Entropy of algebraic maps

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Abstract

In this paper I give upper bounds for the entropy of algebraic maps in terms of certain homological data induced by their graphs.

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§0. Introduction

Let X be a compact metric space and $\Gamma \subset X \times X$ be a closed set. Set

$$X^\infty = \prod_1^\infty X_i, \ X_i = X, \ i = 1, ...,$$
$$\Gamma^\infty = \{(x_i)_1^\infty, \ (x_i, x_{i+1}) \in \Gamma, \ i = 1, 2, ...\}.$$

Then X^∞, Γ^∞ are compact metric spaces in the Tychonoff topology. Let $\sigma : X^\infty \to X^\infty$ be the shift map. Clearly, $\sigma : \Gamma^\infty \to \Gamma^\infty$. The dynamics of $\sigma\big|_{\Gamma^\infty}$ is the the dynamics induced by the graph Γ. Let $h(\Gamma) = h(\sigma\big|_{\Gamma^\infty})$ be the entropy of σ restricted to Γ^∞. Assume that X is a finite set set. Then $\sigma\big|_\Gamma$ is a subshift of a finite type which is a well studied subject, e.g. [**8**]. In this paper I study the case where X is a compact analytic space of complex dimension n and $\Gamma \subset X \times X$ is a graph of a dominating algebraic function. That is, $\Gamma \subset X \times X$ is a closed irreducible complex subspace of dimension n such that the projection of Γ on the its first and the second component is X. I am mainly concerned with upper estimates of $h(\Gamma)$. Following Gromov [**6**] and Friedland [**3**] I show $h(\Gamma) \le lov(\Gamma)$. Here $lov(\Gamma)$ is the volume growth of the projections of Γ^∞ on the first k coordinates, $k = 2, ...,$. In the case that X is Kähler and Γ is the graph of a holomorphic map $F : X \times X$ I prove rigorously that $h(F) = h(\Gamma(F)) = log\rho(F)$ where $\rho(F)$ is the spectral radius of the induced action of F on the homology groups of X. (A sketchy proof is given in [**3**].) Next I consider finite algebraic maps, i.e. where Γ, X are projective varieties and the projections of Γ on the first and the second factor is finite to one map. I show that Γ induces a linear operator on the analytic homology of X, i.e. the homology groups generated by analytic cycles. Let $\rho_a(\Gamma)$ be the spectral radius of this linear operator. I then show that $lov(\Gamma) \le log\rho_a(\Gamma)$. I conjecture that as in the Kähler case $h(\Gamma) = lov(\Gamma) = log\rho_a(\Gamma)$. In the last section I discuss the case where the projections of Γ on the first and the second coordinates are branched nonfinite to one covering. One still has the inequality $h(\Gamma) \le lov(\Gamma) \le log\rho_a(\Gamma)$. By iterating this inequality one can improve the upper bound on $h(\Gamma)$ as in the case of rational maps $F : X \to X$ discussed in [**2**].

§1. Volumes of certain graphs

Let X be a compact metric space with a metric $d : X \times X \to \mathbf{R}_+$. Then X^∞ is a compact metric space with respect to the metric:

$$\delta((x_i)_1^\infty, (y_i)_1^\infty) = \max_{1 \le i} \frac{d(x_i, y_i)}{2^{i-1}}, \ (x_i)_1^\infty, (y_i)_1^\infty \in X^\infty.$$

Let $\pi_m : X^\infty \to X^m = \prod_1^m X_i$ be the projection on the first m components. Recall that the shift map $\sigma : X^\infty \to X^\infty$ is a continuous map given by $\sigma((x_i)_1^\infty) = (x_i)_2^\infty$. Assume that $\Gamma \subset X \times X$ is an arbitrary closed set. It then follows that Γ^∞ is a compact set such that $\sigma : \Gamma^\infty \to \Gamma^\infty$. In what follows I exclude the noninteresting case $\Gamma^\infty = \emptyset$. Let $h(\Gamma) = h(\sigma, \Gamma^\infty)$ be the entropy of $\sigma\big|\Gamma$. Assume that $F : X \to X$ is a continuous map. Set $\Gamma(F) = \{(x, y) : y = F(x), x \in X\}$ to be the graph of F. It then follows that $h(F)$-the entropy of is equal to $h(\Gamma(F))$ [**2**].

Assume that X is a compact quasi Riemannian manifold. That is, there exists an open finite cover $\mathcal{U} = \cup_1^p U_i, U_i \subset X, i = 1, ..., p$, such that the following conditions hold. Each U_i is a Riemannian manifold which induces a metric $d_i(\cdot,\cdot) : U_i \times U_i \to \mathbf{R}_+$. U_i is locally complete with respect to d_i. On $U_i \cap U_j$ the metrics d_i, d_j are equivalent. That is,

$$d_j(x, y) \le A_{ji} d_i(x, y), d_i(x, y) \le A_{ij} d_j(x, y), \forall x, y \in U_i \cap U_j, 0 < A_{ij}, A_{ji}.$$

Thus X is a compact Riemannian manifold if $p = 1$. For $I = \{i_1, ..., i_k\}, 1 \le i_1 < i_2 < \cdots < i_k \le p$ let $U_I = \cap_{j \in I} U_j$. Assume that $x, y \in X$. Set $I(x, y) = \{j : 1 \le j \le p, x, y \in U_j\}$. Let $I(x) = I(x, x)$. If $I(x, y)$ is nonempty define

$$d(x, y) = \min_{j \in I(x,y)} d_j(x, y).$$

A straightforward argument shows that our assumptions yield that the above metric can be extended to $X \times X$. Note that

$$d(x, y) \le d_i(x, y) \le A d(x, y), x, y \in U_i, A = \max_{1 \le i \ne j \le p} A_{ij}.$$

Let $Y \subset X$. Then, for $\emptyset \ne I \subset \{1, ..., p\}$, set $Y_I = \{y : y \in Y, I(y) = I\}$. Note that Y_I may be empty. Observe that

$$Y_I \cap Y_J = \emptyset \ for \ I \ne J, Y = \cup_{\emptyset \ne I \subset \{1,...,p\}} Y_I.$$

It then follows that the sets $\emptyset \neq Y_I, \emptyset \neq I \subset \{1, ..., p\}$, forms a partition of Y. Consider a nonempty set Y_I. For $i \in I$ let $dim_i(Y_I)$ be the Hausdorff dimension of Y_I with respect to the Riemannian metric on U_i. As on U_I all the Riemannian metrics are equivalent we have that $dim_i(Y_I) = t, i \in I$. Thus, $dim(Y_I) = t$ is the Hausdorff dimension of Y_I. Let $d \geq dim(Y_I)$. For $d > dim(Y_I)$ let $vol_d(Y_I) = 0$. For $d = dim(Y_I)$ denote by $vol_d^{(i)}(Y_I)$ the d-volume of Y_I with respect to the Riemannian metric on U_i. Set

$$vol_d(Y_I) = min_{i \in I} vol_d^{(i)}(Y_I).$$

Define $dim(Y) = max_{Y_I \neq \emptyset} dim(Y_I)$. Assume that $d = dim(Y)$. Then the volume of Y is given by

$$vol(Y) = \sum_{Y_I \neq \emptyset} vol_d(Y_I).$$

Consider the space X^k. Clearly, X^k is a quasi Riemannian manifold with an open induced by the product $\mathcal{U}^p = \mathcal{U} \times \cdots \times \mathcal{U}$. Each element of the open cover $U_{j_1} \times \cdots \times U_{j_k}, 1 \leq j_i \leq p, i = 1, ..., k$, is a Riemannian manifold endowed with the Riemannian product metric. Let $B_k(a, r) \subset X^k$ be an open ball of radius r centered at a with respect to the induced metric on X^k by X:

$$B_k(a, r) = \{x, \ x = (x_i)_1^k, a = (a_i)_1^k \in X^k, \ \sum_1^k d(x_i, a_i)^2 < r^2.\}$$

In what follows I assume that $\Gamma \subset X \times X$ is a closed set with an integer Hausdorff dimension $n > 0$. Let $vol(\Gamma^k) \leq \infty$ be the n dimensional volume of Γ^k. I shall assume:

$$vol(\Gamma^k) < \infty, \ k = 2, ...,.$$

This assumption imply that Γ^k has Hausdorff dimension n for $k = 2, ...,$. Set

$$\begin{aligned}
lov(\Gamma) &= \limsup_{k \to \infty} \frac{log\ vol(\Gamma^k)}{k}, \\
Dens_\epsilon(\Gamma^k) &= \inf_{a \in \Gamma^k} vol(\Gamma^k \cap B_k(a, \epsilon)), \\
lodn_\epsilon(\Gamma) &= \liminf_{k \to \infty} \frac{log\ Dens_\epsilon(\Gamma^k)}{k}, \\
lodn(\Gamma) &= \lim_{\epsilon \to 0} lodn_\epsilon(\Gamma).
\end{aligned}$$

Lemma 1.1 *Let X be a compact quasi Riemannian manifold, $\Gamma \subset X \times X$ a closed set of integer Hausdorff dimension n satisfying $vol(\Gamma^k) < \infty, k = 2, ...,$. Then*

$$h(\Gamma) \leq lov(\Gamma) - lodn(\Gamma).$$

Proof. Let

$$\delta_j(\xi,\eta) = \max_{0\le l\le j-1} \delta(\sigma^{\circ l}(\xi), \sigma^{\circ l}(\eta)) =$$

$$\max_{1\le i} \frac{d(x_i,y_i)}{2^{(i-j)^+}},\ \xi = (x_i)_1^\infty, \eta = (y_i)_1^\infty \in X^\infty, j = 1, \dots .$$

Here, $a^+ = max(a,0),\ a \in \mathbf{R}$. Fix $\epsilon > 0$. Let $L(k,\epsilon,\Gamma^\infty)$ be the maximal size of (k,ϵ) separated set in Γ^∞. That is for any finite set $E \subset \Gamma^\infty$ with the property $\xi,\eta \in E,\ \xi \neq \eta \Rightarrow \delta_k(\xi,\eta) > \epsilon$ we have the inequality $Card(E) \le L(k,\epsilon,\Gamma^\infty)$. Furthermore, the equality sign holds for at least one such a set E. The standard definition of $h(\sigma,\Gamma)$ is [**8**, Ch.7]:

$$h(\sigma,\Gamma) = \lim_{\epsilon\to 0} \limsup_{k\to\infty} \frac{log\ L(k,\epsilon,\Gamma^\infty)}{k}.$$

Let $E(k,\epsilon,\Gamma^\infty)$ be a (k,ϵ) separated set of cardinality $L(k,\epsilon,\Gamma^\infty)$. It then follows that

$$\max_{1\le i\le k+K(\epsilon)} d(x_i,y_i) > \epsilon,\ \xi = (x_i)_1^\infty \neq \eta = (y_i)_1^\infty \in E(k,\epsilon,\Gamma^\infty),\ K(\epsilon) = \lceil log_2 D - log_2 \epsilon \rceil.$$

Here D is the diameter of X. In particular the $L(k,\epsilon,\Gamma^\infty)$ balls

$$B_{k+K(\epsilon)}(\pi_{k+K(\epsilon)}(\xi), \frac{\epsilon}{2}),\ \xi \in E(k,\epsilon,\Gamma^\infty)$$

are disjoint. Hence:

$$vol(\Gamma^{k+K(\epsilon)}) \ge L(k,\epsilon,\Gamma^\infty) Dens_{\frac{\epsilon}{2}}(\Gamma^{k+K(\epsilon)}).$$

Take the logarithm of this inequality, divide by $k + K(\epsilon)$, take limsup of the both sides of this inequality and let ϵ tend to zero to deduce the lemma. $\diamond$

For a compact Riemannian manifold this lemma is due to Gromov [**Gro**]. Our proof is the proof given in [**3**].

Let M be a complex Kähler manifold with corresponding $(1,1)$ form ω induced by the Hermitian metric $d\rho^2$. Assume that $X \subset M$ be an irreducible analytic variety of dimension d. If X is smooth then X is Kähler whose $(1,1)$ form ω' and the corresponding Hermitian metric $d\rho'^2$ are the restriction of ω and $d\rho^2$ respectively. Let $Sing(X)$ be the set of singular points of X. Then $X\backslash Sing(X)$ is Kähler with $(1,1)$ form ω' and $d\rho'^2$ its Hermitian metric. Since $Sing(X)$ is a proper subvariety of X it folows that for all purposes needed here X behaves as Riemannian manifold. First note that the induced metric $d : X \times X \to \mathbf{R}_+$ by the metric $d\rho^2$ in M is the metric induced by $d\rho'^2$ in $X\backslash Sing(X)$ obtained by the completion of this metric to X. Consider next the following stratification of X

$$X_0 = X, X_i = Sing(X_{i-1}), i = 1,...,k, X_k \neq \emptyset, Sing(X_k) = \emptyset, X = \cup_0^k X_i \backslash Sing(X_i).$$

Thus, each $X_i\backslash Sing(X_i)$ is Kähler and has the corresponding $(1,1)$ form ω_i and the Hermitian metric $d\rho_i^2$ which are the restrictions of ω and $d\rho^2$ respectively to $X_i\backslash Sing(X_i)$.

By the abuse of notation I consider ω_i and $d\rho_i^2$ as the restrictions of ω' and $d\rho'^2$. The following lemma is needed in the sequel.

Lemma 1.2. *Let M be a Kähler manifold, $X \subset M$ be an irreducible analytic subvariety. Then, for any positive integer $n, n \leq dim(X)$ there exists a constant $C(n, X)$ so that the following condition is satisfied. Let $\Gamma \subset X \times X$ be an irreducible analytic subvariety such that $\Gamma = \Gamma^2, \Gamma^k, k = 3, ...,$ have all complex dimension n . Then*

$$vol(\Gamma^k \cap B_k(a, \epsilon)) \geq C(n, X)\epsilon^{2n}, k = 2, ...,.$$

Proof. Assume first that X is smooth. Clearly, Γ^k is an irreducible analytic subvariety of (complex) dimension n. According to [**1**, Sec. 5.4.19] the above inequality holds. If X is not smooth we trivially have that $\Gamma^k \subset M^k$ and the above inequality still holds. $\diamond$

Let $X, \mathcal{O}$ be a compact analytic space. Consult with [**4**] for the properties of complex spaces and with [**5**] for the properties of complex manifolds and projective varieties needed here. Then one has a finite cover $\mathcal{U} = \cup_1^p U_i$ of X such that each $(U_i, \mathcal{O}_i), \mathcal{O}_i = \mathcal{O}|\mathcal{U}_i$ is isomorphic to the model complex space $(\tilde{U}_i, \tilde{O}_i)$ which is the sheaf of holomorphic functions over a complex variety $U_i \subset \mathbf{C}^{n_i}$. (This definition of a complex space is Serre's definition which is not the most general definition, i.e. [**4**, p'13].) For simplicity of notation I will suppress the reference to the sheaf of a complex space and no ambiguity will arise. I shall assume that X is irreducible, i.e. $X\backslash Sing(X)$ is connected. As explained above one can view each $\tilde{U}_i \subset \mathbf{C}^{n_i}$ as a Riemannian manifold. It then follows that one can view X as a quasi Riemannian manifold. (To see that consider a cover $\hat{\mathcal{U}} = \cup_1^p \hat{U}_i, Closure(\hat{U}_i) \subset U_i, i = 1, ..., p$.) Hence, it is possible to aplly all the results obtained so far. Recall that $Y \subset X$ is a complex subspace of X if $\tilde{Y}_i$ - the isomorphic image of $Y \cap U_i$ in $\tilde{U}_i$ is an analytic subvariety of $\tilde{U}_i$. Let $\Gamma \subset X \times X$ be a complex irreducible subspace of dimension n. Define the quantities

$$vol(\Gamma^k), lov(\Gamma), Dens_\epsilon(\Gamma^k), lodn_\epsilon(\Gamma), lodn(\Gamma)$$

as above. Combine Lemma 1.1 and Lemma 1.2 to deduce

Theorem 1.3. *Let X be a compact complex irreducible space. Assume that $\Gamma \subset X \times X$ is a compact complex irreducible subspace. Then $lodn(\Gamma) \geq 0$. Hence $h(\Gamma) \leq lov(\Gamma)$.*

In the case X is Kähler the above theorem is due to Gromov [**6**]. Assume that the assumptions of Theorem 1.3 hold. Suppose furthermore that

$$dim(\Gamma) = dim(X), \pi_1(\Gamma) = \pi_2(\Gamma) = X.$$

Then I view $\Gamma \subset X \times X$ as a graph of an algebraic function. Indeed, the projections $\pi_i : \Gamma \to X, i = 1, 2,$ are branched covers of degree $d_i, i = 1, 2$. That is, there exists a complex subspace $Y_i \subset X$ such that $\pi_i : X\backslash\pi_i^{-1}(Y_i) \to X\backslash Y_i$ is d_i covering for $i = 1, 2$.

§2. Entropy of holomorphic selfmaps of a compact Kähler manifold

Let X be a compact Kähler manifold and let ω be the corresponding closed $(1,1)$ form of X. Denote by

$$H_*(X,\mathbf{F}) = \sum_0^{2n} \oplus H_i(X,\mathbf{F}), H^*(X,\mathbf{F}) = \sum_0^{2n} \oplus H^i(X,\mathbf{F})$$

be the total homology and cohomology groups of X over a field $\mathbf{F} = \mathbf{Z}, \mathbf{Q}, \mathbf{R}$. Let $[X]$ the fundamental class of X, i.e. the generator of the one dimensional free group $H_{2n}(X,\mathbf{Z})$. Assume that $F : X \to X$ is a holomorphic map. Then

$$F_* : H_*(X,\mathbf{F}) \to H_*(X,\mathbf{F}), F^* : H^*(X,\mathbf{F}) \to H^*(X,\mathbf{F})$$

be the linear operators induced by F. I assume that

$$F_* = Id : H_0(X,\mathbf{F}) \to H_0(X,\mathbf{R}).$$

Let $\rho(F)$ be the spectral radius of $F_*(F^*)$ for $\mathbf{F} = \mathbf{R}$. The above assumptions yield that $\rho(F) \ge 1$. Set $\phi_m = (F^{\circ m})^* \omega,\ m = 0, 1, ...,$ to be the the pull back of ω by the $F^{\circ m}$.

Theorem 2.1. *Let X be a compact Kähler manifold of complex dimension n and assume that $F : X \to X$ is a holomorphic map. Then*

$$lov(\Gamma(F)) = \limsup_{j\to\infty} \frac{log|(\sum_{i=0}^{i=j-1} \phi_i)^n([X])|}{j}. \tag{2.2}$$

Moreover

$$lov(\Gamma(F)) = h(F) = log\rho(F).$$

Proof. Let ω_k be the induced $(1,1)$ form on X^k. Set $\Gamma = \Gamma(F)$. Then

$$\Gamma^k = \{(x, F(x), ..., F^{\circ(k-1)}(x)) : x \in X\}.$$

Denote by θ_k the restriction of ω_k to Γ^k. Hence, in terms of the variable x, the restriction of θ_k to the $j-th$ coordinate of Γ^k is ϕ_{j-1} - the pull back of $\omega = \phi_0$ by $F^{\circ j-1}$. Thus

$$\theta_k(x) = \sum_{j=0}^{k-1} \phi_j(x),\ x \in X,\ k = 0, 1, ...,. \tag{2.3}$$

So $vol(\Gamma^k) = \frac{1}{n!}\theta_k^n([X])$. We now prove the inequality $lov(F) \le log\rho(F)$. Clearly

$$\theta_k^n([X]) \le k^n \max_{0\le m_1\le m_2\le\cdots\le m_n<k} |\phi_{m_1}\phi_{m_2}\cdots\phi_{m_n}[X]|.$$

Let $\|\cdot\|_j$ be a norm on $H^j(X,\mathbf{R})$ and denote $\|F^*\|_j$ the induced norm of the operator $F^*: H^j(X,\mathbf{R}) \to H^j(X,\mathbf{R})$ for $j = 1,...,2n$. It then follows

$$\|\phi_{m_j}\cdots\phi_{m_n}\|_{2(n-j+1)} = \|(F^*)^{m_j}(\phi_0\cdots\phi_{m_n-m_j})\|_{2(n-j+1)} \leq$$
$$\|(F^*)^{m_j}\|_{2(n-j+1)}\|\phi_0\cdots\phi_{m_n-m_j}\|_{2(n-j+1)}, m_j \leq m_p, p = j+1,...,n.$$

Clearly, there exists a constant K_j depending only on the norms $\|\cdot\|_i, i = 2, 2(j-1), 2j$ so that

$$\|xy\|_{2j} \leq K_j\|x\|_2\|y\|_{2(j-1)}, x \in H^2(X,\mathbf{R}), y \in H^{2(j-1)}(X,\mathbf{R})$$

for $j = 2,...,n$. The above inequalites yield

$$|\phi_{m_1}\cdots\phi_{m_n}[X]| \leq K\prod_{i=1}^{n}\|(F^*)^{m_i-m_{i-1}}\|_{2(n-i+1)}, m_0 = 0 \leq m_1 \leq \cdots \leq m_n < k$$

for some fixed K. Let $\rho_i(F)$ be the spectral radius of $F^*: H^i(X,\mathbf{R}) \to H^i(X,\mathbf{R})$ for $i = 0,...,2n$. Note that $\rho(F) = \max_{0\leq i\leq 2n}\rho_i(F)$. Observe next the that for any $\epsilon > 0$ there exists $\kappa(\epsilon)$ so that

$$\|(F^*)^m\|_i \leq \kappa(\epsilon)(\rho(F)+\epsilon)^m, m = 0,1,..., i = 1,...,2n.$$

Combine all the above inequalities with (2.2) to get the inequality $lov(\Gamma(F)) \leq log(\rho(F)+\epsilon)$. As $\epsilon > 0$ was arbitrary small we deduce that $lov(\Gamma(F)) \leq log\rho(F)$. Combine this inequality with Theorem 1.3 to deduce that $h(F) \leq lov(\Gamma(F)) \leq log\rho(F)$. Yomdin's inequality $h(F) \geq log\rho(F)$ [**9**] yields the equality $h(F) = lov(\Gamma(F) = log\rho(F)$. $\diamond$.

A sketchy proof of Theorem 2.1 was given in [**3**]. Assume that $X \subset M$ is an irreducible complex subvariety of dimension n in a compact Kähler manifold M. Let $F: X \to X$ be a continuous map so that the graph $\Gamma(F) \subset X \times X$ is an irreducible complex variety of dimension n. Let $\rho(F)$ be the spectral radius of $F^*: H^*(X,\mathbf{R}) \to H^*(X,\mathbf{R})$. We then can apply all the arguments of Theorem 2.1 except Yomdin's theorem. Hence, we deduce

Theorem 2.4. *Let M be a Kähler manifold and $X \subset M$ be a complex irreducible variety. Assume that $F: X \to X$ be a continuous map such that $\Gamma(F) \subset X \times X$ is a complex subvariety. Then*

$$h(F) \leq lov(\Gamma(F)) \leq log\rho(F).$$

In [**3**] I proved the above theorem in the case that X is a projective variety and F is a continuous rational map. If in addition F is a regular rational map then $h(F) = log\rho(F)$.

§3. Upper bounds on the entropy of finite algebraic maps

Let $\mathbf{CP}^N$ be the N dimenisonal complex projective space and $\Gamma \subset \mathbf{CP}^N \times \mathbf{CP}^N$ be an irreducible subvariety. Denote by $\pi_i'(\Gamma^\infty)$ the projection of Γ^∞ on the $i-th$ component. Clearly, $\pi_i'(\Gamma^\infty) \supset \pi_{i+1}'(\Gamma^\infty), i = 2, ...,$. Hence, $\pi_i'(\Gamma^\infty) = X, i = k, k+1, ...,$ for some $k \geq 1$. Here X is an irreducible subvariety of $\mathbf{CP}^N$. Let $\Gamma_1 = \Gamma \cap X \times X$. It then follows that Γ_1 is an irreducible subvariety and

$$h(\Gamma) = h(\sigma\big|_{\Gamma^\infty}) = h(\sigma\big|_{\Gamma_1^\infty}) = h(\Gamma_1).$$

Since I am interested in $h(\Gamma)$ in what follows I assume that $\pi_1(\Gamma) = \pi_2(\Gamma) = X$ and the complex dimension of X and Γ is n. In order to use Theorem 1.3 one needs to estimate $vol(\Gamma^k)$. For that purpose it is convenient to view Γ^k as a subvariety of $(\mathbf{CP}^N)^k$.

Let $U \subset \mathbf{CP}^N$ be an irreducible variety of dimension d. Then $vol(U) = deg(U)$ is the number of intersection points of the zero dimensional variety $U \cap H^d$. Here $H^j \subset \mathbf{CP}^N$ is a hyperplane of codimension j in general position for $j = 0, ..., n$. Thus $vol(U) = [U] \cdot [H^d]$. As H^d is an intersection of d H^1 in general position we have also the formula $vol(U) = [U] \cdot [H^1] \cdots [H^1]$. Let $1 \leq k, 1 \leq i \leq k$ be given. Set

$$H^{i,k} = \mathbf{CP}^N \times \cdots \times \mathbf{CP}^N \times H^1 \times \mathbf{CP}^N \times \cdots \mathbf{CP}^N \subset (\mathbf{CP}^N)^k$$

to be a codimension 1 variety with the factor H^1 on the $i-th$ component in general position. Let $U \subset (\mathbf{CP}^N)^k$ be an irreducible variety of dimension d. For $1 \leq i_1 \leq i_2 \leq \cdots \leq i_d \leq k$ let $[U] \cdot [H^{i_1,k}] \cdots [H^{i_d,k}]$ be the number of points in the intersection $U \cap H^{i_1,k} \cap \cdots \cap H^{i_d,k}$. This number can be zero. For example, if some number j appears more than N times in the sequence $i_1, ..., i_d$ then the above intersection is empty since $H^{i_1,k} \cap \cdots \cap H^{i_d,k} = \emptyset$.

Lemma 3.1. *Let $U \subset (\mathbf{CP}^N)^k$ be an irreducible variety of dimension d. Then*

$$vol(U) = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_d \leq k} [U] \cdot [H^{i_1,k}] \cdots [H^{i_d,k}].$$

Proof. Assume first that $U = U_1 \times U_2 \times \cdots U_k$ where $U_i \subset \mathbf{CP}^N$ is an irreducible variety of dimension d_i for $i = 1, ..., k$. Then $vol(U) = vol(U_1) \cdots vol(U_k)$. A straightforward computation shows that the lemma holds in this case. I claim that this simple case implies the lemma in general. Indeed, recall that

$$H_{2j}(\mathbf{CP}^N, \mathbf{Z}) \sim \mathbf{Z}, j = 0, ..., N, H_{2j-1}(\mathbf{CP}^N, \mathbf{Z}) = 0, j = 1, ..., N.$$

Now use the standard product formula for $H_*((\mathbf{CP}^N)^k, \mathbf{Z})$ to deduce that any any analytic cycle in $H_{2d}((\mathbf{CP}^N)^k, \mathbf{Z})$ is a sum of cycles of the form $U_1 \times \cdots U_k$. $\diamond$

Corollary 3.2. *Let $\Gamma \subset \mathbf{CP}^N \times \mathbf{CP}^N$ an irreducible complex variety so that Γ^k is an irreducible variety of dimension $n \leq N$ for $k = 2, ...,$. Then*

$$vol(\Gamma^k) = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_n \leq k} [\Gamma^k] \cdot [H^{i_1,k}] \cdots [H^{i_n,k}].$$

Theorem 3.3. *Let* $\Gamma \subset \mathbf{CP}^1 \times \mathbf{CP}^1$ *be an irreducible curve whose projection on the first and second coordinate gives* $\mathbf{CP}^1$. *Assume that in some chart* $\mathbf{C}^2 \subset \mathbf{CP}^1 \times \mathbf{CP}^1$ *the curve* Γ *is given by* $p(x,y) = 0, (x,y) \in \mathbf{C}^2$, *where* $p(x,y)$ *is an irreducible polynomial depending explicitly on* x *and* y. *Then*

$$h(\Gamma) \le lov(\Gamma) = log \max(deg_x(p), deg_y(p)).$$

Proof. Let $d_1 = deg_y(p), d_2 = deg_x(p)$. Thus, the projection of $\tau_i : \Gamma \to \mathbf{CP}^1$ on the $i-th$ coordinate is d_i branched covering for $i = 1, 2$. Next observe that $\Gamma^k \cap H^{i,k}$ means that we specify the $i-th$ coordinate of Γ^k. Then we have d_1, d_2 possible choices for the coordinate $i+1, i-1$ respectively for Γ^k. Continuing in this manner one deduces that $[\Gamma^k] \cdot [H^{i,k}] = d_1^{k-i} d_2^{i-1}$. Thus,

$$vol(\Gamma^k) = \sum_{i=1}^{k} d_1^{k-i} d_2^{i-1}.$$

In particular,

$$(\max(d_1, d_2))^{k-1} < vol(\Gamma^k) \le k(\max(d_1, d_2))^{k-1}$$

and the equality for $lov(\Gamma)$ is established. Use Theorem 1.3. to complete the proof of the theorem. $\diamond$

I conjecture that under the assumptions of Theorem 3.3 the equality $h(\Gamma) = lov(\Gamma)$ holds. I now show how to generalize Theorem 3.3 to proper graphs Γ.

Definition 3.4. *Let* $\Gamma \subset \mathbf{CP}^N \times \mathbf{CP}^N$ *be an irreducible variety of dimension* n. *Then* Γ *is called proper if the following conditions hold. There exist an irreducible smooth variety* $X \subset \mathbf{CP}^N$ *of dimension* n *so that the projections* $\tau_i : \Gamma \to X$ *on the* $i-th$ *component of* $\mathbf{CP}^N \times \mathbf{CP}^N$ *is finite to one branched covering of degree* d_i *for* $i = 1, 2$.

Note that Γ satisfying the assumptions of Theorem 3.3 is proper. Assume the assumptions of Definition 3.4. I call $\Gamma \subset X \times X$ a graph of a finite algebraic function. As X is triangulable, e.g. [**7**], it follows that X is a finite CW complex. As in the previous section I denote by $H_*(X, \mathbf{F}), H^*(X, \mathbf{F}), \mathbf{F} = \mathbf{Z}, \mathbf{Q}, \mathbf{R}$ the total homology and cohomology groups of X. Let $H_{2j,a}(X, \mathbf{F}) \subset H_{2j}(X, \mathbf{F}), j = 0, ..., n$, be the subgroup generated by all varieties $Y \subset X$ of complex dimension j. Let $\mathbf{R}_+$ be the semiring of nonnegative reals. For the one of the above rings $\mathbf{F}$ set $\mathbf{F}_+ = \mathbf{F} \cap \mathbf{R}_+$ to be the corresponding semirings. Let $K_{2j,a}(X, \mathbf{F}_+)$ be the cone generated by the subvarieties of complex dimension j with coefficients in $\mathbf{F}_+$. Thus

$$H_{2j,a}(X, \mathbf{F}) = K_{2j,a}(X, \mathbf{F}_+) - K_{2j,a}(X, \mathbf{F}_+), j = 0, ..., n, H_{*,a}(X, \mathbf{F}) = \sum_{j=0}^{n} \oplus H_{2j,a}(X, \mathbf{F}).$$

I now show that Γ induces a linear operator $\Gamma^* : H_{*,a}(X, \mathbf{F}) \to H_{*,a}(X, \mathbf{F})$. More precisely Γ^* is positive with respect to the cone $K_{*,a}(X, \mathbf{F}_+)$. That is

$$\Gamma^* : K_{2j,a}(X, \mathbf{F}_+) \to K_{2j,a}(X, \mathbf{F}_+), j = 0, ..., n.$$

Let $V \subset X$ be an irreducible subvariety of dimension j. Then $\tau_2(\tau_1^{-1}(V)) \subset X$ is a variety whose each irreducible component is of dimension j. Set

$$\Gamma^*([V]) = [\tau_2(\tau_1^{-1}(V)].$$

It is straightforward to show that Γ^* is linear. Let $\rho_{2j,a}(\Gamma)$ be the spectral radius of $\Gamma^* : H_{2j,a}(X, \mathbf{R}) \to H_{2j,a}(X, \mathbf{R}), j = 0, ..., n$. Note that

$$\rho_{0,a}(\Gamma) = d_1, \rho_{2n,a}(\Gamma) = d_2.$$

Set $\rho_a(\Gamma) = \max_{0 \le j \le n} \rho_{2j,a}(\Gamma)$. Finally I define $L : H_{2j,a}(X, \mathbf{F}) \to H_{2j-2,a}(X, \mathbf{F})$ to be the Lefschetz map which is induced by the hyperplane section. That is, let $V \subset X$ be an irreducible variety of dimension j. Then $L([V]) = [V \cap H^1]$. Note that L is positive with respect to the cone $K_{*,a}(X, \mathbf{F}_+)$.

Theorem 3.5. *Let $\Gamma \subset \mathbf{CP}^N \times \mathbf{CP}^N$ be a proper irreducible variety. Then*

$$lov(\Gamma) \le log \rho_a(\Gamma).$$

Proof. Assume the notations of Definition 3.4. I claim that

$$[\Gamma^k] \cdot [H^{i_1,k}] \cdots [H^{i_n,k}] =$$
$$d_2^{i_1-1} d_1^{k-i_n} L(\Gamma^*)^{i_n - i_{n-1}} \cdots L(\Gamma^*)^{i_3-i_2} L(\Gamma^*)^{i_2-i_1}([X \cap H^1]), 1 \le< i_1 < i_2 < \cdots < i_n \le k.$$

Indeed, the above formula without the factor $d_2^{i_1-1} d_1^{k-i_n}$ determines the number of points when we project this intersection on the components $i_1, ..., i_n$. As all the hyperplanes are in general positions this is exactly the number of distinct points of the above intersection when we project it on the $i_n - i_1 + 1$ consequitive components $i_1, i_1 + 1, ..., i_n$. When we advance from the component i_n to the $k - th$ component we pick the factor $d_1^{k-i_n}$. When we decrease from the $i_1 - th$ component to the first component we pick up the factor $d_2^{i_1-1}$. This proves the above formula for the distinct $i_1, ..., i_n$. Similar formulas hold if some indices coincide. As in the proof of Theorem 2.1 introduce norms on the spaces $H_{2j,a}(X, \mathbf{R}), j = 0, ..., n$. The arguments given in the proof of Theorem 2.1 yield that for any $\epsilon > 0$ there exists $\kappa(\epsilon)$ so that

$$[\Gamma^k] \cdot [H^{i_1,k}] \cdots [H^{i_n,k}] \le \kappa(\epsilon)(\rho_a(\Gamma) + \epsilon)^k.$$

Hence

$$vol(\Gamma^k) \le k^n \kappa(\epsilon)(\rho_a(\Gamma) + \epsilon)^k$$

and the theorem follows. $\diamond$

I conjecture that under the assumptions of Theorem 3.5

$$h(\Gamma) = lov(\Gamma) = log\rho_a(\Gamma).$$

§4. Upper bounds on the entropy of nonfinite algebraic maps

In this section I assume that $\Gamma \subset \mathbf{CP}^N \times \mathbf{CP}^N$ is an irreducible variety of dimension n so that there exists an irreducible variety $X \subset \mathbf{CP}^N$ of dimension n such that $\tau_i : \Gamma \to X$ on the $i-th$ component of $\mathbf{CP}^N \times \mathbf{CP}^N$ is a branched convering of degree d_i for $i = 1, 2$. I call Γ the graph of an algebraic function in X. Assume first that τ_2 is finite to one. Then the linear operator $\Gamma^* : H_{*,a}(X, \mathbf{F}_+) \to H_{*,a}(X, \mathbf{F}_+)$ is well defined and it is straightforward to show that Theorem 3.5 applies in this case. Assume now that τ_1 is finite to one. Then one can define $\tilde{\Gamma}^* : H_{*,a}(X, \mathbf{F}_+) \to H_{*,a}(X, \mathbf{F}_+)$ by pushing from the second factor of $X \times X$ to the first. Let $\tilde{\rho}_a(\Gamma)$ be the spectral radius of $\tilde{\Gamma}^*$. It then follows that one has an analogous inequality

$$lov(\Gamma) \leq log\tilde{\rho}_a(\Gamma).$$

It is not hard to show (by pulling back) that if τ_1, τ_2 are finite to one then $\tilde{\rho}_a(\Gamma) = \rho_a(\Gamma)$. In what follows I assume that neither τ_1 nor τ_2 are finite to one branched covering.

It is still possible to define $\Gamma^* : K_{2j,a}(\mathbf{F}_+) \to K_{2j,a}(\mathbf{F}_+) \to$ by pushing forward varieties $V \subset X$ in general position. More precisely, assume that there exist subvarieties $S_1, S_2 \subset X$ so that

$$\tau_i : \Gamma \backslash \tau_i^{-1}(S_i) \to X \backslash S_i$$

are d_i covering for $i = 1, 2$. Let $V \subset X, V \backslash S_1 \neq \emptyset$ be an irreducible variety of dimension j. I say that V is in general position with respect to S_1. It then follows that $V' = Closure(\tau_2(\tau_1^{-1}(V \backslash S_1)))$ is a subvariety whose each irreducible component is of dimension j. I then let $\Gamma^*([V]) = [V']$. Note that Γ^* is a linear functional on the subcone $K' \subset K_{2j,a}(\mathbf{F}_+)$ generated by all V which are in general position with respect S_1. V is said to be a special irreducible variety with respect to S_1 if the homology class $[V]$ is not contained in the cone K'. I let $\Gamma^*([V]) = 0$ for all special irreducible varieties with respect to S_1. This defines Γ^* on $H_{*,a}(X, \mathbf{F})$. Let $\rho_{2j,a}(\Gamma), j = 0, ..., n, \rho_a(\Gamma)$ be defined as in the previous section. For $k \geq 1$ define $\Gamma_k \subset X \times X$ to be the graph obtained by projecting Γ^{k+1} on the first and the last coordinate. (Note that $\Gamma_1 = \Gamma = \Gamma^2$.) Let $\Gamma_k^* : H_{*,a}(X, \mathbf{F}) \to H_{*,a}(X, \mathbf{F})$ be defined as above. I claim that

$$\Gamma_k^* \leq (\Gamma^*)^k, k = 2, ...,$$

where the inequalities are with respect to the cone $K_{*,a}(X, \mathbf{R}_+)$. This follows from the fact that Γ_k^* picks up more special irreducible varieties on which Γ_k^* vanishes. See more detailed discussion on this matter in [**2**]. The same argument yields

$$\Gamma_{p+k}^* \leq \Gamma_p^* \Gamma_k^*, p, k = 1, 2, ..., . \tag{4.1}$$

In particular

$$\rho(\Gamma_{pk}^*) \le \rho(\Gamma_p^*)^k, p, k = 1, 2, ...,. \tag{4.2}$$

Apply the arguments of the proof of Theorem 3.5 to deduce.

Theorem 4.3. *Let $\Gamma \subset X \times X, X \subset \mathbf{CP}^N$ be a graph of an algebraic function on X. Then*

$$vol(\Gamma^\infty) \le log \rho_a(\Gamma).$$

Let $\sigma : \Gamma^\infty \to \Gamma^\infty$ be the shift map. Then $\sigma^k : \Gamma^\infty \to \Gamma^\infty$ splits to k copies of the shift map applied to the graph Γ_k^∞. Therefore

$$h(\sigma^k | \Gamma^\infty) = kh(\sigma | \Gamma^\infty) = h(\sigma | \Gamma_k^\infty).$$

See for example [**8**]. Observe next that $\rho_a(\Gamma_k) = \rho(\Gamma_k^*)$. Combine (4.2) with Theorems 4.3 and 1.3 to deduce

Corollary 4.4. *Let the assumptions of Theorem 4.2 hold. Then*

$$h(\Gamma) \le \liminf_{k \to \infty} \frac{log \rho_a(\Gamma_k)}{k}.$$

Actually, the inequality (4.1) yields that lim inf can be replaced by lim. I conjecture that $h(\Gamma)$ is equal to the lim inf.

I close this section with another estimate on $lov(\Gamma)$. Assume that X and Γ are contained in the following complete intersections

$$X \subset \tilde{X} = \{x : x \in \mathbf{C}^{N+1}, f_i(x) = 0, i = 1, ..., N-n\}, \Gamma \subset \tilde{\Gamma} =$$
$$\{(x,y) : (x,y) \in \mathbf{C}^{N+1}, f_i(x) = f_i(y) = 0, i = 1, ..., N-n, g_j(x,y) = 0, j = 1, ..., n\}. \tag{4.5}$$

I assume that $f_1(x), ..., f_{N-n}(x)$ are homogeneous polynomials in x and $g_1(x,y), ..., g_n(x,y)$ are bihomogeneous polynomials in (x,y). Note that if $X = \mathbf{CP}^N$ then $\tilde{\Gamma}$ is given only by the polynomials $g_1, ..., g_N$. Let $f(x), g(x,y)$ be arbitrary polynomials in the variables $x, y \in \mathbf{C}^k$. Then

$$deg(f), deg_x(g), deg_y(g), deg(g) = \max(deg_x(g), deg_y(g))$$

are the corresponding degrees the above polynomials.

Theorem 4.6. *Let $\Gamma \subset X \times X, X \subset \mathbf{CP}^N$ be a graph of an algebraic function on X. Assume that X, Γ are contained in the complete intersections given in* (4.5). *Then*

$$lov(\Gamma) \le \sum_{i=1}^{N-n} log deg(f_i) + \sum_{j=1}^{n} log deg(g_j).$$

Proof. Note that Γ^k are contained in the complete intersection given by

$f_i(x^p) = 0, g_j(x^q, x^{q+1}), x^p \in \mathbf{C}^{N+1}, p = 1, ..., k, q = 1, ..., k-1, i = 1, ..., N-n, j = 1, ..., n.$

Observe next that each $H^{i,k}$ is given by one linear equation. Bezout theorem yields that

$$[\Gamma^k] \cdot [H^{i_1,k}] \cdots [H^{i_n,k}] \le \big(\prod_{i=1}^{N-n} deg(f_i)\big)^k \big(\prod_{j=1}^{n} deg(g_j)\big)^{k-1}.$$

Hence, $vol(\Gamma^k)$ is at most k^n times the right-hand side of the above inequality. The proof of the theorem is completed.

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