3)}$

$R_1 := \mathrm{rank}\, A_1$:
dimension of row or column subspace spanned in direction 1

$T_{i,1} := [t_{i,j,k}]_{j,k=1}^{m_2,m_3} \in \mathbb{F}^{m_2 \times m_3}, i = 1, \ldots, m_1$
$\mathcal{T} = \sum_{i=1}^{m_1} T_{i,1} \mathbf{e}_{i,1}$ (convenient notation)
$R_1 := \dim \mathrm{span}(T_{1,1}, \ldots, T_{m_1,1})$.

Similarly, unfolding in directions $2, 3$

$\mathrm{rank}\, \mathcal{T}$ minimal r:

$$\mathcal{T} = f_r(\mathbf{x}_1, \mathbf{y}_1, \mathbf{z}_1, \ldots, \mathbf{x}_r, \mathbf{y}_r, \mathbf{z}_r) := \sum_{i=1}^{r} \mathbf{x}_i \otimes \mathbf{y}_i \otimes \mathbf{z}_i,$$

Ranks of tensors

Unfolding tensor: in direction 1:
$\mathcal{T} = [t_{i,j,k}]$ view as a matrix $A_1 = [t_{i,(j,k)}] \in \mathbb{F}^{m_1 \times (m_2 \cdot m_3)}$

$R_1 := \mathrm{rank}\, A_1$:
dimension of row or column subspace spanned in direction 1

$T_{i,1} := [t_{i,j,k}]_{j,k=1}^{m_2,m_3} \in \mathbb{F}^{m_2 \times m_3}, i = 1, \ldots, m_1$
$\mathcal{T} = \sum_{i=1}^{m_1} T_{i,1} \mathbf{e}_{i,1}$ (convenient notation)
$R_1 := \dim \mathrm{span}(T_{1,1}, \ldots, T_{m_1,1})$.

Similarly, unfolding in directions $2, 3$

$\mathrm{rank}\, \mathcal{T}$ minimal r:

$\mathcal{T} = f_r(\mathbf{x}_1, \mathbf{y}_1, \mathbf{z}_1, \ldots, \mathbf{x}_r, \mathbf{y}_r, \mathbf{z}_r) := \sum_{i=1}^{r} \mathbf{x}_i \otimes \mathbf{y}_i \otimes \mathbf{z}_i$,
(CANDEC, PARFAC)

Basic facts

Basic facts

FACT I: $\operatorname{rank} \mathcal{T} \geq \max(R_1, R_2, R_3)$

Basic facts

FACT I: $\mathrm{rank}\ \mathcal{T} \geq \max(R_1, R_2, R_3)$
Reason $\mathbb{U}_2 \otimes \mathbb{U}_3 \sim \mathbb{F}^{m_2 \times m_3} \equiv \mathbb{F}^{m_2 m_3}$

Basic facts

FACT I: $\operatorname{rank} \mathcal{T} \geq \max(R_1, R_2, R_3)$
Reason $\mathbb{U}_2 \otimes \mathbb{U}_3 \sim \mathbb{F}^{m_2 \times m_3} \equiv \mathbb{F}^{m_2 m_3}$

Note:

- R_1, R_2, R_3 are easily computable
- It is possible that $R_1 \neq R_2 \neq R_3$

Basic facts

FACT I: $\mathrm{rank}\, \mathcal{T} \geq \max(R_1, R_2, R_3)$
Reason $\mathbb{U}_2 \otimes \mathbb{U}_3 \sim \mathbb{F}^{m_2 \times m_3} \equiv \mathbb{F}^{m_2 m_3}$

Note:

- R_1, R_2, R_3 are easily computable
- It is possible that $R_1 \neq R_2 \neq R_3$

FACT II : For $\tau = \mathcal{T} = [t_{i,j,k}]$ let $T_{k,3} := [t_{i,j,k}]_{i,j=1}^{m_1,m_2} \in \mathbb{F}^{m_1 \times m_2}, k = 1, \ldots, m_3$. Then $\mathrm{rank}\, \mathcal{T} =$ minimal dimension of subspace $L \subset \mathbb{F}^{m_1 \times m_2}$ spanned by rank one matrices containing $T_{1,3}, \ldots, T_{m_3,3}$.

Basic facts

FACT I: $\operatorname{rank} \mathcal{T} \geq \max(R_1, R_2, R_3)$
Reason $\mathbb{U}_2 \otimes \mathbb{U}_3 \sim \mathbb{F}^{m_2 \times m_3} \equiv \mathbb{F}^{m_2 m_3}$

Note:

- R_1, R_2, R_3 are easily computable
- It is possible that $R_1 \neq R_2 \neq R_3$

FACT II : For $\tau = \mathcal{T} = [t_{i,j,k}]$ let $T_{k,3} := [t_{i,j,k}]_{i,j=1}^{m_1,m_2} \in \mathbb{F}^{m_1 \times m_2}, k = 1, \ldots, m_3$. Then $\operatorname{rank} \mathcal{T} =$ minimal dimension of subspace $L \subset \mathbb{F}^{m_1 \times m_2}$ spanned by rank one matrices containing $T_{1,3}, \ldots, T_{m_3,3}$.

COR $\operatorname{rank} \mathcal{T} \leq \min(mn, ml, nl)$

Basic facts

FACT I: $\mathrm{rank}\, \mathcal{T} \geq \max(R_1, R_2, R_3)$
Reason $\mathbb{U}_2 \otimes \mathbb{U}_3 \sim \mathbb{F}^{m_2 \times m_3} \equiv \mathbb{F}^{m_2 m_3}$

Note:

- R_1, R_2, R_3 are easily computable
- It is possible that $R_1 \neq R_2 \neq R_3$

FACT II : For $\tau = \mathcal{T} = [t_{i,j,k}]$ let
$T_{k,3} := [t_{i,j,k}]_{i,j=1}^{m_1,m_2} \in \mathbb{F}^{m_1 \times m_2}, k = 1, \ldots, m_3$. Then $\mathrm{rank}\, \mathcal{T} =$ minimal dimension of subspace $L \subset \mathbb{F}^{m_1 \times m_2}$ spanned by rank one matrices containing $T_{1,3}, \ldots, T_{m_3,3}$.

COR $\mathrm{rank}\, \mathcal{T} \leq \min(mn, ml, nl)$

PROOF: Suppose $\tau = \sum_{i=1}^{p} \mathbf{x}_i \otimes \mathbf{y}_i \otimes \mathbf{z}_i$ (1)

Basic facts

FACT I: $\mathrm{rank}\ \mathcal{T} \ge \max(R_1, R_2, R_3)$
Reason $\mathbb{U}_2 \otimes \mathbb{U}_3 \sim \mathbb{F}^{m_2 \times m_3} \equiv \mathbb{F}^{m_2 m_3}$

Note:

- R_1, R_2, R_3 are easily computable
- It is possible that $R_1 \ne R_2 \ne R_3$

FACT II : For $\tau = \mathcal{T} = [t_{i,j,k}]$ let
$T_{k,3} := [t_{i,j,k}]_{i,j=1}^{m_1,m_2} \in \mathbb{F}^{m_1 \times m_2}, k = 1, \ldots, m_3$. Then $\mathrm{rank}\ \mathcal{T} =$ minimal dimension of subspace $L \subset \mathbb{F}^{m_1 \times m_2}$ spanned by rank one matrices containing $T_{1,3}, \ldots, T_{m_3,3}$.

COR $\mathrm{rank}\ \mathcal{T} \le \min(mn, ml, nl)$

PROOF: Suppose $\tau = \sum_{i=1}^{p} \mathbf{x}_i \otimes \mathbf{y}_i \otimes \mathbf{z}_i$ (1)
Write $\mathbf{z}_i = \sum_{j=1}^{m_3} z_{i,j} \mathbf{e}_{j,3}$ then each $T_{k,3} \in \mathrm{span}(\mathbf{x}_1 \otimes \mathbf{y}_1, \ldots, \mathbf{x}_p \otimes \mathbf{y}_p)$.

Basic facts

FACT I: $\mathrm{rank}\ \mathcal{T} \geq \max(R_1, R_2, R_3)$
Reason $\mathbb{U}_2 \otimes \mathbb{U}_3 \sim \mathbb{F}^{m_2 \times m_3} \equiv \mathbb{F}^{m_2 m_3}$

Note:

- R_1, R_2, R_3 are easily computable
- It is possible that $R_1 \neq R_2 \neq R_3$

FACT II : For $\tau = \mathcal{T} = [t_{i,j,k}]$ let
$T_{k,3} := [t_{i,j,k}]_{i,j=1}^{m_1,m_2} \in \mathbb{F}^{m_1 \times m_2}, k = 1, \ldots, m_3$. Then $\mathrm{rank}\ \mathcal{T} =$ minimal dimension of subspace $L \subset \mathbb{F}^{m_1 \times m_2}$ spanned by rank one matrices containing $T_{1,3}, \ldots, T_{m_3,3}$.

COR $\mathrm{rank}\ \mathcal{T} \leq \min(mn, ml, nl)$

PROOF: Suppose $\tau = \sum_{i=1}^{p} \mathbf{x}_i \otimes \mathbf{y}_i \otimes \mathbf{z}_i$ (1)
Write $\mathbf{z}_i = \sum_{j=1}^{m_3} z_{i,j} \mathbf{e}_{j,3}$ then each $T_{k,3} \in \mathrm{span}(\mathbf{x}_1 \otimes \mathbf{y}_1, \ldots, \mathbf{x}_p \otimes \mathbf{y}_p)$.
Vise versa suppose $T_{k,3} = \sum_{i=1}^{p} a_{k,i} \mathbf{x}_i \otimes \mathbf{y}_i, k = 1, \ldots, m_3$.

Basic facts

FACT I: $\text{rank}\,\mathcal{T} \geq \max(R_1, R_2, R_3)$
Reason $\mathbb{U}_2 \otimes \mathbb{U}_3 \sim \mathbb{F}^{m_2 \times m_3} \equiv \mathbb{F}^{m_2 m_3}$

Note:

- R_1, R_2, R_3 are easily computable
- It is possible that $R_1 \neq R_2 \neq R_3$

FACT II : For $\tau = \mathcal{T} = [t_{i,j,k}]$ let
$T_{k,3} := [t_{i,j,k}]_{i,j=1}^{m_1,m_2} \in \mathbb{F}^{m_1 \times m_2}, k = 1, \ldots, m_3$. Then $\text{rank}\,\mathcal{T} =$ minimal dimension of subspace $L \subset \mathbb{F}^{m_1 \times m_2}$ spanned by rank one matrices containing $T_{1,3}, \ldots, T_{m_3,3}$.

COR $\text{rank}\,\mathcal{T} \leq \min(mn, ml, nl)$

PROOF: Suppose $\tau = \sum_{i=1}^{p} \mathbf{x}_i \otimes \mathbf{y}_i \otimes \mathbf{z}_i$ (1)
Write $\mathbf{z}_i = \sum_{j=1}^{m_3} z_{i,j}\mathbf{e}_{j,3}$ then each $T_{k,3} \in \text{span}(\mathbf{x}_1 \otimes \mathbf{y}_1, \ldots, \mathbf{x}_p \otimes \mathbf{y}_p)$.
Vise versa suppose $T_{k,3} = \sum_{i=1}^{p} a_{k,i}\mathbf{x}_i \otimes \mathbf{y}_i, k = 1, \ldots, m_3$.
Then (1) holds with $\mathbf{z}_i := \sum_{k=1}^{m_3} a_{k,i}\mathbf{e}_{k,3}$.

Complexity of rank of 3-tensor

Complexity of rank of 3-tensor

Hastad 1990: Tensor rank is NP-complete for any finite field
Is rank of a tensor is at most k:
provide the decomposition of $\mathcal{T}$ as a sum of at most k rank one tensors

Complexity of rank of 3-tensor

Hastad 1990: Tensor rank is NP-complete for any finite field
Is rank of a tensor is at most k:
provide the decomposition of $\mathcal{T}$ as a sum of at most k rank one tensors

PRF: 3-sat with n variables m clauses
satisfiable iff $\mathrm{rank}\ \mathcal{T} = 4n + 2m, \mathcal{T} \in \mathbb{F}^{(2n+3m)\times(3n)\times(3n+m)})$
otherwise rank is larger

Generic and typical ranks

Generic and typical ranks

$\mathcal{R}_r(m,n,l) \subset \mathbb{F}^{m\times n\times l}$: all tensors of rank $\leq r$

Generic and typical ranks

$\mathcal{R}_r(m, n, l) \subset \mathbb{F}^{m \times n \times l}$: all tensors of rank $\leq r$

$\mathcal{R}_r(m, n, l)$ not closed variety for $r \geq 2$

Generic and typical ranks

$\mathcal{R}_r(m,n,l) \subset \mathbb{F}^{m\times n\times l}$: all tensors of rank $\le r$

$\mathcal{R}_r(m,n,l)$ not closed variety for $r \ge 2$

Border rank of $\mathcal{T}$ the minimum k s.t. $\mathcal{T}$ is a limit of $\mathcal{T}_j, j \in \mathbb{N}, \text{rank } T_j = k$.

Generic and typical ranks

$\mathcal{R}_r(m,n,l) \subset \mathbb{F}^{m\times n\times l}$: all tensors of rank $\le r$

$\mathcal{R}_r(m,n,l)$ not closed variety for $r \ge 2$

Border rank of $\mathcal{T}$ the minimum k s.t. $\mathcal{T}$ is a limit of $\mathcal{T}_j, j \in \mathbb{N}$, $\mathrm{rank}\ T_j = k$.

generic rank is the rank of a random tensor $\mathcal{T} \in \mathbb{C}^{m\times n\times l}$: $\mathrm{grank}(m,n,l)$

Generic and typical ranks

$\mathcal{R}_r(m,n,l) \subset \mathbb{F}^{m\times n\times l}$: all tensors of rank $\leq r$

$\mathcal{R}_r(m,n,l)$ not closed variety for $r \geq 2$

Border rank of $\mathcal{T}$ the minimum k s.t. $\mathcal{T}$ is a limit of $\mathcal{T}_j, j \in \mathbb{N}$, $\mathrm{rank}\ T_j = k$.

generic rank is the rank of a random tensor $\mathcal{T} \in \mathbb{C}^{m\times n\times l}$: $\mathrm{grank}(m,n,l)$

typical rank is a rank of a random tensor $\mathcal{T} \in \mathbb{R}^{m\times n\times l}$.

Generic and typical ranks

$\mathcal{R}_r(m,n,l) \subset \mathbb{F}^{m\times n\times l}$: all tensors of rank $\le r$

$\mathcal{R}_r(m,n,l)$ not closed variety for $r \ge 2$

Border rank of $\mathcal{T}$ the minimum k s.t. $\mathcal{T}$ is a limit of $\mathcal{T}_j, j \in \mathbb{N}$, $\mathrm{rank}\ T_j = k$.

generic rank is the rank of a random tensor $\mathcal{T} \in \mathbb{C}^{m\times n\times l}$: $\mathrm{grank}(m,n,l)$

typical rank is a rank of a random tensor $\mathcal{T} \in \mathbb{R}^{m\times n\times l}$.

typical rank takes all the values $k = \mathrm{grank}(m,n,l), \ldots, \mathrm{mtrank}(m,n,l)$

Generic and typical ranks

$\mathcal{R}_r(m, n, l) \subset \mathbb{F}^{m \times n \times l}$: all tensors of rank $\leq r$

$\mathcal{R}_r(m, n, l)$ not closed variety for $r \geq 2$

Border rank of $\mathcal{T}$ the minimum k s.t. $\mathcal{T}$ is a limit of $\mathcal{T}_j, j \in \mathbb{N}$, $\mathrm{rank}\ T_j = k$.

generic rank is the rank of a random tensor $\mathcal{T} \in \mathbb{C}^{m \times n \times l}$: $\mathrm{grank}(m, n, l)$

typical rank is a rank of a random tensor $\mathcal{T} \in \mathbb{R}^{m \times n \times l}$.

typical rank takes all the values $k = \mathrm{grank}(m, n, l), \ldots, \mathrm{mtrank}(m, n, l)$

In all the examples we know $\mathrm{mtrank}(m, n, l) \leq \mathrm{grank}(m, n, l) + 1$

Examples

$\mathbf{U} \subset \mathbb{F}^{m \times n}$: $\mathrm{mrank}\mathbf{U} := \max\{\mathrm{rank}\ A, A \in \mathbf{U}\}$
$\mathrm{rank}\ \mathcal{T} \geq \mathrm{mrank}\mathbf{T}_p(\mathcal{T})$.
$\mathrm{grank}(2, m, m) = m$
$\mathrm{mtrank}(2, 2, 2) = 3$
$grank(2, m, n) = min(n, 2m)$ for $2 \leq m \leq n$

Order of presentation from the paper On the generic and typical ranks of 3-tensors

1. Appendix: Complex and real algebraic geometry
(first the complex case).
2. Generic rank.
3. Matrices and the rank of 3 tensors
4. Maximal rank
5. Known results on rank of tensors
6. Typical rank of real 3 tensors
(First rudiments of real algebraic geometry.)

Supersymmetric tensors

$\mathcal{F} = [f_{i_1,\ldots,i_d}] \in (\mathbb{C}^m)^{\otimes d}$ supersymmetric if
$\mathcal{F}$ invariant under permutations of indices
the entries of $\mathcal{F}$ are d-mixed derivative of
homogeneous polynomial $f(\mathbf{x})$ of degree d in $\mathbf{x} = (x_1, \ldots, x_m)^\top$
$f(\mathbf{x}) = \sum_{i=1}^{r} l_i(\mathbf{x})^d$ where each $l_i(\mathbf{x}) = \sum_{j=1}^{m} l_{ij} x_j$
the minimal r-is the supersymmetric rank of $\mathcal{F}$
Sylvester's theorem: for $d = 2$ the symmetric rank of symmetric matrix is the rank of symmetric matrix
$d \geq 3$
Counting parameters: $f(\mathbf{x})$ has $\binom{m+d-1}{d}$ coefficients
to each sequence $1 \leq i_1 \leq m_2 \leq \ldots \leq i_d \leq m$ corresponds a unique sequence $1 \leq m_1 < m_2 + 1 < \ldots < m_d + d - 1 \leq m + d - 1$
$\mathrm{symgrank}(\mathcal{F}) \geq \lceil \frac{\binom{m+d-1}{d}}{m} \rceil$
Alexander-Hirschowitz theorem:
Equality holds except for a finite number of exceptions