§0. Introduction

Let $X$ be a compact metric space and $T : X \to X$ is continuous transformation. Then the dynamics of $T$ is a widely studied subject. In particular, $h(T)$ - the entropy of $T$ is a well understood object. Let $\Gamma \subset X \times X$ be a closed set. Then $\Gamma$ induces certain dynamics and entropy $h(\Gamma)$. If $X$ is a finite set then $\Gamma$ can be naturally viewed as a directed graph. That is, if $X = \{1, ..., n\}$ then $\Gamma$ consists of all directed arcs $i \to j$ so that $(i, j) \in \Gamma$. Then $\Gamma$ induces a subshift of finite type which is a widely studied subject. However, in the case that $X$ is infinite, the subject of dynamic of $\Gamma$ and its entropy are relatively new. The first paper treating the entropy of a graph is due to [Gro]. In that context $X$ is a compact Riemannian manifold and $\Gamma$ can be viewed as a Riemannian submanifold. (Actually, $\Gamma$ can have singularities.) We treated this subject in [Fri1-3]. See Bullet [Bul1-2] for the dynamics of quadratic correspondences and [M-R] for iterated algebraic functions.

The object of this paper is to study the entropy of a corresponding map induced by $\Gamma$. We now describe briefly the main results of the paper. Let $X$ be a compact metric space and assume that $\Gamma \subset X \times X$ is a closed set. Set

$$\Gamma^\infty = \{(x_i)_i : (x_i, x_{i+1}) \in \Gamma, i = 1, ..., \}.$$ 

Let $\sigma : \Gamma^\infty \to \Gamma^\infty$ be the shift map. Denote by $h(\Gamma)$ be the topological entropy of $\sigma|\Gamma^\infty$. It then follows that $\sigma$ unifies in a natural way the notion of a (continuous) map $T : X \to X$ and a (finitely generated) semigroup or group of (continuous) transformations $S : X \to X$. Indeed, let $T_i : X \to X, i = 1, ..., m$, be $m$ continuous transformations. Denote by $\Gamma(T_i)$ the graphs corresponding to $T_i, i = 1, ..., m$. Set $\Gamma = \bigcup_{i=1}^m \Gamma(T_i)$. Then the dynamics of $\sigma$ is the dynamics of the semigroup generated by $T = \{T_1, ..., T_m\}$. If $T$ is a set of homeomorphisms and $T^{-1} = T$ then the dynamics of $\sigma$ is the dynamics of the group $G(T)$ generated by $T$.

In particular, we let $h(G(T)) = h(\Gamma)$ be the entropy of $G(T)$ using the particular set of generators $T$. For a finitely generated group $G$ of homeomorphisms of $X$ we define

$$h(G) = \inf_{T, G = G(T)} h(G(T)).$$

In the second section we study the entropy of graphs, semigroups and groups acting on the finite space $X$. The results of this section give a good motivation for the general case. In particular we have the following simple inequality

$$h(\bigcup_{i=1}^m \Gamma_i) \leq h(\bigcup_{i=1}^m (\Gamma_i \cup \Gamma_i^T)) \leq \log \sum_{i=1}^m e^{h(\Gamma_i \cup \Gamma_i^T)}.$$ (0.1)
Here $\Gamma^T = \{(y,x) : (x,y) \in \Gamma\}$. Let $\text{Card}(X) = n$. Then any group of homeomorphisms $\mathcal{G}$ of $X$ is a subgroup of the symmetric group $S_n$, acting on $X$ as a group of permutations. We then show that if $\mathcal{G}$ is commutative then $h(\mathcal{G}) = \log k$ for some integer $k$. If $\mathcal{G}$ acts transitively on $X$ then $k$ is the minimal number of generators for $\mathcal{G}$. Moreover, $h(\mathcal{G}) = 0$ iff $\mathcal{G}$ is a cyclic group. For each $n \geq 3$ we produce a group $\mathcal{G}$ generated by two elements so that $0 < h(\mathcal{G}) < \log 2$.

In §3 we discuss the entropy of graphs on compact metric spaces. We show that if $T_i : X \to X, i = 1, \ldots, m$, is a set of Lipschitzian transformations of a compact Riemannian manifold $X$ of dimension $n$ then

$$h(\bigcup_1^m \Gamma(T_i)) \leq \log \sum_{1}^{m} L_+(T_i)^n. \quad (0.2)$$

Here, $L_+(T_i)$ is the maximum of the Lipschitz constant of $T_i$ and 1. Thus, $L_+(T_i)^n$ is analogous to the norm of a graph on a finite space $X$. The above inequality generalizes to semi-Riemannian manifolds which have a Hausdorff dimension $n \in \mathbb{R}_+$ and a finite volume with respect to a given metric $d$ on $X$. Thus, if $X$ is a compact smooth Riemannian manifold and $\mathcal{G}$ is a finitely generated group of diffeomorphisms (0.2) yields that $h(\mathcal{G}) < \infty$. Let $X$ be a compact metric space and $T : X \to X$ a noninvolutive homeomorphism $(T^2 \neq Id)$. We then show that $h(\Gamma(T) \cup \Gamma(T^{-1})) \geq \log 2$. The following example due to M. Boyle shows that (0.1) does not apply in general. Let $X$ be a compact metric space for which there exists a homeomorphism $T : Y \to Y$ with $h(T) = h(T^2) = \infty$. (See for example [Wal, p. 192].) Set

$$X = X_1 \cup X_2, X_1 = Y, X_2 = Y, T_i(X_1) = X_2, T_i(X_2) = X_1,$$
$$T_1(x_1) = T x_1, T_1(x_2) = T^{-1} x_2, T_2(x_1) = T^{-1} x_1, T_2(x_2) = T x_2, x_1 \in X_1, x_2 \in X_2.$$

As $T_1^2 = T_2^2 = Id$ it follows that $2h(T_1) = 2h(T_2) = h(Id) = 0$. Clearly, $T_2T_1|X_1 = T^2$ and $h(\Gamma) = \infty$. The last section discusses mainly the entropy of semigroups and groups of Möbius transformations on the Riemann sphere. Let $\mathcal{T} = \{T_1, \ldots, T_k\}$ is a set of Möbius transformations. Inequality (0.2) yield that $h(\mathcal{G}(\mathcal{T})) \leq \log k$. Let $T_i(z) = z + a_i, i = 1, 2$, be two translations of $\mathcal{C}$. Assume that $\frac{a_1}{a_2}$ is a negative rational number. We then show that

$$h(\Gamma(T_1) \cup \Gamma(T_2)) = - \frac{|a|}{|a| + |b|} \log \frac{|a|}{|a| + |b|} - \frac{|b|}{|a| + |b|} \log \frac{|b|}{|a| + |b|}. $$

Assume now that $a_1$ and $a_2$ are linearly independent over $\mathbb{R}$. We then show that

$$h(\bigcup_1^2 (\Gamma(T_1) \cup \Gamma(T_1^{-1}))) = \log 4.$$ 

It is of great interest to see if $h(\mathcal{G})$ has any geometric meaning for a finitely generated Kleinian group $\mathcal{G}$. Consult with [G-L-W], [L-W], [N-P], [L-P] and [Hur] for other definitions of the entropy of relations and foliations.
§1. Basic definitions

Let $X$ be a compact metric space and assume that $\Gamma \subset X \times X$ is a closed set. Set

$$X^k = \prod_{1}^{k} X_i, X^{\infty} = \prod_{1}^{\infty} X_i, X_i = X, i \in \mathbb{Z},$$

$$\Gamma^k = \{(x_i)_1^{k} : (x_i, x_{i+1}) \in \Gamma, i = 1, ..., k - 1\}, k = 2, ...,$$

$$\Gamma^{\infty} = \{(x_i)_1^{\infty} : (x_i, x_{i+1}) \in \Gamma, i = 1, ..., \}, \Gamma^{\infty} = \{(x_i)_{i \in \mathbb{Z}} : (x_i, x_{i+1}) \in \Gamma, i \in \mathbb{Z}\}.$$

We shall assume that $\Gamma^k \neq \emptyset, k = 2, ...$, unless stated otherwise. (In any case, if this assumption does not hold we set $h(\Gamma) = 0$.) This in particular implies that $\Gamma^{\infty} \neq \emptyset$. Let

$$\pi_{l}^{p,q} : X^l \to X^{q-p+1}, \{x_i\}_1^{l} \mapsto \{x_i\}_p^{q}, 1 \leq p \leq q \leq l.$$

If no ambiguity arise we shall denote $\pi_{l}^{p,q}$ by $\pi_{p,q}$. The maps $\pi_{p,q}$ are well defined for $X^{\infty}, X^{\infty}$. For $p \leq 0, p \leq q$ we let $\pi_{p,q} : X^{\infty} \to X^{q-p+1}$. Similarly, for a finite $p$ we have the obvious maps $\pi_{-\infty,p}, \pi_{p,\infty}$ whose range is $\Gamma^{\infty}$. Let $d : X \times X \to \mathbb{R}$ be a metric on $X$. As $X$ is compact we have that $X$ is a bounded diameter $0 < D < \infty$. That is, $d(x, y) \leq D, \forall x, y \in X$. On $X^k, X^{\infty}, X^{\infty}$ one has the induced metric

$$d(\{x_i\}_1^{k}, \{y_i\}_1^{k}) = \max_{1 \leq i \leq k} \frac{d(x_i, y_i)}{\rho^{i-1}},$$

$$d(\{x_i\}_1^{\infty}, \{y_i\}_1^{\infty}) = \sup_{1 \leq i} \frac{d(x_i, y_i)}{\rho^{i-1}},$$

$$d(\{x_i\}_{i \in \mathbb{Z}}, \{y_i\}_{i \in \mathbb{Z}}) = \sup_{i \in \mathbb{Z}} \frac{d(x_i, y_i)}{\rho^{i-1}}.$$

Here $\rho > 1$ to be fixed later. Since $X$ is compact it follows that $X^k, X^{\infty}, X^{\infty}$ are compact metric spaces where the infinite products have the Tychonoff topology. Let

$$\sigma : X^{\infty} \to X^{\infty}, \sigma((x_i)_1^{\infty}) = (x_{i+1})_1^{\infty},$$

$$\sigma : X^{\infty} \to X^{\infty}, \sigma((x_i)_{i \in \mathbb{Z}}) = (x_{i+1})_{i \in \mathbb{Z}}$$

be the one sided shift and two sided shift respectively. We refer to Walters [Wal] for the definitions and properties of dynamical systems used here. Note that $\Gamma^{\infty}, \Gamma^{\infty}$ are invariant subsets of one sided and two sided shifts, i.e.

$$\sigma : \Gamma^{\infty} \to \Gamma^{\infty}, \sigma : \Gamma^{\infty} \to \Gamma^{\infty}.$$

We call the above restrictions of $\sigma$ as the dynamics (maps) induced by $\Gamma$. As $\Gamma$ was assumed to be closed it follows that $\Gamma^{\infty}, \Gamma^{\infty}$ are closed too. Hence, we can define the topological entropies $h(\sigma|\Gamma^{\infty}), h(\sigma|\Gamma^{\infty})$ of the corresponding restrictions. We shall show that these two entropies are equal. The above entropy is $h(\Gamma)$.

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Denote by $C(X)$ the Banach space of all continuous functions $f : X \to \mathbb{R}$. For $f \in C(X)$ it is possible to define the topological pressure $P(\Gamma, f)$ as follows. First observe that $f$ induces the following continuous functions

$$
\begin{align*}
  f_1 : \Gamma^\infty_+ &\to \mathbb{R}, f_1((x_i)_{i=1}^\infty) = f(x_1), \\
  f_2 : \Gamma^\infty &\to \mathbb{R}, f_2((x_i)_{i\in\mathbb{Z}}) = f(x_1).
\end{align*}
$$

Let $P(\sigma, f_1), P(\sigma, f_2)$ be the topological pressures of $f_1, f_2$ with respect to the map $\sigma$ acting on $\Gamma^\infty_+, \Gamma^\infty$ respectively. We shall show that the above topological pressures coincide. We then let $P(\Gamma, f) = P(\sigma, f_1) = P(\sigma, f_2)$.

Let $T : X \to X$ be a continuous map. Set $\Gamma = \Gamma(T) = \{(x, y) : x \in X, y = T(x)\}$ be the graph of $T$. Denote by $h(T)$ the topological entropy of $T$. It then follows that $h(T) = h(\Gamma)$. Indeed, observe that $x \mapsto orb_T(x) = (T^{i-1}(x))_{i=1}^\infty$ induces a homeomorphism $\phi : X \to \Gamma(T)^\infty_+$ such that $T = \phi^{-1} \circ \sigma \circ \phi$ and the equality $h(T) = h(\sigma|_\Gamma^\infty)$ follows. Similarly, for $f \in C(X)$ we have the equality $P(T, f) = P(\sigma, f_1) = P(\Gamma(T), f)$.

Let $\Gamma_\alpha, \alpha \in A$ be a family of closed graphs in $X \times X$. Set

$$
\forall_{\alpha \in A} \Gamma_\alpha = \text{Closure}(\cup_{\alpha \in A} \Gamma_\alpha).
$$

Note that if $A$ is finite then $\forall \Gamma_\alpha = \bigcup \Gamma_\alpha$. The dynamics of $\Gamma = \forall \Gamma_\alpha$ is called the product dynamics induced by $\Gamma_\alpha, \alpha \in A$. Let $T_\alpha : X \to X, \alpha \in A$ be a set of continuous maps. Set

$$
T = \cup_{\alpha \in A} T_\alpha, \Gamma(T) = \text{Closure}(\cup_{\alpha \in A} \Gamma(T_\alpha)).
$$

Then the dynamics of $\Gamma(T)$ is the dynamics of a semigroup $S(T)$ generated by $T$. If each $T_\alpha, \alpha \in A$ is a homeomorphism and $T^{-1} = T$ then the dynamics of $\Gamma(T)$ is the dynamics of a group $G(T)$ generated by $T$. Note that for a fixed $x \in X$ the orbit of $x$ is given by the formula

$$
 orb_T(x) = \{(x_i)_{i=1}^\infty, x_1 = x, x_i \in \text{Closure}(T_{\alpha_1} \circ \cdots \circ T_{\alpha_1}(x)), \alpha_1, \ldots, \alpha_{i-1} \in A, i = 2, \ldots, \}.
$$

If $A$ is finite then we can drop the closure in the above definition.

Let $T$ be a set of continuous transformations of $X$ as above. We then define

$$
 h(S(T)) = h(\Gamma(T)), \quad P(S(T), f) = P(\Gamma(T), f), f \in C(X)
$$

to be the entropy of $S(T)$ and the topological pressure of $f$ with respect to the set of generators $T$. In order to ensure that the above quantities are finite we shall assume that $T$ is a finite set. Given a finitely generated semigroup $S$ of $T : X \to X$ let

$$
 h(S) = \inf_{\Gamma, S = \Gamma(T)} h(S(T)), \quad P(S, f) = \inf_{\Gamma, S = \Gamma(T)} P(S(T), f), f \in C(X).
$$

Here, the infimum is taken over all finite generators of $S$. 

4
§2. Entropy of graphs on finite spaces

Let $X$ be a finite space. We assume that $X = \{1, ..., n\}$. Then each $\Gamma \subset X \times X$ is in one to one correspondence with a $n \times n$ $0$–$1$ matrix $A = (a_{ij})_1^n$. That is $(i, j) \in \Gamma \iff a_{ij} = 1$. As usual we let $M_n(\{0 – 1\})$ be the set of $0 – 1$ $n \times n$ matrices. For $\Gamma \subset X \times X$ we let $A(\Gamma) \in M_n(\{0 – 1\})$ to be the matrix induced by $\Gamma$ and for $A \in M_n(\{0 – 1\})$ we let $\Gamma(A)$ to be the graph induced by $A$. The assumption that $\Gamma^k \neq \emptyset, k = 1, 2, ...$, is equivalent to $\rho(A(\Gamma)) > 0 \iff \rho(A(\Gamma)) \geq 1$. Here, for any $A$ in the set of $n \times n$ complex valued matrices $M_n(\mathbb{C})$ we let $\rho(A)$ to be the spectral radius of $A$. For $\Gamma \subset X \times X$ consider the sets $X_l = \pi_l, l = 2, ...$. It easily follows that $X_2 \supset X_3 \supset \cdots X_n = X_{n+1} = \cdots = X'$. Then $\Gamma_l \neq \emptyset, l = 2, ...$, iff $X' \neq \emptyset$. Set $\Gamma' = \Gamma \cap X' \times X'$. It then follows that $\Gamma' = \Gamma''$. Moreover,

$$\pi_{1,\infty}(\Gamma) = \pi_{1,\infty}(\Gamma') = \Gamma'' \subset \Gamma.$$

Here the containment is strict iff $X' \neq X$. It is well known fact in symbolic dynamics that if $X' \neq \emptyset$ then

$$h(\sigma |_{\Gamma''} = h(\sigma |_{\Gamma'}) = \log \rho(A(\Gamma)) = \log \rho(A(\Gamma')) = h(\sigma |_{\Gamma''} = h(\sigma |_{\Gamma'}).$$

See for example [Wal]. We thus let $h(\Gamma)$ - the entropy of the graph $\Gamma$ to be any of the above numbers. In fact, $X'$ can be viewed as a limit set of the "transformation" induced by $\Gamma$ on $X'$. If $\rho(A(\Gamma)) = 0$, i.e. $X' = \emptyset$ then let $h(\Gamma) = \log^+ \rho(A(\Gamma))$. Here, $\log^+ x = \log \max(x, 1)$.

Let $\Gamma_0 \subset X \times X, \alpha \in \mathcal{A}$ be a family of graphs. Set $A_\alpha = (a_{ij}^{(\alpha)})_1^n = A(\Gamma_\alpha), \alpha \in \mathcal{A}$. It then follows that

$$\forall \alpha \in \mathcal{A} A_\alpha = (\max_{\alpha \in \mathcal{A}} a_{ij}^{(\alpha)})_1^n = A(\forall \alpha \in \mathcal{A} \Gamma_\alpha).$$

The Perron-Frobenius theory of nonnegative matrices yields straightforward that $\rho(A_\alpha) \leq \rho(\forall A_\beta)$. This is equivalent to the obvious inequality $h(\Gamma_\alpha) \leq h(\forall \Gamma_\beta)$. We now point out that we can not obtain an upper bound on $h(\forall \Gamma_\alpha)$ as a function of $h(\Gamma_\alpha), \alpha \in \mathcal{A}$. It suffices to pass to the corresponding matrices and their spectral radii. Let $A = (a_{ij})_1^n \in M_n(\{0 – 1\})$ matrix such that $a_{ij} = 1 \iff i \leq j$. Assume that $B = A^T$. Then $\rho(A) = \rho(B) = 1, \rho(A \vee B) = n$.

Let $\| \cdot \| : \mathbb{C}^n \rightarrow \mathbb{R}_+$ be a norm on $\mathbb{C}^n$. Denote by $\| \cdot \| : M_n(\mathbb{C}) \rightarrow \mathbb{R}_+$ the induced operator norm. Clearly, $\rho(A) \leq \|A\|$. Hence

$$\rho(\forall \alpha \in \mathcal{A} A_\alpha) \leq \rho(\sum_{\alpha \in \mathcal{A}} A_\alpha) \leq \sum_{\alpha \in \mathcal{A}} \|A_\alpha\|.$$

Thus

$$h(\forall \alpha \in \mathcal{A} \Gamma_\alpha) \leq \log^+ \sum_{\alpha \in \mathcal{A}} \|A_\alpha\|.$$

(2.1)

In the next section we shall consider analogs of $\|A(\Gamma)\|$ for which we have the inequality (2.1) for any set $\mathcal{A}$. For a graph $\Gamma \subset X \times X$ let $\Gamma^T = \{(x, y) : (y, x) \in \Gamma\}$. That is, $A(\Gamma^T) = A^T(\Gamma)$. A graph $\Gamma$ is symmetric if $\Gamma^T = \Gamma$. Assume that $\Gamma$ is symmetric.
It then follows that \( \rho(A(\Gamma)) = \|A(\Gamma)\| \) where \( \| \cdot \| \) is the spectral norm on \( M_n(C) \), i.e. \( \|A\| = \rho(AA^*)^{\frac{1}{2}}. \) Thus, for a family \( \Gamma_{\alpha}, \alpha \in \mathcal{A} \) of symmetric graphs we have the inequalities
\[
h(\bigvee_{\alpha \in \mathcal{A}} \Gamma_{\alpha}) \leq \log \sum_{\alpha \in \mathcal{A}} e^{h(\Gamma_{\alpha})}. \tag{2.2}
\]

More generally, for any family of graphs we have the inequalities
\[
h(\bigvee_{\alpha \in \mathcal{A}} \Gamma_{\alpha}) \leq h(\bigvee_{\alpha \in \mathcal{A}} (\Gamma_{\alpha} \vee \Gamma_{\alpha}^T)) \leq \log \sum_{\alpha \in \mathcal{A}} e^{h(\Gamma_{\alpha} \vee \Gamma_{\alpha}^T)}. \tag{2.3}
\]

Let \( T : X \to X \) be a transformation. Then \( A(T) = A(\Gamma(T)) \) is a \( 0-1 \) stochastic matrix, i.e. each row of \( A(T) \) contains exactly one 1. Vice versa, if \( A \in M_n(\{0,1\}) \) is a stochastic matrix then \( A = A(T) \) for some transformation \( T : X \to X \). Furthermore, \( T : X \to X \) is a homeomorphism iff \( A(T) \) is a permutation matrix. For \( T = \{T_1, \ldots, T_k\} \) \( S(T) \) is a group iff each \( T_i \) is a homeomorphism, i.e. \( A(T_i) \) is a permutation matrix for \( i = 1, \ldots, k \). Clearly, any group of homeomorphisms \( S \) of \( X \) is a subgroup of the symmetric group \( S_n \), \( n = \text{Card}(X) \).

\textbf{(2.4) Theorem.} Let \( X \) be a finite space and assume that \( T_i : X \to X, i = 1, \ldots, k \), be a set of transformation. Set
\[
T = \{T_1, \ldots, T_k\}, \Gamma = \Gamma(T) = \bigcup_i \Gamma(T_i), A = A(\Gamma).
\]

Then \( h(S(T)) \leq \log k. \) Furthermore, \( h(S(T)) = 0 \) iff \( A(\Gamma) \) is a permutation matrix. Assume that \( k \geq 2. \) Then \( h(S(T)) = \log k \) iff there exists an irreducible component \( \hat{X} \subset X' \) on which \( S(T) \) acts transitively such that \( A(\Gamma \cap \hat{X} \times \hat{X}) \) is a \( 0-1 \) matrix with \( k \) ones in each row. In particular, \( h(S(T, T^{-1})) = \log 2 \) for \( T^2 \neq Id. \) Assume finally that \( S(T) \) is a commutative group. Then \( h(S(T)) = \log k' \) for some integer \( 1 \leq k' \leq k \).

\textbf{Proof.} Recall that \( h(S(T)) = \log \rho(A). \) As \( A(T_i) \) is a stochastic matrix it follows that \( \rho(A(T_i)) = 1, i = 1, \ldots, k. \) Since \( A \geq A(T_i) \) we deduce that \( \rho(A) \geq 1. \) Thus, \( X' \neq \emptyset. \) Then \( X' = \bigcup_i X_i, X_i \cap X_j = \emptyset, 1 \leq i < j \leq m. \) Here, \( A \) acts transitively on each \( X_i. \) Set \( \Gamma_i = \Gamma \cap X_i \times X_i, A_i = A(\Gamma_i), i = 1, \ldots, m. \) Note that each \( A_i \) is an irreducible matrix. It then follow that \( h(\Gamma) = \max \log \rho(A_i). \) If \( u_i : X_i \to \{1\}. \) Then \( A_i u_i \leq ku_i. \) The minmax characterization of Wielandt for an irreducible \( A_i \) yields that \( \rho(A_i) \leq k. \) The equality holds iff each row of \( A_i \) has exactly \( k \) ones. Thus, \( h(\Gamma) = \log k, k \geq 1 \) iff each row of some \( A_i \) has \( k \) ones.

Assume next that \( T \) is a homeomorphism such that \( T^2 \neq Id. \) Set \( \Gamma = \Gamma(T) \cup \Gamma(T^{-1}). \) Then \( X' = X = \bigcup_i X_i \) and least one \( X_i \) contains more then one point. Clearly, this \( A_i \) has two ones in each row and column. Hence, \( h(\Gamma) = \log 2. \)

Assume now that \( G = S(T) \) is a commutative group. Then \( X = X' = \bigcup_i X_i. \) We claim that the following dichotomy holds for each pair \( T_i, T_j, i \neq j. \) Either \( T_i(x) \neq T_j(x) \forall x \in X_i \) or \( T_i(x) = T_j(x) \forall x \in X_i. \) Indeed, assume that \( T_i(x) = T_j(x) \) for some \( x \in X_i. \) As \( G \) acts transitively on \( X_i \) and is commutative we deduce that \( T_i(x) = T_j(x) \forall x \in X_i. \)
Thus \( \Gamma(T_i) \cap X_i \times X_i, i = 1, \ldots, k \), consists of \( k \) distinct permutation matrices which do not have any 1 in common. That is \( \Gamma_i = \Gamma \cap X_i \times X_i \) is a matrix with \( k \) ones in each row and column. Hence,

\[
h(\Gamma_i) = \log k_i, l = 1, \ldots, m, h(\Gamma) = \log \max_{1 \leq l \leq m} k_l.
\]

\( \diamond \)

(2.5) Theorem. Let \( X \) be a finite space of \( n \) points. If \( \mathcal{G} \) is commutative then \( h(\mathcal{G}) = \log k \) for some integer \( k \) which is not greater then the number of the minimal generators of \( \mathcal{G} \). If \( \mathcal{G} \) acts transitively on \( X \) or the restrictiton of \( \mathcal{G} \) to one of the irreducible (transitive) components is faithful then \( k \) is the minimal number of generators of \( \mathcal{G} \). In particular, for any \( \mathcal{G} \) \( h(\mathcal{G}) = 0 \) iff \( \mathcal{G} \) is cyclic. For each \( n \geq 3 \) there exists a group \( \mathcal{G} \) which acts transitively on \( X \) so that \( 0 < h(\mathcal{G}) < \log 2 \).

Proof. Assume first that \( \mathcal{G} \) is commutative. Let \( T = \{T_1, \ldots, T_p\} \) be a set of generators. Theorem 2.4 yields that \( h(\mathcal{G}(T)) = \log k(T), k(T) \leq p \). Choose a minimal subset of generators \( T' \subset T \). Clearly, \( h(\mathcal{G}(T')) \leq h(\mathcal{G}(T)) \). Thus, to compute \( h(\mathcal{G}) \) it is enough to assume that \( T \) consists of a minimal set of generators of \( \mathcal{G} \). Hence, \( h(\mathcal{G}) = \log k \) and \( k \) is at most the number of the minimal generators of \( \mathcal{G} \).

Assume now that \( \mathcal{G} \) acts transitively on \( X \). The arguments of the proof of Theorem 2.4 yield that \( x \in X, T_i(x) \neq T_j(x) \) for \( i \neq j \). Therefore, \( h(\mathcal{G}(T)) = \log p \). In particular, \( h(\mathcal{G}) = \log k \) where \( k \) is the minimal number of generators for \( \mathcal{G} \). Suppose now that \( X \) is reducible under the action of \( \mathcal{G} \) and the restriction of \( \mathcal{G} \) to one of its irreducible components is faithful. Then the above results yield that \( h(\mathcal{G}) = \log k \) where \( k \) is the minimal number of generators of \( \mathcal{G} \).

Assume now that \( h(\mathcal{G}) = 0 \). Let \( h(\mathcal{G}) = h(\mathcal{G}(T)) \). Assume first that \( \mathcal{G} \) acts irreducibly on \( X \). If \( T \) consists of one element \( T \) we are done. Assume to the contrary that \( T = \{T_1, \ldots, T_q\}, q > 1 \). Then \( A(\Gamma) \geq A(T_1) \). Since \( A(\Gamma) \) is irreducible as \( \mathcal{G} \) acts transitively, and \( A(\Gamma) \neq A(T_1) \) we deduce that \( \rho(A(\Gamma)) > 1 \). See for example [Gau]. This contradicts our assumption that \( h(\mathcal{G}) = 0 \). Hence, \( \mathcal{G} \) is generated by one element, i.e. \( \mathcal{G} \) is cyclic. Assume now that \( X = \bigcup_{i}^{n} X_i \) is the decomposition of \( X \) to its irreducible components. According to the above arguments \( \Gamma(T) \cap X_i \times X_i \) is a permutation matrix. Hence \( \Gamma(T) \) is a permutation matrix corresponding to the homeomorphism \( T \). Thus \( \mathcal{G} \) is generated by \( T \).

Assume that \( \text{Card}(X) = n \geq 3 \). Let \( T : X \rightarrow X \) be a homeomorphism that acts transitively on \( X \), i.e. \( T^n = \text{Id}, T^{n-1} \neq \text{Id} \). Let \( Q : X \rightarrow X, Q \neq T \) be another homeomorphism so that \( Q(x) = T(x) \) for some \( x \in X \). Set \( \mathcal{G} = \mathcal{G}([T, Q]) \). According to Theorem 2.4 \( h(\mathcal{G}([T, Q])) < \log 2 \). Hence, \( h(\mathcal{G}) < \log 2 \). As \( \mathcal{G} \) is not cyclic it follows that \( h(\mathcal{G}) > 0 \). \( \diamond \)

It is an interesting problem to determine the entropy of a commutative group in the general case.
§3. Entropy of graphs on compact spaces

Let $X$ be a compact metric space and $\Gamma \subset X \times X$ be a closed graph. As in the previous section set $X_1 = \pi_{1,1}(\Gamma^t), t = 2, \ldots$. Then $\{X_1\}_2$ is a sequence of decreasing closed spaces. Let $X' = \cap_2^\infty X_t, \Gamma' = \Gamma \cap X' \times X'$. Clearly,

$$ \Gamma_+ = \Gamma_+^\infty, \pi_1,\infty(\Gamma_+) = \pi_1,\infty(\Gamma_+^\infty) = \Gamma'_+ \subset \Gamma_+^\infty. $$

(3.1) Theorem. Let $X$ be a compact metric space and $\Gamma \subset X \times X$ be a closed set. Then

$$ h(\sigma|\Gamma_+^\infty) = h(\sigma|\Gamma_+^\infty) = h(\sigma|\Gamma_+), $$

$$ P(\Gamma_+^\infty, f) = P(\Gamma_+^\infty, f) = P(\Gamma_+, f), f \in C(X). $$

Proof. The equality $h(\sigma|\Gamma_+^\infty) = h(\sigma|\Gamma_+^\infty)$ follows from the observation that $\Gamma_+^\infty = \cap_0^\infty \sigma'(\Gamma_+^\infty)$. See [Wal, Cor. 8.6.1]. We now prove the equality $h(\sigma|\Gamma_+^\infty) = h(\sigma|\Gamma_+^\infty)$

It is enough to assume that $X' = X$. Set $X_1 = \Gamma_+^\infty, X_2 = \Gamma_+^\infty$. Let $\pi : X_2 \rightarrow X_1$ be the projection $\pi_1,\infty$. It then follows that $\pi(X_2) = X_1, \pi \circ \sigma_2 = \sigma_1 \circ \pi$. Denote by $\sigma_1$ the restriction of $\sigma$ to $X_1$ and let $h_i = h(\sigma_i)$ be the topological entropy of $\sigma_i$. As $\sigma_1$ is a factor of $\sigma_2$ one deduces $h_1 \leq h_2$.

We now prove the reversed inequality $h_1 \geq h_2$. Let $Y$ be a compact metric space and assume that $T : Y \rightarrow Y$ is a continuous transformation. Denote by $\Pi(Y)$ the set of all probability measures on the Borel $\sigma$-algebra generated by all open sets of $Y$. Let $\mathcal{M}(T) \subset \Pi(Y)$ be the set of all $T$-invariant probability measures. Assume that $\mu \in \mathcal{M}(T)$. Then one defines the Kolmogorov-Sinai entropy $h_\mu(T)$. The variational principle states that

$$ h(T) = \sup_{\mu \in \mathcal{M}(T)} h_\mu(T), P(T, f) = \sup_{\mu \in \mathcal{M}(T)} (h_\mu(T) + \int f d\mu), f \in C(X). $$

Let $\mathcal{B}_2$ be the $\sigma$-algebra generated by open sets in $X_2$. An open set $A \subset X_2$ is called cylindrical if there exist $p \leq q$ with the following property. Let $y \in \pi_{i,1}(A)$. Then for $i \leq p$ we have the property $\pi_{2,1}^2((\pi_{2,2}^2)^{-1}(y)) \subset \pi_{i-1,1,i-1}(A)$. For $i \geq q$ we have the property $\pi_{2,2}^2((\pi_{1,1}^2)^{-1}(y)) \subset \pi_{i+1,i+1}(A)$. Let $\mathcal{C} \subset \mathcal{B}_2$ be the finite Borel subalgebra generated by open cylindrical sets. Note that each set in $\mathcal{C}$ is cylindrical. Since $\sigma_2$ is a homeomorphism it follows that for any $\mu \in \mathcal{M}(\sigma_2) \mathcal{B}(\mathcal{C})^\sigma = \mathcal{B}_2$. That is up a set of zero $\mu$-measure every set in $\mathcal{B}_2$ can be presented as a set in $\sigma$-Borel algebra generated by $\mathcal{C}$. Let $\alpha \subset \mathcal{C}$ be a finite partition of $X_2$. One then can define the entropy $h(\sigma_2, \alpha)$ with respect to the measure $\mu$ [Wal, Ch.4]. Since $\sigma_2$ is a homeomorphism and $\mu$ is $\sigma_2$ invariant it follows that $h(\sigma_2, \alpha) = h(\sigma_2, \sigma_2^m(\alpha))$ for any $m \in \mathbb{Z}$. The assumption that $\mathcal{B}(\mathcal{C})^\sigma = \mathcal{B}_2$ implies that $\sup_{\alpha \subset \mathcal{C}} h(\sigma_2, \alpha) = h_\mu(\sigma_2)$. Taking $m$ big enough in the previous equality we deduce that it is enough to consider all finite partitions $\alpha \subset \mathcal{C}$ with the following property. For each $A \in \alpha$ and each $i \leq 1, y \in \pi_{i,1}(A)$ we have the condition $\pi_{1,1}^2((\pi_{2,2}^2)^{-1}(y)) \subset \pi_{i-1,1,i-1}(A)$. It then follows that $\mu$ projects on $\mu' \in \mathcal{M}(\sigma_1)$ and $h_\mu(\sigma_2) = h_{\mu'}(\sigma_1)$. The variational principle yields $h_2 \leq h_1$ and the equalities of all three entropies are established.
To prove the three equalities on the topological pressure we use the analogous arguments for the topological pressure. ♦

Let \( h(\Gamma) \) be one of the entropies in Theorem 3.1. We call \( h(\Gamma) \) the entropy of \( \Gamma \). For \( f \in C(X) \) we denote by \( P(\Gamma, f) \) to be one of the topological in Theorem 3.1. Let \( X \) be a complete metric space with a metric \( d \). Denote by \( B(x, r) \) the open ball of radius \( r \) centered in \( x \). Let \( \bar{B}(x, r) = \text{Closure}(B(x, r)) \). We say that \( X \) is semi-Riemannian of Hausdorff dimension \( n \geq 0 \) if for every open ball \( B(x, r) \), \( 0 < r < \delta \) the Hausdorff dimension of \( \bar{B}(x, r) \) is \( n \) and its Hausdorff volume \( \text{vol}(\bar{B}(x, r)) \) satisfies the inequality

\[
\alpha r^n \leq \text{vol}(\bar{B}(x, r))
\]

for some \( 0 < \alpha \). Recall that if the Hausdorff dimension of a compact set \( Y \subset X \) is \( m \) then its Hausdorff volume is defined as follows.

\[
\text{vol}(Y) = \lim_{\epsilon \to 0} \liminf_{x_i, 0 < \epsilon_i \leq \epsilon, i=1,...,k, \cup B(x_i, \epsilon_i) \supset Y} \sum_{1}^{k} \epsilon_i^m.
\]

The following lemma is a straightforward generalization of Bowen’s inequality \([\text{Bow}], [\text{Wal, Thm. 7.15}]\).

**Lemma.** Let \( X \) be a semi-Riemannian compact metric space of Hausdorff dimension \( n \). Assume that \( T : X \to X \) is Lipschitzian - \( d(T(x), T(y)) \leq \lambda d(x, y) \) for all \( x, y \in X \) and some \( \lambda \geq 1 \). Suppose furthermore that \( X \) has a finite \( n \) dimensional Hausdorff volume. Then \( h(T) \leq \log \lambda^n \).

**Proof.** As \( X \) is compact and semi-Riemannian it follows that \( X \) has the Hausdorff dimension \( n \). Let \( N(k, \epsilon) \) be the cardinality of the maximal \((k, \epsilon)\) separated set. Assume that \( \{x_1, \ldots, x_{N(k, \epsilon)}\} \) is a maximal \((k, \epsilon)\) separated set. That is for \( i \neq j \)

\[
\max_{0 \leq l \leq k-1} d(T^l(x_i), T^l(x_j)) > \epsilon.
\]

We claim that

\[
\bar{B}(x_i, \epsilon_k) \cap \bar{B}(x_j, \epsilon_k) = \emptyset, \ i \neq j, \ \epsilon_k = \frac{\epsilon}{3\lambda^{k-1}}.
\]

This is immediate from the inequality \( d(T^l(x), T^l(y)) \leq \lambda^l d(x, y) \) and the \((k, \epsilon)\) separability of \( \{x_1, \ldots, x_{N(k, \epsilon)}\} \). We thus deduce the obvious inequality

\[
\sum_{l=1}^{N(k, \epsilon)} \text{vol}(\bar{B}(x_l, \epsilon_k)) \leq \text{vol}(X).
\]

In the above inequality assume that \( \epsilon \leq \delta \). Then the lower bound on \( \text{vol}(\bar{B}(x_l, \epsilon_k)) \) yields

\[
N(k, \epsilon) \leq \frac{\text{vol}(X)3^n \lambda^{n(k-1)}}{\alpha \epsilon^n}.
\]

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Thus
\[ h(T) = \lim_{\epsilon \to 0} \limsup_{k \to \infty} \frac{\log N(k, \epsilon)}{k} \leq n \log \lambda \]
and the proof of the lemma is completed. ⋄

The above estimate can be improved as follows. Let \( X \) be a compact metric space and \( T : X \to X \). Set
\[ L(T) = \sup_{x \neq y \in X} \frac{d(T(x), T(y))}{d(x, y)}, \quad L_+(T) = \max(L(T), 1). \]
Thus \( T \) is Lipschitzian iff \( L(T) < \infty \).

\[ l(T) = \liminf_{k \to \infty} L_{+1}^k(T^k). \]

Note that \( T^k \) is Lipschitzian for some \( k \geq 1 \) iff \( l(T) < \infty \). \( l(T) \) can be considered as a generalization of the maximal Lyapunov exponent for the mapping \( T \). As \( h(T^k) = kh(T), k \geq 0 \) from Lemma 3.2 we obtain.

(3.3) Theorem. Let \( X \) be a semi-Riemannian compact metric space of Hausdorff dimension \( n \). Assume that \( T : X \to X \) is a continuous map. Suppose furthermore that \( X \) has a finite \( n \) dimensional Hausdorff measure. Then \( h(T) \leq n \log l(T) \).

We have in mind the following application. Let \( T : \mathbb{CP}^1 \to \mathbb{CP}^1 \) be a rational map of the Riemann sphere \( \mathbb{CP}^1 \). Let \( X = J(T) \) be its Julia set. It is plausible to assume that \( \log l(T) \) on \( X \) is the Lyapunov exponent corresponding to \( T \) and the maximal \( T \)-invariant measure on \( X \). Suppose that the Hausdorff dimension of \( X \) is \( n \) and \( X \) has a finite Hausdorff volume. Assume furthermore that \( X \) is semi-Riemannian of Hausdorff dimension \( n \). We then can apply Theorem 3.3. As \( h(T) = \log \deg(T) \) we have the inequality \( \deg(f) \leq l(f)^n \).

(3.4) Theorem. Let \( X \) be a semi-Riemannian compact metric space of Hausdorff dimension \( n \). Assume that \( T_i : X \to X, i = 1, \ldots, m, \) are continuous maps. Let \( \Gamma(T_i) \) be the graph of \( T_i = 1, \ldots, m \). Set \( \Gamma = \cup_1^m \Gamma(T_i) \). Suppose furthermore that \( X \) has a finite \( n \) dimensional Hausdorff volume. Then
\[ h(\Gamma) \leq \log \sum_1^m L_+(T_i)^n. \]

Proof. It is enough to consider the nontrivial case where each \( T_i \) is Lipschitzian. In the definitions of the metrics on \( \Gamma^k, \Gamma_+^\infty \) set
\[ \rho > \max_{1 \leq i \leq m} L_+(T_i). \]
Let \( M = \{1, \ldots, m\} \). Then for \( \omega = (\omega_1, \ldots, \omega_{k-1}) \in M^{k-1} \) we let
\[
\Gamma(\omega) = \{(x^j_i)_{i=1}^\infty : x^j_i \in X, x_i = T_{\omega_{i-1}} \circ \cdots \circ T_{\omega_1}(x_1), i = 2, \ldots, k\} \subset \Gamma^k, \omega \in M^{k-1}.
\]

Clearly, each \( \Gamma(\omega) \) is isometric to \( X \). Hence, the Hausdorff dimension of \( \Gamma(\omega) \) is \( n \) and \( \text{vol}(\Gamma(\omega)) = \text{vol}(X) \). Furthermore, \( \cup_{\omega \in M^{k-1}} \Gamma(\omega) = \Gamma^k \). It then follows that each \( \Gamma^k \)
has Hausdorff dimension \( n \), has finite Hausdorff volume not exceeding \( m^{k-1} \text{vol}(X) \) and is semi-Riemannian compact metric space of Hausdorff dimension \( n \). Moreover, the volume of any closed ball \( \bar{B}(y, r) \subset \Gamma^k \) is at least \( \alpha r^n \) where \( \alpha \) is the constant for \( X \). Let \( Y = \Gamma^\infty \) and consider a maximal \((k, \epsilon)\) separated set in \( Y \) of cardinality \( N(k, \epsilon) \), \( y^j \in Y, j = 1, \ldots, N(k, \epsilon) \). That is
\[
y^j = (x^j_i)_{i=1}^\infty, (x^j_i, x^j_{i+1}) \in \Gamma, i = 1, \ldots, j = 1, \ldots, N(k, \epsilon),
\]
\[
\max_{1 \leq i} d(x^j_i, x^l_i) > \epsilon, 1 < j \neq l \leq N(k, \epsilon).
\]

Here, \( a^+ = \max(a, 0), a \in \mathbb{R} \). Fix \( \epsilon, 0 < \epsilon < \delta \). Assume that \( D \) is the diameter of \( X \) and let \( K(\epsilon) = [\log_\rho D - \log_\rho \epsilon] \). It then follows that
\[
\max_{1 \leq i \leq k + K(\epsilon)} d(x^j_i, x^l_i) > \epsilon, 1 < j \neq l \leq N(k, \epsilon).
\]

Set \( z^j = (x^j_i)_{i=1}^{k+K(\epsilon)} \subset \Gamma^{k+K(\epsilon)}, j = 1, \ldots, N(k, \epsilon) \). Clearly,
\[
\{z^j\}_1^{N(k+K(\epsilon))} = \bigcup_{\omega \in M^{k+K(\epsilon)-1}} \{z^j\}_1^{N(k, \epsilon)} \cap \Gamma(\omega) \Rightarrow
N(k, \epsilon) \leq \sum_{\omega \in M^{k+K(\epsilon)-1}} \text{Card}(\{z^j\}_1^{N(k, \epsilon)} \cap \Gamma(\omega)).
\]

We now estimate \( \text{Card}(\{z^j\}_1^{N(k, \epsilon)} \cap \Gamma(\omega)) \) for a fixed \( \omega = (\omega_1, \ldots, \omega_{k+K(\epsilon)-1}) \in M^{k+K(\epsilon)-1} \). For each \( z^j = (x^j_i)_{i=1}^{k+K(\epsilon)} \in \Gamma(\omega) \) consider the closed set ball
\[
\bar{B}(z^j, \epsilon(\omega)) \subset \Gamma(\omega), \epsilon(\omega) = \frac{\epsilon}{3 \prod_{i=1}^{k+K(\epsilon)-1} L_+\left(T_{\omega_i}\right)}.
\]

(We restrict here our discussion to the compact metric space \( \Gamma(\omega) \) with the metric induced from \( \Gamma^{k+K(\epsilon)} \).) Let \( z^j \neq z^l \in \Gamma(\omega) \). The condition (3.5) yields that \( \bar{B}(z^j, \epsilon(\omega)) \cap \bar{B}(z^l, \epsilon(\omega)) = \emptyset \). As \( \Gamma(\omega) \) is isometric to \( X \) we deduce that
\[
\text{Card}(\{z^j\}_1^{N(k, \epsilon)} \cap \Gamma(\omega)) \leq \frac{\text{vol}(X)3^n \prod_{i=1}^{k+K(\epsilon)-1} L_+(T_{\omega_i})^n}{\alpha \epsilon^n}.
\]

Hence,
\[
N(k, \epsilon) \leq \sum_{\omega \in M^{k+K(\epsilon)-1}} \frac{\text{vol}(X)3^n \prod_{i=1}^{k+K(\epsilon)-1} L_+(T_{\omega_i})^n}{\alpha \epsilon^n} = \frac{\text{vol}(X)3^n \left(\sum_{i=1}^m L_+(T_i)^n\right)^{k+K(\epsilon)-1}}{\alpha \epsilon^n}.
\]
Thus
\[ h(\Gamma) = \lim_{\epsilon \to 0} \lim_{k \to \infty} \sup \frac{\log N(k, \epsilon)}{k} \leq \log \sum_{i=1}^{n} L_+(T_i)^n \]
and the theorem is proved. \(\diamondsuit\)

We remark that the inequality of Theorem 3.4 holds if we replace the assumption that
X has a finite n-Hausdorff volume by the following one: the number of points of every
r-separated set in X does not exceed Cr\(^{-n}\) for some positive constant C.

Let X satisfies the assumptions of Theorem 3.4. It then follows that for the Lipschitzian maps f : X \to X the quantity \(L_+(T)^n\) is the ”norm” of the graph \(\Gamma(f)\) discussed in §2.

(3.6) **Lemma.** Let X be a compact metric space and T : X \to X be a noninvolutive homeomorphism (\(T^2 \neq \text{Id}\)). Then \(\log 2 \leq h(\Gamma(T) \cup \Gamma(T^{-1}))\). If \(T, T^{-1} : X \to X\) are noninvolutive isometries then \(h(\Gamma(T) \cup \Gamma(T^{-1})) = \log 2\).

**Proof.** Assume first that T has a periodic orbit \(Y = \{y_1, ..., y_p\}\) of period \(p > 2\). Restrict \(T, T^{-1}\) to this orbit. Theorem 2.4 yields the desired inequality. Assume now that we have an infinite orbit \(y_i = T^i(y), i = 1, 2, ..., \) Fix \(n \geq 3\). Let \(Y_n = \{y_1, ..., y_n\}\). Denote by \(\Gamma_n \subset Y_n \times Y_n\) the graph corresponding to the undirected linear graph on the vertices \(y_1, ..., y_n\). That \((i, j) \in \Gamma_n \iff |i - j| = 1\). Clearly
\[ \Gamma_n^\infty \subset \Gamma^\infty, \Gamma = \Gamma(T) \cup \Gamma(T^{-1}). \]

Hence \(h(\Gamma_n) \leq h(\Gamma)\). Obviously, \(h(\Gamma_n) = \log \rho(A(\Gamma_n))\). It is well known that \(\rho(A(\Gamma_n)) = 2 \cos \frac{\pi}{n+1}\) (The eigenvalues of \(A(\Gamma_n)\) are the roots of the Chebycheff polynomial.) Let \(n \to \infty\) and deduce \(h(\Gamma) \geq \log 2\). Assume now that T and \(T^{-1}\) are noninvolutive isometries. Then Theorem 3.4 and the above inequality implies that \(h(\Gamma(T) \cup \Gamma(T^{-1})) = \log 2\). \(\diamondsuit\)

Thus, Theorem 3.4 is sharp for \(m = 2\). Similar examples using isometries and Theorem 2.4 show that Theorem 3.4 is sharp in general.

Let X be a compact metric space and \(T_i : X \to X, i = 1, ..., m\), be a set of continuous transformations. Let \(T = \{T_1, ..., T_m\}\). Then \(h(S(T))\) was defined to be the entropy of the graph \(\Gamma = \bigcup^m \Gamma(T_i)\). As in the case of \(m = 1\) this entropy can be defined in terms of ”\((k, \epsilon)\)” separated (spanning) sets as follows. Set
\[ d_{k+1}(x, y) = \max \left\{ \max_{1 \leq i_1, j_1, ..., i_k, j_k \leq m} d(T_{i_1} \cdots T_{i_k}(x), T_{j_1} \cdots T_{j_k}(y)), d(x, y) \right\}, k = 1, 2, ..., \]

Let \(M(k, \epsilon)\) be the maximal cardinality the \(\epsilon\) separated set in the metric \(d_k\).

(3.7) **Lemma.** Let X be a compact metric space and assume that \(T_i : X \to X, i = 1, ..., m,\) are continuous transformations. Then
\[ h(S(\{T_1, ..., T_m\})) = \lim_{\epsilon \to 0} \lim_{k \to \infty} \sup \frac{\log M(k, \epsilon)}{k}. \]
Proof. From the definition of the \((k, \epsilon)\) separated set for \(\Gamma\) it immediately follows that
\[
M(k, \epsilon) \leq N(k, \epsilon).
\]
The arguments in the proof of Theorem 3.4 yield that
\[
N(k, \epsilon) \leq M(k + K(\epsilon))
\]
and the lemma follows. \(\diamond\)

§4. Approximating entropy of graphs by entropy of subshifts of finite type

Let \(X\) be a set. \(U = \{U_1, ..., U_m\} \subset 2^X\) is called a finite cover of \(X\) if \(X = \bigcup_{i}^{m} U_i\). The cover \(U\) is called minimal if any strict subset of \(U\) is not a cover of \(X\). Let \(\Gamma \subset X \times X\) be any subset. Introduce the following graph and its corresponding matrix on the space \(< m > = \{1, ..., m\}:
\[
U = \{U_1, ..., U_m\}, \; \Gamma(U) = \{(i, j) : \Gamma \cap U_i \times U_j \neq \emptyset\} \subset < m > \times < m >, \\
A(\Gamma(U)) = (a_{ij})_{1 \leq i, j \leq m} \in M_m(\{0 - 1\}), \; a_{ij} = 1 \iff (i, j) \in \Gamma(U).
\]
Note that \(\Gamma(U)\) induces a subshift of a finite type on \(< m >\). Thus, \(\log^+ \rho(\Gamma(U))\) is the entropy of \(\Gamma\) induced by the cover \(U\). Let \(V\) be also a finite cover of \(X\). Then \(V\) is called a refinement of \(U\), written \(U < V\), if every member of \(V\) is a subset of a member of \(U\). Assume that \(V = \{V_1, ..., V_m\}\) is a refinement of \(U\) such that \(V_i \subset U_i, i = 1, ..., m\). It then follows that \(A(\Gamma(U)) \geq A(\Gamma(V))\) for any \(\Gamma \subset X \times X\). Hence, \(\rho(A(\Gamma(U))) \geq \rho(A(\Gamma(V)))\). If \(U_i \cap U_j = \emptyset, 1 \leq i < j \leq m\), then \(U\) is called a finite partition of \(X\). Given a finite minimal cover \(U = \{U_1, ..., U_m\}\) there always exist a partition \(V = \{V_1, ..., V_m\}\) such that \(V_i \subset U_i, i = 1, ..., m\). Indeed, consider a partition \(U'\) corresponding to the subalgebra generated by \(U\). This partition is a refinement of \(U\). Then each \(U_i\) is union of some sets in \(U'\). Set \(V_1 = U_1\). Let \(V_2 \subset U_2\) be the union of sets of \(U'\) which are subsets of \(U_2 \setminus U_1\). Continue this process to construct \(V\). In particular, \(\rho(A(T, U)) \geq \rho(A(T, V))\).

Let \(U < V\) be finite partitions of \(X\). Assume that \(\Gamma \subset X \times X\). In general, there is no relation between \(\rho(A(\Gamma(U)))\) and \(\rho(A(\Gamma(V)))\). Indeed, if \(A(\Gamma(V))\) is a matrix whose all entries are equal to 1 then \(A(\Gamma(U))\) is also a matrix whose all entries are equal to 1. Hence
\[
\rho(A(\Gamma(V))) = Card(V) > \rho(A(\Gamma(U))) = Card(U) \iff U \neq V.
\]
Assume now that \(Card(V) = n, A(\Gamma(V)) = (\delta_{(i+1)j})^n_0, n + 1 \equiv 1\) be the matrix corresponding to a cyclic graph on \(< n >\). Suppose furthermore that \(n \geq 3\) and let \(U_1 = V_1 \cup V_2, U_i = V_{i+1}, i = 2, ..., n - 1\). It then follows that \(\rho(A(\Gamma(U))) > \rho(A(\Gamma(V))) = 1\).

Let \(F_\epsilon, 0 < \epsilon < 1\) be a family of finite covers of \(X\) increasing in \(\epsilon\). That is, \(F_\delta \subset F_\epsilon, 0 < \delta < \epsilon < 1\). Assume that \(\Gamma \subset X \times X\) be any set. We then set
\[
e(\Gamma, F) = \lim_{\epsilon \to 0^+} \inf_{U \in F_\epsilon} \log^+ \rho(A(\Gamma(U))).
\]
Thus, $e(\Gamma, \mathcal{F})$ can be considered as the entropy of $\Gamma$ induced by the family $\mathcal{F}_\epsilon$. Its definition is reminiscent of the definition of the Hausdorff dimension of a metric space $X$. Let $\mathcal{U}$ be a finite cover of $X$. Clearly, $A(\Gamma^T(\mathcal{U})) = A^T(\Gamma(\mathcal{U}))$. Hence, $\rho(A(\Gamma(\mathcal{U}))) = \rho(A(\Gamma^T(\mathcal{U})))$ and $e(\Gamma, \mathcal{F}) = e(\Gamma^T, \mathcal{F})$.

(4.1) **Lemma.** Let $\mathcal{U} = \{U_1, \ldots, U_m\}$ be a finite cover of compact metric space $X$. Assume that $\operatorname{diam}(\mathcal{U}) \overset{\text{def}}{=} \max \operatorname{diam}(U_i) \leq \frac{\delta}{2}$. Let $\Gamma \subset X \times X$ be a closed set. Assume that $N_k(\delta)$ is the maximal cardinality of $(k, \delta)$ separated set for $\sigma : \Gamma_+ \to \Gamma_+^\infty$. Then

$$
\limsup_{k \to \infty} \frac{\log N_k(\delta)}{k} \leq \log \rho(A(\Gamma(\mathcal{U}))).
$$

**Proof.** Set

$$
A^k(\Gamma(\mathcal{U})) = (a^{(k)}_{ij})_1^m, \nu_k(\mathcal{U}) = \sum_{i=1}^{m} a^{(k-1)}_{ij}.
$$

Then $\nu_k(\mathcal{U})$ is counting the number of distinct point

$$(y_i)_1^k \in <m >^k, (y_i, y_{i+1}) \in \Gamma(\mathcal{U}), i = 1, \ldots, k - 1.
$$

Let $K(\delta)$ be defined as in the proof of Theorem 3.4. We claim that $N(k, \delta) \leq \nu_{k+K(\delta)}(\mathcal{U})$. Indeed, assume that $x^i = (x^i_j)_{j=1}^{\infty}, i = 1, \ldots, N(k, \delta)$, is a $(k, \delta)$ separated set. Then each $x^i$ generates at least one point $y^i = (y^i_1, \ldots, y^i_p) \in <m >^p$ as follows: $x^i_j \in U_{y^i_j}, j = 1, \ldots, p$. From (3.5) and the assumption that $\operatorname{diam}(\mathcal{U}) < \frac{\delta}{2}$ we deduce that for $p = k + K(\delta)$ $i \neq l \Rightarrow y^i \neq y^l$. Hence $N(k, \delta) \leq \nu_{k+K(\delta)}(\mathcal{U})$. As a point $x^i$ may generate more then one point $y^i$ in general we have strict inequality. Since $A(T, \mathcal{U})$ is a nonnegative matrix it is well known that

$$
K_1 \rho(A)^k \leq \nu_k \leq K_2 k^{m-1} \rho(A(T, \mathcal{U}))^k, k = 1, \ldots,
$$

See for example [F-S]. The above inequalities yield the lemma. □

Let $\{\mathcal{U}_1\}_1^\infty$ be sequence of finite open covers such $\operatorname{diam}(\mathcal{U}_1) \to 0$. Assume that $\Gamma \subset X \times X$ is closed. Then $\{\mathcal{U}_1\}_1^\infty$ is called an approximation cover sequence for $\Gamma$ if

$$
\lim_{i \to \infty} \log^+ \rho(A(\Gamma(\mathcal{U}_i))) = h(\Gamma).
$$

Note as $\rho(A^T) = \rho(A), \forall A \in M_n(\mathbb{C})$ and $h(\Gamma) = h(\Gamma^T)$ we deduce that $\{\mathcal{U}_1\}_1^\infty$ is also an approximation cover for $\Gamma^T$. Use Lemma 4.1 and (2.2) for finite graphs to obtain sufficient conditions for the validity of the inequality (2.2) for infinite graphs.

(4.2) **Corollary.** Let $X$ be a compact metric space and $\Gamma_j^T = \Gamma_j \subset X \times X, j = 1, \ldots, m$ be closed sets. Assume that there exist a sequence of open finite covers

$$
\{\mathcal{U}_1\}_1^\infty, \lim_{i \to \infty} \operatorname{diam}(\mathcal{U}_i) = 0,
$$

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which is an approximation cover for $\Gamma_1, \ldots, \Gamma_m$. Then

$$h(\bigcup_1^m \Gamma_j) \leq \log \sum_1^m e^{h(\Gamma_j)}.$$  

Let $Z$ be a compact metric space and $T : Z \to Z$ is a homeomorphism. Then $T$ is called expansive if there exists $\delta > 0$ such that

$$\sup_{n \in \mathbb{Z}} d(T^n(x), T^n(y)) > \delta, \forall x, y \in Z, x \neq y.$$  

A finite open cover $\mathcal{U}$ of $Z$ is called a generator for homeomorphism $T$ if for every bisequence $\{U_n\}_{-\infty}^{\infty}$ of members of $\mathcal{U}$ the set $\bigcap_{n=-\infty}^{\infty} T^{-n} \bar{U}_n$ contains at most one point of $X$. If this condition is replaced by $\bigcap_{n=-\infty}^{\infty} U_n$ then $\mathcal{U}$ is called a weak generator. A basic result due to Keynes and Robertson [K-R] and Reddy [Red] claims that $T$ is expansive iff $T$ has a generator iff $T$ has a weak generator. See [Wal, §5.6]. Moreover, $T$ is a factor of the restriction of a shift $S$ on a finite number of symbols to a closed $S$-invariant set $\Delta$ [Wal, Thm 5.24]. If $\Delta$ is a subshift of a finite type then $T$ is called FP. See [Fr] for the theory of FP maps. In particular, for any expansive $T$, $h(T) < \infty$.

Let $\Gamma \subset X \times X$ be a closed set such that $\Gamma^{\infty} \neq \emptyset$. Then $\Gamma$ is called expansive if

$$\sup_{n \in \mathbb{Z}} d(\sigma^n(x), \sigma^n(y)) > \delta, \forall x, y \in \Gamma^{\infty}, x \neq y$$

for some $\delta > 0$. A finite open cover $\mathcal{U}$ of $X$ is called a generator for $\Gamma$ if for every bisequence $\{U_n\}_{-\infty}^{\infty}$ of members of $\mathcal{U}$ the set

$$x = (x_n)_{-\infty}^{\infty} \in \Gamma^{\infty}, x_n \in \bar{U}_n, n \in \mathbb{Z}$$

contains at most one point of $\Gamma^{\infty}$. If this condition is replaced by $x_n \in U_n$ then $\mathcal{U}$ is called a weak generator. We claim that $\Gamma$ is expansive iff $\Gamma$ has a generator iff $\Gamma$ has a weak generator. Indeed, observe first that the condition that $\Gamma$ is expansive is equivalent to the assumption that $\sigma$ is expansive on $\Gamma^{\infty}$. Let $V_i = \pi_{n-1}^{-1} U_i \subset X^{\infty}, i = 1, \ldots, m$. That is, $V_i$ is an open cylindrical set in $X^{\infty}$ whose projection on the first coordinate is $U_i$ while on all other coordinates is $X$. Set $W_i = V_i \cap \Gamma^{\infty}, i = 1, \ldots, m$. It now follows that $W_1, \ldots, W_m$ is a standard set of generators for the map $\sigma : \Gamma^{\infty} \to \Gamma^{\infty}$.

Assume that $T : X \to X$ is expansive with the expansive constant $\delta$. It is known [Wal, Thm. 7.11] that

$$h(T) = \limsup_{k \to \infty} \frac{\log N(k, \delta_0)}{k}, \delta_0 < \frac{\delta}{4}.$$  

Thus, according to Lemma 4.1 $h(\Gamma) \leq \log \rho(A(\Gamma(\mathcal{U})))$ if $\Gamma$ is expansive with an expansive constant $\delta$ and $\text{diam}(\mathcal{U}) < \frac{\delta}{5}$. Assume that $T_i : X \to X, i = 1, \ldots, m$, are expansive maps. We claim that for $m > 1$ it can happen that $h(\bigcup_1^m \Gamma(T_i))$ is infinite. Let $T_1$ be Anosov
map on the 2-torus \( X \) in the standard coordinates. Now change the coordinates in \( X \) by a homeomorphism and let \( T_2 \) be Anosov with respect to the new coordinates. It is possible to choose a homeomorphism (which is not diffeo!) so that that \( T_2 \circ T_1 \) contains horseshoes of arbitrary many folds. Hence \( h(\Gamma(T_1) \cup \Gamma(T_2)) \geq h(T_2 \circ T_1) = \infty. \)

§5. Entropy of semigroups of Möbius transformations

Let \( X \subset \mathbb{CP}^n \) be an irreducible smooth projective variety of complex dimension \( n \). Assume that \( \Gamma \subset X \times X \) be a projective variety such that the projections \( \pi_{i,2} : \Gamma \to X, i = 1, 2 \) are onto and finite to one. Then \( \Gamma \) can be viewed as a graph of an algebraic function. In algebraic geometry such a graph is called a correspondence. Furthermore, \( \Gamma \) induces a linear operator

\[
\Gamma^* : H_{*,a}(X) \to H_{*,a}(X), \quad H_{*,a}(X) = \sum_{j=0}^{n} H_{2j,a}(X),
\]

\[
\Gamma^* : H_{2j,a}(X) \to H_{2j,a}(X), \quad j = 0, \ldots, n.
\]

Here, \( H_{2j,a}(X) \) is the homology generated by the algebraic cycles of \( X \) of complex dimension \( j \) over the rationals \( \mathbb{Q} \). Indeed, if \( Y \subset X \) is an irreducible projective variety then \( \Gamma^*(Y) = \pi_2^2(\pi_2^1(Y)) \). Let \( \rho(\Gamma^*) \) be the spectral radius of \( \Gamma^* \). Assume that first that \( \Gamma \) is irreducible. In [Fri3] we showed that \( h(\Gamma) \leq \log \rho(\Gamma^*) \). However our arguments apply also to the case \( \Gamma \) is reducible. We also conjectured in [Fri3] that in the case that \( \Gamma \) is irreducible we have the equality \( h(\Gamma) = \log \rho(\Gamma^*) \). We now doubt the validity of this conjecture. We will show that in the reducible case we can have a strict inequality \( h(\Gamma) < \log \rho(\Gamma^*) \). Let \( \Gamma_i \subset X \times X, i = 1, \ldots, m, \) be algebraic correspondences as above. Set \( \Gamma = \cup_m \Gamma_i \). Then

\[
\Gamma^* = \sum_{i=1}^{m} \Gamma_i^*, \quad h(\Gamma) \leq \log \rho(\sum_{i=1}^{m} \Gamma_i^*).
\]

Thus, there is a close analogy between the entropy of algebraic (finite to one) correspondences and entropy of shifts of finite types. Consider the simplest case of the above situation. Let \( X = \mathbb{CP}^1 \) be the Riemann sphere and \( \Gamma \) be an algebraic curve given by a polynomial \( p(x, y) = 0 \) on some chart \( \mathbb{C}^2 \subset \mathbb{CP}^1 \times \mathbb{CP}^1 \). Let \( d_1 = \deg_y(p), d_2 = \deg_x(p), d_1 \geq 1, d_2 \geq 1 \). It then follows that \( \rho(\Gamma^*) = \max(d_1, d_2). \) Note that \( \rho(\Gamma^*) = 1 \) iff \( \Gamma \) is the graph of a Möbius transformation. Observe next that if \( f_i : \mathbb{CP}^1 \to \mathbb{CP}^1, i = 1, \ldots, m, \) are nonconstant rational maps then the correspondence given by \( p(x, y) = \prod_{i=1}^{m} (y - f_i(x)) \) is induced by \( \Gamma = \cup_i \Gamma(f_i) \). In particular,

\[
h(\Gamma) \leq \log \sum_{i=1}^{m} \deg(f_i). \quad (5.1)
\]

Here, by \( \deg(f_i) \) we denote the topological degree of the map \( f_i \). Combine the above inequality with Lemma 3.6 to deduce that for any noninvolutive Möbius transformation \( f \) we have the equality \( h(\Gamma(f) \cup \Gamma(f^{-1})) = \log 2 \).
(5.2) Lemma. Let \( f, g : \mathbb{CP}^1 \to \mathbb{CP}^1 \) be two Möbius transformations such that \( x \) as a common fixed attracting point of \( f \) and \( g \) and \( y \) is a common repelling point of \( f \) and \( g \). Then \( h(\Gamma(f) \cup \Gamma(g)) = 0 \).

Proof. We may assume that

\[ f = az, g = bz, 0 < |a|, |b| < 1. \]

Set \( \Gamma = \Gamma(f) \cup \Gamma(g) \). It the follows that for any point \( \zeta = (z_i)_1^\infty \neq \eta = (\infty)_1^\infty \) converges to the fixed point \( \xi = (0)_1^\infty \). That is, the nonwondering set of \( \sigma \) is the set \( \{\xi, \eta\} \) on which \( \sigma \) acts trivially. Hence \( h(\Gamma) = 0. \diamondsuit \)

(5.3) Lemma. Let \( f, g : \mathbb{CP}^1 \to \mathbb{CP}^1 \) be two parabolic Möbius transformation with the same fixed point \( -\infty \), i.e. \( f = z + a, g = z + b \). If either \( a, b \) are linearly independent over \( \mathbb{R} \) or \( b = aa, a > 0, a \neq 0 \) then for any point \( \zeta \in \Gamma_\infty, \sigma^I(\zeta) \) converges to the fixed point \( \eta \). Hence \( h(\Gamma) = 0 \). Suppose next that \( a = b = 0 \). Then \( \sigma \) is the identity map on \( \Gamma_\infty \) and \( h(\Gamma) = 0 \). Assume finally that \( b = 0, a \neq 0 \). Then \( \Omega \) limit set of \( \sigma \) consists of all points \( \zeta = (z_i)_1^\infty, z_i = z_1, i = 2, \ldots \). So \( \sigma|\Omega \) is identity and \( h(\Gamma) = 0. \diamondsuit \)

(5.4) Theorem. Let \( T = z + a, Q = z + b, ab \neq 0 \) be two Möbius transformations of \( \mathbb{CP}^1 \). Assume that there \( \frac{b}{a} \) is a negative rational number. Then

\[ h(\Gamma) = -\frac{|a|}{|a| + |b|} \log \frac{|a|}{|a| + |b|} - \frac{|b|}{|a| + |b|} \log \frac{|b|}{|a| + |b|}. \]

We first state an approximation lemma which will be used later.

(5.5) Lemma. Let \( X \) be compact metric space and \( T : X \to X \) be a continuous transformation. Assume that we have a sequence of closed subsets \( X_i \subset X, i = 1, \ldots, \) which are \( T \)-invariant, i.e. \( T(X_i) \subset X_i, i = 1, 2, \ldots \). Suppose furthermore that \( \forall \delta > 0 \exists M(\delta) \) with the following property. \( \forall x \in X \setminus X_i \exists y = y(x,i) \in X_i, \sup_{n \geq 0} d(T^n(x), T^n(y)) \leq \delta \) for each \( i > M(\delta) \). Then \( \lim_{i \to \infty} h(T|X_i) = h(T) \).

Proof. Observe first that \( h(T) \geq h(T|X_i) \). Thus it is left to show

\[ \liminf_{i \to \infty} h(T|X_i) \geq h(T). \]

Let \( N(k, \epsilon), N_i(k, \epsilon) \) be the cardinality of maximal \((k, \epsilon)\) separating set of \( X \) and \( X_i \) respectively. Clearly, \( N_i(k, \epsilon) \leq N(k, \epsilon) \). Let \( x_1, \ldots, x_{N(k, \epsilon)} \) be a \((k, \epsilon)\) separating set of \( X \). Then

\[ \forall i > M(\frac{\epsilon}{4}), \forall x_j, \exists y_j, i \in X_i, \sup_{n \geq 0} d(T^n(x_j), T^n(y_j,i)) \leq \frac{\epsilon}{4}. \]
Hence, \(y_{j,i}, j = 1, ..., N(k, \epsilon),\) is \(\frac{\epsilon}{2}\) separated set in \(X_i\). In particular, \(N(k, \epsilon) \leq N_i(k, \frac{\epsilon}{2}), i > M(\frac{\epsilon}{4}).\) Thus
\[
\limsup_{k \to \infty} \frac{\log N(k, \epsilon)}{k} \leq \limsup_{k \to \infty} \frac{\log N_i(k, \frac{\epsilon}{2})}{k} \leq h(T|X_i), i > M(\frac{\epsilon}{4}).
\]

The characterization of \(h(T)\) yields the lemma. \(\diamondsuit\)

**Proof of Theorem 5.4.** W.l.o.g. (without loss of generality) we may assume that \(a = p, b = -q\) where \(p, q\) are two positive coprime integers. First note that \(\mathbb{CP}^1\) is foliated by the invariant lines \(\exists z = \text{Const}.\) Hence, the maximal characterization of \(h(\sigma)\) as the supremum over all measure entropy \(h_\mu(\sigma)\) for all extremal \(\sigma\) invariant measures yields that it enough to restrict ourselves to the action of \(T, Q\) on (closure of) the real line. Using the same argument again it is enough to consider the action on the lattice \(\mathbb{Z} \subset \mathbb{R}\) plus the point at \(\infty.\) We may view \(Y = \mathbb{Z} \cup \{\infty\}\) as a compact subspace of \(S^1 = \{z : |z| = 1\}.
\[
0 \mapsto 1, \infty \mapsto -1, j \mapsto e^{\frac{\sqrt{-1}(1+2j)}{43}}, 0 \neq j \in \mathbb{Z}.
\]
For a positive integer \(i\) let \(Y_i = \{-ipq, -ipq + 1, ..., ipq - 1, ipq\}.\) Set
\[
\Gamma = \Gamma(T) \cup \Gamma(Q) \subset Y \times Y, X = \Gamma^\infty, \Gamma_i = \Gamma \cap Y_i \times Y_i, X_i = (\Gamma_i)^\infty, i = 1, ...,\]

We will view a point \(x = (x_j)^\infty \in X\) a path of a particle who starts at time 1 at \(x_1\) and jumps from the place \(x_i\) at time \(i\) to the place \(x_{i+1}\) at time \(i+1.\) At each point of the lattice \(\mathbb{Z}\) a particle is allowed to jump \(p\) steps forward and \(q\) backwards. The point \(\xi = (\infty)^\infty\) is the fixed point of our random walk. Observe next that \(\Gamma_i\) is a subshift of a finite type on \(2ipq + 1\) points corresponding to the random walk in which a particle stays in the space \(Y_i.\) Note that \(A_i = A(\Gamma_i)\) is a matrix whose almost each row (column) sums to two, except the first and the last \(\max(p, q) - 1\) rows (columns). Moreover, \(h(\sigma|X_i) = \log \rho(A_i)\).

We claim that \(X, X_1, ...,\) satisfy the assumption of Lemma 5.5. That is any point \(x = (x_j)^\infty \in X\) can be approximated up to an arbitrary \(\epsilon > 0\) by \(y_i = (y_{j,i})_j^{\infty} \in X_i\) for \(i > M(\epsilon).\) We assume that \(i > L\) some fixed big \(L.\) Suppose first that \(x_j > ipq, j = 1, ...,\)
That is the path described by the vector \(x\) never enters \(X_i.\) Then consider the following path \(y_i = (y_{j,i})_j^{\infty} \in X_i.\) It starts at the point \(ipq, i.e.\) \(y_{1,i} = ipq.\) Then it jumps \(p\) times to the left to the point \((i-1)ipq.\) Then it the particle jumps \(q\) time to the right back to the the point \(ipq\) and so on. Clearly, \(\sup_{n \geq 0} d(\sigma^n(x), \sigma^n(y_i)) \leq d((i-1)ipq, \infty).\) Hence for \(i\) big enough the above distance is less than \(\epsilon.\) Same arguments apply to the case \(x_j < -ipq, j = 1, ...,\) Consider next a path \(x = (x_j)^\infty\) which starts outside \(X_i\) and then enters \(X_i\) at some time. If the particle enters to \(X_i\) and then stays for a short time, e.g. \(\leq pq,\) every time it enters \(X_i\) then we can approximate this path by a path looping around the vertex \(ipq\) or \(-ipq\) in \(X_i\) as above. Now suppose that we have a path which enters to \(X_i\) at least one time for a longer period of time. We then approximate this path by a path \((y_{j,i})_j^{\infty} \in X_i\) such that this path coincide with \(x\) for all time when \(x\) is in \(X_i\) except the short period when \(x\) leaves \(X_i.\) One can show that such path exists. (Start with the simple example \(p = 1, q = 2.\) It then follows that \(\sup_{n \geq 0} d(\sigma^n(x), \sigma^n(y_i)) \leq d((i - K)ipq, \infty)\) for
some $K = K(p, q)$. If $i$ is big enough then we have the desired approximation. Lemma 5.5 yields
\[ h(\Gamma) = \lim_{i \to \infty} \log \rho(A_i). \]

We now estimate $\log \rho(A_i)$ from above and from below. Recall the well known formula for the spectral radius of a nonnegative $n \times n$ matrix $A$:
\[ \rho = \limsup_{m \to \infty} \left( \frac{\text{trace}(A^m)}{m} \right)^{\frac{1}{m}} = \limsup_{m \to \infty} \left( \max_{1 \leq j \leq n} a^{(m)}_{jj} \right)^{\frac{1}{m}}, A^m = (a^{(m)}_{ij})_{1 \leq i, j \leq n}. \]

Let $A = A_i$. We now estimate $a^{(m)}_{jj}$. Obviously, $a^{(m)}_{jj}$ is positive if $m = (p + q)k$ as we have to move $kq$ times to the right and $kp$ times to the left. Assume that $m = (p + q)k$. To estimate $a^{(m)}_{jj}$ we assume that we have an uncostrained motion on $Z$. Then the number of all possible moves on $Z$ bringing us back to the original point is equal to
\[ \frac{((p + q)k)!}{(qk)!(pk)!} \leq K \sqrt{p + q} \frac{(p + q)^{(p+q)k}}{q^{qk}p^{pk}}. \]

The last part of inequality follows from the Stirling formula for some suitable $K$. The characterization of $\rho(A)$ gives the inequality
\[ \log \rho(A_i) \leq \log \alpha = \log(p + q) - \frac{p}{p + q} \log p - \frac{q}{p + q} \log q. \]

We thus deduce the upper bound on $h(\Gamma) \leq \log \alpha$. Let $0 < \delta < \alpha$. The Stirling formula yields that for $k > M(\delta)$
\[ \frac{((p + q)k)!}{(qk)!(pk)!} \geq (\alpha - \delta)^{(p+q)k}. \]

Fix $k > M(\delta)$ and let $i > k$. Then for $m = (p + q)k$
\[ a^{(m)}_{00} = \frac{((p + q)k)!}{(qk)!(pk)!}. \]

Clearly,
\[ \rho(A)^m = \rho(A^m) \geq a^{(m)}_{00}. \]

Thus, $h(\Gamma) \geq \log \rho(A_i) \geq \log(\alpha - \delta)$. Let $\delta \to 0$ and deduce the theorem. \(\diamond\)

Note that $h(\Gamma)$ is the entropy of the Bernoulli shift on two symbols with the distribution $\left( \frac{p}{p+q}, \frac{q}{p+q} \right)$. This can be explained by the fact that to have a closed orbit of length $k(p + q)$ we need move to the right $kq$ times and to the left $kp$. That is, the frequency of the right motion is $\frac{q}{p+q}$ and the left motion is $\frac{p}{p+q}$. It seems that Theorem 5.4 remains valid as long as $\frac{q}{p}$ is a real negative number.

(5.6) Theorem. Let $f, g : \mathbb{CP}^1 \to \mathbb{CP}^1$ be two parabolic Möbius transformations with the same fixed point $- \infty$, i.e. $f = z + a, g = z + b$ where $a, b$ are linearly independent over $\mathbb{R}$. Let $\Gamma = \Gamma(f) \cup \Gamma(f^{-1}) \cup \Gamma(g) \cup \Gamma(g^{-1})$. Then $h(\Gamma) = \log 4$. 

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Proof. The orbit of any fixed point \( z \in \mathbb{C} \) under the action of the group generated by \( f, g \) is a lattice in \( \mathbb{C} \) which has one accumulation point \( \infty \in \mathbb{CP}^1 \). Let \( Y \) is defined in the proof of Theorem 5.4. Consider the dynamics of \( \sigma \times \sigma \) on \( Y_j \times Y_j \), for \( j = 1, ..., \) as in the proof Theorem 5.4. It then follows that \( h(\Gamma) = 2h(\sigma | X) = 2 \log 2. \)

Let \( T = \{ f_1, ..., f_k \} \) be a set of \( k \) - Möbius transformations. Set \( \Gamma = \bigcup_{1}^{k} \Gamma(f_i) \). Then (5.1) yields \( h(\Gamma) \leq \log k \). Our examples show that we may have a strict inequality even for the case \( k = 2 \). Let \( \Gamma \) be the correspondence of the Gauss arithmetic-geometric mean \( y^2 = \frac{(x+1)^2}{4x} \) [Bul2]. Our inequality in [Fri3] yield that \( h(\Gamma) \leq \log 2 \). According to Bullet [Bul2] it is possible to view the dynamics of \( \Gamma \) as a factor of the dynamics of \( \tilde{\Gamma} = \Gamma(f_1) \cup \Gamma(f_2) \) for some two Möbius transformations \( f_1, f_2 \). Hence, \( h(\Gamma) \leq h(\tilde{\Gamma}) \). If \( h(\tilde{\Gamma}) < \log 2 \) we will have the inequality \( h(\Gamma) < \log 2 \) as the dynamics of \( \Gamma \) is a subfactor of the dynamics of \( \tilde{\Gamma} \). Thus, it would be very interesting to compute \( h(\Gamma) \).

Assume that \( T \) generates nonelementary Kleinian group. Theorem 2.5 suggests that \( e^{h(\Gamma)} \) may have a noninteger value. It would be very interesting to find such a Kleinian group.

We now state an open problem which is inspired by Furstenberg’s conjecture [Fur]. Assume that \( 1 < p < q \) are two co-prime integers. (More generally \( p^m = q^n \Rightarrow m = n = 0 \).) Let

\[
f, g : \mathbb{CP}^1 \to \mathbb{CP}^1, T_1(z) = z^p, T_2(z) = z^q, z \in \mathbb{C}, f(\infty) = g(\infty). = \infty
\]

Note that for \( f \) and \( g \) 0, \( \infty \) are two attractive points with the interior and the exterior of the unit disk as basins of attraction respectively. Thus, the nontrivial dynamics takes place on the unit circle \( S^1 \). Note that \( f \circ g = g \circ f \). Hence \( f \) and \( g \) have common invariant probability measures. Let \( \mathcal{M} \) be the convex set of all probability measures invariant under \( f, g \). Denote by \( \mathcal{E} \subset \mathcal{M} \) the set of the extreme points of \( \mathcal{M} \) in the standard \( w^* \) topology. Then \( \mathcal{E} \) is the set of ergodic measures with respect to \( f, g \). (For a recent discussion on the common invariant measure of a semigroup of commuting transformation see [Fri4]). Furstenberg’s conjecture (for \( p = 2, q = 3 \)) is that any ergodic measure \( \mu \in \mathcal{E} \) is either supported on a finite number of points or is the Lebesgue (Haar) measure on \( S^1 \). See [Rud] and [K-S] for the recent results on this conjecture. Let \( \mathcal{G} \) be the semigroup generated by \( T = \{ f, g \} \). Then (0.2) for \( X = S^1 \) or the results of [Fri3] yield the inequality \( h(\mathcal{G}(T)) \leq \log(p + q) \). What is the value of \( h(\mathcal{G}(T)) \)? It is plausible to conjecture equality in this inequality.

References


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