Topics in Tensors II
A set theoretic solution of the salmon conjecture

A Summer School by Shmuel Friedland[1]

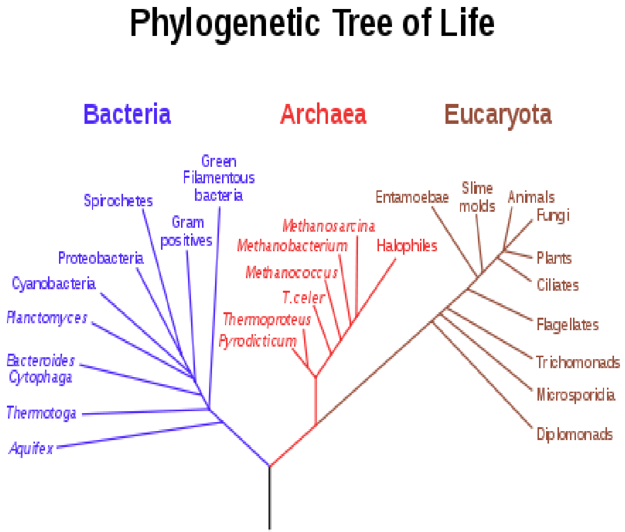
July 6-8, 2011
given in Department of Mathematics
University of Coimbra, Portugal

[1]Department of Mathematics, Statistics and Computer Science, University of Illinois at Chicago, Chicago, Illinois 60607-7045, USA, e-mail:friedlan@uic.edu

Summary

1. Phylogenetic trees and their invariants
2. Statement of the problem
3. Border rank
4. Known results
5. New conditions
6. Outline of the complete solution

Phylogenetic tree



Reconstruction of the Phylogenetic tree with n taxa

Reconstruction of the Phylogenetic tree with n taxa

Given n leaves of a tree, taxa
Find a best tree with internal vertices of degree 3 with given taxa

Reconstruction of the Phylogenetic tree with n taxa

Given n leaves of a tree, taxa
Find a best tree with internal vertices of degree 3 with given taxa
Branching pattern of the tree is called the topology

Reconstruction of the Phylogenetic tree with n taxa

Given n leaves of a tree, taxa
Find a best tree with internal vertices of degree 3 with given taxa
Branching pattern of the tree is called the topology

The evolution of the tree is modeled by Markov chain

Reconstruction of the Phylogenetic tree with n taxa

Given n leaves of a tree, taxa
Find a best tree with internal vertices of degree 3 with given taxa
Branching pattern of the tree is called the topology

The evolution of the tree is modeled by Markov chain
Evolution- random substitution of one nucleotide of DNA A, G, C, T at individual sites

Reconstruction of the Phylogenetic tree with n taxa

Given n leaves of a tree, taxa
Find a best tree with internal vertices of degree 3 with given taxa
Branching pattern of the tree is called the topology

The evolution of the tree is modeled by Markov chain
Evolution- random substitution of one nucleotide of DNA A, G, C, T at individual sites

The topology of the tree gives rise to the joint distribution of the taxa
$\mathbf{X} = (X_1, \ldots, X_n)$, $X_i \in \{A, G, C, T\} = \{1, 2, 3, 4\}$, $i = 1, \ldots, n$

Reconstruction of the Phylogenetic tree with n taxa

Given n leaves of a tree, taxa
Find a best tree with internal vertices of degree 3 with given taxa
Branching pattern of the tree is called the topology

The evolution of the tree is modeled by Markov chain
Evolution- random substitution of one nucleotide of DNA A, G, C, T at individual sites

The topology of the tree gives rise to the joint distribution of the taxa
$\mathbf{X} = (X_1, \ldots, X_n)$, $X_i \in \{A, G, C, T\} = \{1, 2, 3, 4\}$, $i = 1, \ldots, n$

Joint distribution of $\mathbf{X}$ is tensor $\mathcal{T} = [t_{i_1 \ldots i_n}] \in \otimes^n [0, 1]$

Reconstruction of the Phylogenetic tree with n taxa

Given n leaves of a tree, taxa
Find a best tree with internal vertices of degree 3 with given taxa
Branching pattern of the tree is called the topology

The evolution of the tree is modeled by Markov chain
Evolution- random substitution of one nucleotide of DNA A, G, C, T at individual sites

The topology of the tree gives rise to the joint distribution of the taxa
$\mathbf{X} = (X_1, \ldots, X_n)$, $X_i \in \{A, G, C, T\} = \{1, 2, 3, 4\}$, $i = 1, \ldots, n$

Joint distribution of $\mathbf{X}$ is tensor $\mathcal{T} = [t_{i_1 \ldots i_n}] \in \otimes^n [0, 1]$
Basic problem of algebraic statistics:
Characterize the variety which is a closure of all $\mathcal{T}$ corresponding to a given tree

Specific problem

One parent, (the root) 3 descendants, (taxa): x, y, z

Specific problem

One parent, (the root) 3 descendants, (taxa): x, y, z

Main technical assumption on the joint distribution of X, Y, Z

$\mathcal{T} = \pi_A \mathbf{x}_A \otimes \mathbf{y}_A \otimes \mathbf{z}_A + \pi_C \mathbf{x}_C \otimes \mathbf{y}_C \otimes \mathbf{z}_C + \pi_G \mathbf{x}_G \otimes \mathbf{y}_G \otimes \mathbf{z}_G + \pi_T \mathbf{x}_T \otimes \mathbf{y}_T \otimes \mathbf{z}_T$

$\mathbf{x}_A, \ldots, \mathbf{z}_T, \boldsymbol{\pi} = (\pi_A, \pi_C, \pi_G, \pi_T)^\top$ probability vectors in $\mathbb{R}^4$

Specific problem

One parent, (the root) 3 descendants, (taxa): x, y, z

Main technical assumption on the joint distribution of X, Y, Z

$$\mathcal{T} = \pi_A \mathbf{x}_A \otimes \mathbf{y}_A \otimes \mathbf{z}_A + \pi_C \mathbf{x}_C \otimes \mathbf{y}_C \otimes \mathbf{z}_C + \pi_G \mathbf{x}_G \otimes \mathbf{y}_G \otimes \mathbf{z}_G + \pi_T \mathbf{x}_T \otimes \mathbf{y}_T \otimes \mathbf{z}_T$$

$\mathbf{x}_A, \ldots, \mathbf{z}_T, \boldsymbol{\pi} = (\pi_A, \pi_C, \pi_G, \pi_T)^\top$ probability vectors in $\mathbb{R}^4$

Problem: Characterize the variety of all tensors in $\mathbb{C}^{4\times4\times4} = \mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$ of border rank 4 at most

Specific problem

One parent, (the root) 3 descendants, (taxa): x, y, z

Main technical assumption on the joint distribution of X, Y, Z

$$\mathcal{T} = \pi_A \mathbf{x}_A \otimes \mathbf{y}_A \otimes \mathbf{z}_A + \pi_C \mathbf{x}_C \otimes \mathbf{y}_C \otimes \mathbf{z}_C + \pi_G \mathbf{x}_G \otimes \mathbf{y}_G \otimes \mathbf{z}_G + \pi_T \mathbf{x}_T \otimes \mathbf{y}_T \otimes \mathbf{z}_T$$

$\mathbf{x}_A, \ldots, \mathbf{z}_T, \boldsymbol{\pi} = (\pi_A, \pi_C, \pi_G, \pi_T)^\top$ probability vectors in $\mathbb{R}^4$

Problem: Characterize the variety of all tensors in $\mathbb{C}^{4\times4\times4} = \mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$ of border rank 4 at most

$\mathcal{T} \in \mathbb{C}^{4\times4\times4}$ has a border at most k
if it is a limit of tensors of rank k at most

Ranks of tensor 1

$\mathcal{T} = [t_{ijk}]_{i=j=k=1}^{m,n,l} \in \mathbb{C}^{m\times n\times l}$ general 3-tensor

Ranks of tensor 1

$\mathcal{T} = [t_{ijk}]_{i=j=k=1}^{m,n,l} \in \mathbb{C}^{m\times n\times l}$ general 3-tensor

$T_{k,3} = [t_{ijk}]_{i=j=1}^{m,n} \in \mathbb{C}^{m\times n}$, $k = 1, \ldots, l$ called k-3-sections of $\mathcal{T}$.

Ranks of tensor 1

$\mathcal{T} = [t_{ijk}]_{i=j=k=1}^{m,n,l} \in \mathbb{C}^{m\times n\times l}$ general 3-tensor

$T_{k,3} = [t_{ijk}]_{i=j=1}^{m,n} \in \mathbb{C}^{m\times n}$, $k = 1, \ldots, l$ called k-3-sections of $\mathcal{T}$.

$\mathbf{W} \subset \mathbb{C}^{4\times 4}$ subspace spanned by four sections of $\mathcal{T} \in \mathbb{C}^{4\times 4\times 4}$
$\mathrm{rank}\ \mathcal{T}$ is the minimal dimension of a subspace containing $\mathbf{W}$ and spanned by rank one matrices

Ranks of tensor 1

$\mathcal{T} = [t_{ijk}]_{i=j=k=1}^{m,n,l} \in \mathbb{C}^{m\times n\times l}$ general 3-tensor

$T_{k,3} = [t_{ijk}]_{i=j=1}^{m,n} \in \mathbb{C}^{m\times n}$, $k = 1, \ldots, l$ called k-3-sections of $\mathcal{T}$.

$\mathbf{W} \subset \mathbb{C}^{4\times 4}$ subspace spanned by four sections of $\mathcal{T} \in \mathbb{C}^{4\times 4\times 4}$
$\mathrm{rank}\ \mathcal{T}$ is the minimal dimension of a subspace containing $\mathbf{W}$ and spanned by rank one matrices

$\mathrm{grank}(m, n, l)$ the rank of most of tensors in $\mathbb{C}^{m\times n\times l}$.

Ranks of tensor 1

$\mathcal{T} = [t_{ijk}]_{i=j=k=1}^{m,n,l} \in \mathbb{C}^{m\times n\times l}$ general 3-tensor

$T_{k,3} = [t_{ijk}]_{i=j=1}^{m,n} \in \mathbb{C}^{m\times n}$, $k = 1, \ldots, l$ called k-3-sections of $\mathcal{T}$.

$\mathbf{W} \subset \mathbb{C}^{4\times 4}$ subspace spanned by four sections of $\mathcal{T} \in \mathbb{C}^{4\times 4\times 4}$
$\mathrm{rank}\ \mathcal{T}$ is the minimal dimension of a subspace containing $\mathbf{W}$ and spanned by rank one matrices

$\mathrm{grank}(m, n, l)$ the rank of most of tensors in $\mathbb{C}^{m\times n\times l}$.

$\mathrm{grank}(m, n, l) = l$ for $l \in [(m-1)(n-1)+1, l]$

Ranks of tensor 1

$\mathcal{T} = [t_{ijk}]_{i=j=k=1}^{m,n,l} \in \mathbb{C}^{m\times n\times l}$ general 3-tensor

$T_{k,3} = [t_{ijk}]_{i=j=1}^{m,n} \in \mathbb{C}^{m\times n}$, $k = 1, \ldots, l$ called k-3-sections of $\mathcal{T}$.

$\mathbf{W} \subset \mathbb{C}^{4\times 4}$ subspace spanned by four sections of $\mathcal{T} \in \mathbb{C}^{4\times 4\times 4}$
$\mathrm{rank}\ \mathcal{T}$ is the minimal dimension of a subspace containing $\mathbf{W}$ and spanned by rank one matrices

$\mathrm{grank}(m, n, l)$ the rank of most of tensors in $\mathbb{C}^{m\times n\times l}$.

$\mathrm{grank}(m, n, l) = l$ for $l \in [(m-1)(n-1)+1, l]$

Reason: A generic space $\mathbf{W} \subset \mathbb{C}^{m\times n}$, $\dim \mathbf{W} = (m-1)(n-1)+1$ intersects the variety of all matrices of rank 1: $\mathbb{C}^m \times \mathbb{C}^n \subset \mathbb{C}^{m\times n}$ at least at $(m-1)(n-1)+1$ linearly independent rank one matrices

Ranks of tensors 2

Generic subspace $\mathbf{W} \subset \mathrm{S}(m, \mathbb{C}), \dim \mathbf{W} = \frac{m(m-1)}{2} + 1$ intersects variety of symmetric matrices of rank 1 at least at $\frac{m(m-1)}{2} + 1$ lin. ind. mat.

Cor.: generic $\mathcal{T} \in \mathbb{C}^{3 \times 3 \times 4}$ symmetric in the first two indices has rank 4

Ranks of tensors 2

Generic subspace $\mathbf{W} \subset \mathrm{S}(m, \mathbb{C}), \dim \mathbf{W} = \frac{m(m-1)}{2} + 1$ intersects variety of symmetric matrices of rank 1 at least at $\frac{m(m-1)}{2} + 1$ lin. ind. mat.

Cor.: generic $\mathcal{T} \in \mathbb{C}^{3\times3\times4}$ symmetric in the first two indices has rank 4

Strassen 1983: a. $\mathrm{grank}(3,3,3) = 5$

Ranks of tensors 2

Generic subspace $\mathbf{W} \subset \mathrm{S}(m, \mathbb{C}), \dim \mathbf{W} = \frac{m(m-1)}{2} + 1$ intersects variety of symmetric matrices of rank 1 at least at $\frac{m(m-1)}{2} + 1$ lin. ind. mat.

Cor.: generic $\mathcal{T} \in \mathbb{C}^{3\times 3\times 4}$ symmetric in the first two indices has rank 4

Strassen 1983: a. $\mathrm{grank}(3, 3, 3) = 5$

b. variety of all tensors in $\mathbb{C}^{3\times 3\times 3}$ of at most rank 4 is a hypersurface of degree 9

$$\frac{1}{\det\ Z} \det\left(X(\mathrm{adj}\, Z)Y - Y(\mathrm{adj}\, Z)X\right) = 0$$

X, Y, Z are three sections of $\mathcal{T}$

Tensors of rank m in $\mathbb{C}^{m \times m \times l}$

Tensors of rank m in $\mathbb{C}^{m\times m\times l}$

$\mathcal{T} \in \mathbb{C}^{m\times m\times l}$, rank $\mathcal{T} = m$, $\mathbf{W} = \mathrm{span}(T_{1,3}, \ldots, T_{l,3}) \in \mathbb{C}^{m\times m}$
spanned by $\mathbf{u}_1\mathbf{v}_1^\top, \ldots, \mathbf{u}_m\mathbf{v}_m^\top$.

Tensors of rank m in $\mathbb{C}^{m\times m\times l}$

$\mathcal{T} \in \mathbb{C}^{m\times m\times l}$, $\mathrm{rank}\ \mathcal{T} = m$, $\mathbf{W} = \mathrm{span}(T_{1,3}, \ldots, T_{l,3}) \in \mathbb{C}^{m\times m}$ spanned by $\mathbf{u}_1\mathbf{v}_1^\top, \ldots, \mathbf{u}_m\mathbf{v}_m^\top$.

generic case: $\exists P, Q \in \mathbf{GL}(m, \mathbb{C})$ $P\mathbf{W}Q$ subspace of commuting diagonal matrices $\iff$ $Z^{-1}\mathbf{W}$ a subspace of commuting matrices

Tensors of rank m in $\mathbb{C}^{m\times m\times l}$

$\mathcal{T} \in \mathbb{C}^{m\times m\times l}$, $\mathrm{rank}\ \mathcal{T} = m$, $\mathbf{W} = \mathrm{span}(T_{1,3}, \ldots, T_{l,3}) \in \mathbb{C}^{m\times m}$ spanned by $\mathbf{u}_1\mathbf{v}_1^\top, \ldots, \mathbf{u}_m\mathbf{v}_m^\top$.

generic case: $\exists P, Q \in \mathbf{GL}(m, \mathbb{C})$ $P\mathbf{W}Q$ subspace of commuting diagonal matrices $\iff$ $Z^{-1}\mathbf{W}$ a subspace of commuting matrices

If $\mathbf{W}$ contains an invertible matrix Z then any other $X, Y \in \mathbf{W}$ satisfy $X(\mathrm{adj}Z)Y = Y(\mathrm{adj}Z)X$ - equations of degree 5 for $m = 4$

Tensors of rank m in $\mathbb{C}^{m\times m\times l}$

$\mathcal{T} \in \mathbb{C}^{m\times m\times l}$, $\mathrm{rank}\ \mathcal{T} = m$, $\mathbf{W} = \mathrm{span}(T_{1,3}, \ldots, T_{l,3}) \in \mathbb{C}^{m\times m}$ spanned by $\mathbf{u}_1\mathbf{v}_1^\top, \ldots, \mathbf{u}_m\mathbf{v}_m^\top$.

generic case: $\exists P, Q \in \mathbf{GL}(m, \mathbb{C})$ $P\mathbf{W}Q$ subspace of commuting diagonal matrices $\iff$ $Z^{-1}\mathbf{W}$ a subspace of commuting matrices

If $\mathbf{W}$ contains an invertible matrix Z then any other $X, Y \in \mathbf{W}$ satisfy $X(\mathrm{adj} Z)Y = Y(\mathrm{adj} Z)X$ - equations of degree 5 for $m = 4$

similarly $\mathrm{C}_2(X)\widetilde{\mathrm{C}_2(Z)}\mathrm{C}_2(Y) = \mathrm{C}_2(Y)\widetilde{\mathrm{C}_2(Z)}\mathrm{C}_2(Z)$- equations of degree 6 for $m = 4$

Tensors of rank m in $\mathbb{C}^{m\times m\times l}$

$\mathcal{T} \in \mathbb{C}^{m\times m\times l}$, $\mathrm{rank}\ \mathcal{T} = m$, $\mathbf{W} = \mathrm{span}(T_{1,3},\dots,T_{l,3}) \in \mathbb{C}^{m\times m}$ spanned by $\mathbf{u}_1\mathbf{v}_1^\top,\dots,\mathbf{u}_m\mathbf{v}_m^\top$.

generic case: $\exists P, Q \in \mathbf{GL}(m,\mathbb{C})$ $P\mathbf{W}Q$ subspace of commuting diagonal matrices $\iff$ $Z^{-1}\mathbf{W}$ a subspace of commuting matrices

If $\mathbf{W}$ contains an invertible matrix Z then any other $X, Y \in \mathbf{W}$ satisfy $X(\mathrm{adj} Z)Y = Y(\mathrm{adj} Z)X$ - equations of degree 5 for $m = 4$

similarly $\mathrm{C}_2(X)\widetilde{\mathrm{C}_2(Z)}\mathrm{C}_2(Y) = \mathrm{C}_2(Y)\widetilde{\mathrm{C}_2(Z)}\mathrm{C}_2(Z)$- equations of degree 6 for $m = 4$

Strassen's condition hold for any $3\times 3\times 3$ subtensor of $\mathcal{T} \in \mathbb{C}^{4\times 4\times 4}$: equations of degree 9

Tensors of rank m in $\mathbb{C}^{m\times m\times l}$

$\mathcal{T} \in \mathbb{C}^{m\times m\times l}$, $\operatorname{rank} \mathcal{T} = m$, $\mathbf{W} = \operatorname{span}(T_{1,3}, \ldots, T_{l,3}) \in \mathbb{C}^{m\times m}$ spanned by $\mathbf{u}_1\mathbf{v}_1^\top, \ldots, \mathbf{u}_m\mathbf{v}_m^\top$.

generic case: $\exists P, Q \in \mathbf{GL}(m, \mathbb{C})$ $P\mathbf{W}Q$ subspace of commuting diagonal matrices $\iff Z^{-1}\mathbf{W}$ a subspace of commuting matrices

If $\mathbf{W}$ contains an invertible matrix Z then any other $X, Y \in \mathbf{W}$ satisfy $X(\operatorname{adj}Z)Y = Y(\operatorname{adj}Z)X$ - equations of degree 5 for $m = 4$

similarly $\mathrm{C}_2(X)\widetilde{\mathrm{C}_2(Z)}\mathrm{C}_2(Y) = \mathrm{C}_2(Y)\widetilde{\mathrm{C}_2(Z)}\mathrm{C}_2(Z)$- equations of degree 6 for $m = 4$

Strassen's condition hold for any $3 \times 3 \times 3$ subtensor of $\mathcal{T} \in \mathbb{C}^{4\times 4\times 4}$: equations of degree 9

[3] one needs equations of degree 16

16 degree conditions 1

16 degree conditions 1

Manivel-Landsberg: Cor. 5.6: to determine completely the variety of tensors of border rank at most 4 in $C^{4\times4\times4}$ one needs in addition to above conditions to determine the the variety of tensors of border rank at most 4 in $C^{3\times3\times4}$

Prf. is wrong as Prop. 5.4 wrong. I had to reprove Cor. 5.6

16 degree conditions 1

Manivel-Landsberg: Cor. 5.6: to determine completely the variety of tensors of border rank at most 4 in $C^{4\times4\times4}$ one needs in addition to above conditions to determine the the variety of tensors of border rank at most 4 in $C^{3\times3\times4}$

Prf. is wrong as Prop. 5.4 wrong. I had to reprove Cor. 5.6

generic subspace spanned by four rank one matrices in $\mathbb{C}^{4\times4}$: $\mathrm{span}(\mathbf{u}_1\mathbf{v}_1^\top,\ldots,\mathbf{u}_4\mathbf{v}_4^\top)$ where any three vectors out of $\mathbf{u}_1,\ldots,\mathbf{u}_4, \mathbf{v}_1,\ldots,\mathbf{v}_4$ linearly independent

16 degree conditions 1

Manivel-Landsberg: Cor. 5.6: to determine completely the variety of tensors of border rank at most 4 in $C^{4\times4\times4}$ one needs in addition to above conditions to determine the the variety of tensors of border rank at most 4 in $C^{3\times3\times4}$

Prf. is wrong as Prop. 5.4 wrong. I had to reprove Cor. 5.6

generic subspace spanned by four rank one matrices in $\mathbb{C}^{4\times4}$: $\mathrm{span}(\mathbf{u}_1\mathbf{v}_1^\top,\ldots,\mathbf{u}_4\mathbf{v}_4^\top)$ where any three vectors out of $\mathbf{u}_1,\ldots,\mathbf{u}_4,\mathbf{v}_1,\ldots,\mathbf{v}_4$ linearly independent

$\exists P,Q\in\mathbf{GL}(3,\mathbb{C})$: $P\mathbf{u}_i=Q\mathbf{v}_i=(\delta_{i1},\delta_{i2},\delta_{i3})^\top, i=1,2,3$, $P\mathbf{u}_4=Q\mathbf{v}_4$ $\iff PWQ\subset \mathrm{S}(3,\mathbb{C})\iff$

16 degree conditions 1

Manivel-Landsberg: Cor. 5.6: to determine completely the variety of tensors of border rank at most 4 in $C^{4\times4\times4}$ one needs in addition to above conditions to determine the the variety of tensors of border rank at most 4 in $C^{3\times3\times4}$

Prf. is wrong as Prop. 5.4 wrong. I had to reprove Cor. 5.6

generic subspace spanned by four rank one matrices in $\mathbb{C}^{4\times4}$: $\mathrm{span}(\mathbf{u}_1\mathbf{v}_1^\top,\ldots,\mathbf{u}_4\mathbf{v}_4^\top)$ where any three vectors out of $\mathbf{u}_1,\ldots,\mathbf{u}_4,\mathbf{v}_1,\ldots,\mathbf{v}_4$ linearly independent

$\exists P,Q\in\mathbf{GL}(3,\mathbb{C})$: $P\mathbf{u}_i=Q\mathbf{v}_i=(\delta_{i1},\delta_{i2},\delta_{i3})^\top, i=1,2,3$, $P\mathbf{u}_4=Q\mathbf{v}_4$ $\iff$ $P\mathbf{W}Q\subset \mathrm{S}(3,\mathbb{C})$ $\iff$

$\exists 0\neq S,T\in\mathbb{C}^{3\times3}$ s.t. $S\mathbf{W},\mathbf{W}T\subset\mathrm{S}(3,\mathbb{C})$

16 degree conditions 2

$\mathbf{W} = \mathrm{span}(W_1, \ldots, W_4)$

$$SW_i - W_i^\top S^\top = 0,\ i = 1, \ldots, 4, \quad W_i T - T^\top W_i^\top = 0,\ i = 1, \ldots, 4$$

existence of nontrivial solutions S, T, each system in 9 variables, (entries of) S, T implies that any 9×9 minor of the coefficient matrix of two systems vanishes

16 degree conditions 2

$\mathbf{W} = \mathrm{span}(W_1, \ldots, W_4)$

$$SW_i - W_i^\top S^\top = 0,\ i = 1, \ldots, 4, \quad W_i T - T^\top W_i^\top = 0,\ i = 1, \ldots, 4$$

existence of nontrivial solutions S, T, each system in 9 variables, (entries of) S, T implies that any 9×9 minor of the coefficient matrix of two systems vanishes

generic case: T, S determined uniquely up to nonzero multiplicative scalar, invertible: $ST = TS = tI$.

16 degree conditions 2

$\mathbf{W} = \mathrm{span}(W_1, \ldots, W_4)$

$$SW_i - W_i^\top S^\top = 0,\ i = 1, \ldots, 4, \quad W_i T - T^\top W_i^\top = 0,\ i = 1, \ldots, 4$$

existence of nontrivial solutions S, T, each system in 9 variables, (entries of) S, T implies that any 9×9 minor of the coefficient matrix of two systems vanishes

generic case: T, S determined uniquely up to nonzero multiplicative scalar, invertible: $ST = TS = tI$.

expressing all possible solutions S, T in terms of 8×8 minors of coefficient matrices, the conditions $ST = TS = \lambda I$ are given by vanishing of the corresponding $16 - th$ degree polynomials

Sufficiency of all conditions

If $\mathbf{W} \subset \mathbb{C}^{4\times 4}, \dim \mathbf{W} = 4$ contains an invertible matrix then commutativity conditions $X(\mathrm{adj}\, Z)Y - Y\,\mathrm{adj}(Z)X = 0$ imply that border rank of $\mathcal{T} \in \mathbb{C}^{4\times 4\times 4}$ at most 4.
need to use fact: variety of commuting matrices $(A_1, A_2, A_3) \subset (\mathbb{C}^{3\times 3})^3$ is irreducible [5]

Sufficiency of all conditions

If $\mathbf{W} \subset \mathbb{C}^{4\times 4}, \dim \mathbf{W} = 4$ contains an invertible matrix then commutativity conditions $X(\operatorname{adj} Z)Y - Y \operatorname{adj}(Z)X = 0$ imply that border rank of $\mathcal{T} \in \mathbb{C}^{4\times 4\times 4}$ at most 4.
need to use fact: variety of commuting matrices $(A_1, A_2, A_3) \subset (\mathbb{C}^{3\times 3})^3$ is irreducible [5]

If subspace spanned by each $p = 1, 2, 3$ sections of $\mathcal{T}$ does not contain an invertible matrix then by change of basis in each factor and possibly permute the factors $\mathcal{T} \in \mathbb{C}^{3\times 3\times 4}$.

Sufficiency of all conditions

If $\mathbf{W} \subset \mathbb{C}^{4\times 4}, \dim \mathbf{W} = 4$ contains an invertible matrix then commutativity conditions $X(\mathrm{adj}\, Z)Y - Y\,\mathrm{adj}(Z)X = 0$ imply that border rank of $\mathcal{T} \in \mathbb{C}^{4\times 4\times 4}$ at most 4.
need to use fact: variety of commuting matrices $(A_1, A_2, A_3) \subset (\mathbb{C}^{3\times 3})^3$ is irreducible [5]

If subspace spanned by each $p = 1, 2, 3$ sections of $\mathcal{T}$ does not contain an invertible matrix then by change of basis in each factor and possibly permute the factors $\mathcal{T} \in \mathbb{C}^{3\times 3\times 4}$.

$\mathbf{W} = \mathrm{span}(T_{1,3}, \ldots, T_{4,3}) \subset \mathbb{C}^{3\times 3}$. If $\dim \mathbf{W} \leq 3$ use Strassen's condition

Sufficiency of all conditions

If $\mathbf{W} \subset \mathbb{C}^{4\times 4}, \dim \mathbf{W} = 4$ contains an invertible matrix then commutativity conditions $X(\operatorname{adj} Z)Y - Y \operatorname{adj}(Z)X = 0$ imply that border rank of $\mathcal{T} \in \mathbb{C}^{4\times 4\times 4}$ at most 4.
need to use fact: variety of commuting matrices $(A_1, A_2, A_3) \subset (\mathbb{C}^{3\times 3})^3$ is irreducible [5]

If subspace spanned by each $p = 1, 2, 3$ sections of $\mathcal{T}$ does not contain an invertible matrix then by change of basis in each factor and possibly permute the factors $\mathcal{T} \in \mathbb{C}^{3\times 3\times 4}$.

$\mathbf{W} = \mathrm{span}(T_{1,3}, \ldots, T_{4,3}) \subset \mathbb{C}^{3\times 3}$. If $\dim \mathbf{W} \le 3$ use Strassen's condition

$\dim \mathbf{W} = 4$ use symmetrization condition. If S or T invertible $\mathrm{brank} \mathcal{T} \le 4$.

Sufficiency of all conditions

If $\mathbf{W} \subset \mathbb{C}^{4\times 4}, \dim \mathbf{W} = 4$ contains an invertible matrix then commutativity conditions $X(\mathrm{adj}\, Z)Y - Y\,\mathrm{adj}(Z)X = 0$ imply that border rank of $\mathcal{T} \in \mathbb{C}^{4\times 4\times 4}$ at most 4.
need to use fact: variety of commuting matrices $(A_1, A_2, A_3) \subset (\mathbb{C}^{3\times 3})^3$ is irreducible [5]

If subspace spanned by each $p = 1, 2, 3$ sections of $\mathcal{T}$ does not contain an invertible matrix then by change of basis in each factor and possibly permute the factors $\mathcal{T} \in \mathbb{C}^{3\times 3\times 4}$.

$\mathbf{W} = \mathrm{span}(T_{1,3}, \ldots, T_{4,3}) \subset \mathbb{C}^{3\times 3}$. If $\dim \mathbf{W} \le 3$ use Strassen's condition

$\dim \mathbf{W} = 4$ use symmetrization condition. If S or T invertible $\mathrm{brank}\mathcal{T} \le 4$.

If S, T singular, analyze different cases to show that $\mathrm{brank}\mathcal{T} \le 4$. Some of them use the 16 degree condition

$5, 6, 9$ degree equations suffice: Friedland-Gross

Degree 16 needed in condition A.I.3 to eliminate the case:
R, L rank one and either $R^\top L \neq 0$ or $LR^\top \neq 0$

FG: after change of bases in $\mathbb{C}^3$ frontal section of $\mathcal{T}$ $L = \mathbf{e}_3\mathbf{e}_3^\top$
$R \in \{\mathbf{e}_3\mathbf{e}_3^\top, \mathbf{e}_3\mathbf{e}_2^\top, \mathbf{e}_2\mathbf{e}_3^\top\}$

For $R = \mathbf{e}_3\mathbf{e}_2^\top, \mathbf{e}_2\mathbf{e}_3^\top$ border rank $\mathcal{T} \leq 4$.

For $R = \mathbf{e}_3\mathbf{e}_3^\top$ 4 frontal section of $\mathcal{T}$ are $\begin{bmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{bmatrix}$

$$T_{k,3} = \begin{bmatrix} x_{11,k} & x_{12,k} & 0 \\ x_{21,k} & x_{22,k} & 0 \\ 0 & 0 & x_{33,k} \end{bmatrix} = \mathrm{diag}(X_k, x_{33,k}) \; i = 1, 2, 3, 4$$

10 invariant pol. degree 6: $\det(X_1, X_2, X_3, X_4) x_{33,p} x_{33,q}$ $1 \leq p \leq q \leq 4$

Their vanishing yields bd $\mathcal{T} \leq 4$.

Details from papers

1. [3]: Thm. 4.5
2. [4]: §3
3. [3] §5, §3

References I

- E.S. Allman and J.A. Rhodes, Phylogenic ideals and varieties for general Markov model, *Advances in Appl. Math.*, 40 (2008) 127-148.
- S. Friedland, On the generic rank of 3-tensors, *Linear Algebra Appl.* to appear, arXiv: 0805.3777
- S. Friedland, On tensors of border rank l in $\mathbb{C}^{m\times n\times l}$, *Linear Algebra Appl.* to appear, arXiv:1003.1968
- S. Friedland and E. Gross, A proof of the set-theoretic version of the salmon conjecture, arXiv:1104.1776, *submitted*.
- R.M. Guralnick and B.A. Sethuraman, Commuting pairs and triples of matrices and related varieties *Linear Algebra and its Applications*, 310 (2000), 139-148.

References II

- D.J. Bates and L. Oeding, Toward a salmon conjecture, arXiv:1009.6181.
- J.M. Landsberg and L. Manivel, Generalizations of Strassen's equations for secant varieties of Segre varieties, *Comm. Algebra* 36 (2008), 405–422.
- L. Pachter and B. Sturmfels, *Algebraic Statistics for Computational Biology*, Cambridge University Press, 2005.
- V. Strassen, Rank and optimal computations of generic tensors, *Linear Algebra Appl.* 52/53: 645-685, 1983.