Entropy of graphs, semigroups and groups

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August 7, 1995

§0. Introduction

Let X be a compact metric space and $T : X \to X$ is continuous transformation. Then the dynamics of T is a widely studied subject. In particular, $h(T)$ - the entropy of T is a well understood object. Let $\Gamma \subset X \times X$ be a closed set. Then Γ induces certain dynamics and entropy $h(\Gamma)$. If X is a finite set then Γ can be naturally viewed as a directed graph. That is, if $X = \{1, ..., n\}$ then Γ consists of all directed arcs $i \to j$ so that $(i, j) \in \Gamma$. Then Γ induces a subshift of finite type which is a widely studied subject. However, in the case that X is infinite, the subject of dynamic of Γ and its entropy are relatively new. The first paper treating the entropy of a graph is due to [**Gro**]. In that context X is a compact Riemannian manifold and Γ can be viewed as a Riemannian submanifold. (Actually, Γ can have singularities.) We treated this subject in [**Fri1-3**]. See Bullet [**Bul1-2**] for the dynamics of quadratic correspondences and [**M-R**] for iterated algebraic functions.

The object of this paper is to study the entropy of a corresponding map induced by Γ. We now describe briefly the main results of the paper. Let X be a compact metric space and assume that $\Gamma \subset X \times X$ is a closed set. Set

$$\Gamma_+^\infty = \{(x_i)_1^\infty : (x_i, x_{i+1}) \in \Gamma, i = 1, ..., \}.$$

Let $\sigma : \Gamma_+^\infty \to \Gamma_+^\infty$ be the shift map. Denote by $h(\Gamma)$ be the topological entropy of $\sigma\big|\Gamma_+^\infty$. It then follows that σ unifies in a natural way the notion of a (continuous) map $T : X \to X$ and a (finitely generated) semigroup or group of (continuous) transformations $\mathcal{S} : X \to X$. Indeed, let $T_i : X \to X, i = 1, ..., m$, be m continuous transformations. Denote by $\Gamma(T_i)$ the graphs corresponding to $T_i, i = 1, ..., m$. Set $\Gamma = \cup_1^m \Gamma(T_i)$. Then the dynamics of σ is the dynamics of the semigroup generated by $\mathcal{T} = \{T_1, ..., T_m\}$. If $\mathcal{T}$ is a set of homeomorphisms and $\mathcal{T}^{-1} = \mathcal{T}$ then the dynamics of σ is the dynamics of the group $\mathcal{G}(\mathcal{T})$ generated by $\mathcal{T}$. In particular, we let $h(\mathcal{G}(\mathcal{T})) = h(\Gamma)$ be the entropy of $\mathcal{G}(\mathcal{T})$ using the particular set of generators $\mathcal{T}$. For a finitely generated group $\mathcal{G}$ of homeomorphisms of X we define

$$h(\mathcal{G}) = \inf_{\mathcal{T}, \mathcal{G} = \mathcal{G}(\mathcal{T})} h(\mathcal{G}(\mathcal{T})).$$

In the second section we study the entropy of graphs, semigroups and groups acting on the finite space X. The results of this section give a good motivation for the general case. In particular we have the following simple inequality

$$h(\cup_{i=1}^m \Gamma_i) \le h(\cup_{i=1}^m (\Gamma_i \cup \Gamma_i^T)) \le \log \sum_{i=1}^m e^{h(\Gamma_i \cup \Gamma_i^T)}. \tag{0.1}$$

Here $\Gamma^T = \{(y,x) : (x,y) \in \Gamma\}$. Let $Card(X) = n$. Then any group of homeomorphisms $\mathcal{G}$ of X is a subgroup of the symmetric group S_n acting on X as a group of permutations. We then show that if $\mathcal{G}$ is commutative then $h(\mathcal{G}) = \log k$ for some integer k. If $\mathcal{G}$ acts transitively on X then k is the minimal number of generators for $\mathcal{G}$. Moreover, $h(\mathcal{G}) = 0$ iff $\mathcal{G}$ is a cyclic group. For each $n \geq 3$ we produce a group $\mathcal{G}$ generated by two elements so that $0 < h(\mathcal{G}) < \log 2$.

In §3 we discuss the entropy of graphs on compact metric spaces. We show that if $T_i : X \to X, i = 1, ..., m$, is a set of Lipschitzian transformations of a compact Riemannian manifold X of dimension n then

$$h(\cup_1^m \Gamma(T_i)) \leq \log \sum_1^m L_+(T_i)^n. \tag{0.2}$$

Here, $L_+(T_i)$ is the maximum of the Lipschitz constant of T_i and 1. Thus, $L_+(T_i)^n$ is analogous to the norm of a graph on a finite space X. The above inequality generalizes to semi-Riemannian manifolds which have a Hausdorff dimension $n \in \mathbf{R}_+$ and a finite volume with respect to a given metric d on X. Thus, if X is a compact smooth Riemannian manifold and $\mathcal{G}$ is a finitely generated group of diffeomorphisms (0.2) yields that $h(\mathcal{G}) < \infty$. Let X be a compact metric space and $T : X \to X$ a noninvolutive homeomorphism ($T^2 \neq Id$). We then show that $h(\Gamma(T) \cup \Gamma(T^{-1})) \geq \log 2$. The following example due to M. Boyle shows that (0.1) does not apply in general. Let X be a compact metric space for which there exists a homeomorphism $T : Y \to Y$ with $h(T) = h(T^2) = \infty$. (See for example [**Wal**, p. 192].) Set

$$X = X_1 \cup X_2, X_1 = Y, X_2 = Y, T_i(X_1) = X_2, T_i(X_2) = X_1,$$
$$T_1(x_1) = Tx_1, T_1(x_2) = T^{-1}x_2, T_2(x_1) = T^{-1}x_1, T_2(x_2) = Tx_2, x_1 \in X_1, x_2 \in X_2.$$

As $T_1^2 = T_2^2 = Id$ it follows that $2h(T_1) = 2h(T_2) = h(Id) = 0$. Clearly, $T_2T_1\big|X_1 = T^2$ and $h(\Gamma) = \infty$. The last section discusses mainly the entropy of semigroups and groups of Möbius transformations on the Riemann sphere. Let $\mathcal{T} = \{T_1, ..., T_k\}$ is a set of Möbius transformations. Inequality (0.2) yield that $h(\mathcal{G}(\mathcal{T})) \leq \log k$. Let $T_i(z) = z + a_i, i = 1, 2$, be two translations of $\mathbf{C}$. Assume that $\frac{a_1}{a_2}$ is a negative rational number. We then show that

$$h(\Gamma(T_1) \cup \Gamma(T_2)) = -\frac{|a|}{|a|+|b|} \log \frac{|a|}{|a|+|b|} - \frac{|b|}{|a|+|b|} \log \frac{|b|}{|a|+|b|}.$$

Assume now that a_1 and a_2 are linearly independent over $\mathbf{R}$. We then show that

$$h(\cup_1^2(\Gamma(T_i) \cup \Gamma(T_i^{-1})) = \log 4.$$

It is of great interest to see if $h(\mathcal{G})$ has any geometric meaning for a finitely generated Kleinian group $\mathcal{G}$. Consult with [**G-L-W**], [**L-W**], [**N-P**], [**L-P**] and [**Hur**] for other definitions of the entropy of relations and foliations.

§1. Basic definitions

Let X be a compact metric space and assume that $\Gamma \subset X \times X$ is a closed set. Set

$$X^k = \prod_1^k X_i, X_+^\infty = \prod_1^\infty X_i, X^\infty = \prod_{i \in \mathbf{Z}} X_i, X_i = X, i \in \mathbf{Z},$$
$$\Gamma^k = \{(x_i)_1^k : (x_i, x_{i+1}) \in \Gamma, i = 1, ..., k-1, \}, k = 2, ...,$$
$$\Gamma_+^\infty = \{(x_i)_1^\infty : (x_i, x_{i+1}) \in \Gamma, i = 1, ..., \}, \ \Gamma^\infty = \{(x_i)_{i \in \mathbf{Z}} : (x_i, x_{i+1}) \in \Gamma, i \in \mathbf{Z}\}.$$

We shall assume that $\Gamma^k \neq \emptyset, k = 2, ...,$ unless stated otherwise. (In any case, if this assumption does not hold we set $h(\Gamma) = 0$.) This in particular implies that $\Gamma_+^\infty \neq \emptyset, \Gamma^\infty \neq \emptyset$. Let

$$\pi_{p,q}^l : X^l \to X^{q-p+1}, \{x_i\}_1^l \mapsto \{x_i\}_p^q, 1 \leq p \leq q \leq l.$$

If no ambiguity arise we shall denote $\pi_{p,q}^l$ by $\pi_{p,q}$. The maps $\pi_{p,q}$ are well defined for X_+^∞, X^∞. For $p \leq 0, p \leq q$ we let $\pi_{p,q} : X^\infty \to X^{q-p+1}$. Similarly, for a finite p we have the obvious maps $\pi_{-\infty,p}, \pi_{p,\infty}$ whose range is Γ_+^∞. Let $d : X \times X \to \mathbf{R}_+$ be a metric on X. As X is compact we have that X is a bounded diameter $0 < D < \infty$. That is, $d(x,y) \leq D, \forall x, y \in X$. On $X^k, X_+^\infty, X^\infty$ one has the induced metric

$$d(\{x_i\}_1^k, \{y_i\}_1^k) = \max_{1 \leq i \leq k} \frac{d(x_i, y_i)}{\rho^{i-1}},$$
$$d(\{x_i\}_1^\infty, \{y_i\}_1^\infty) = \sup_{1 \leq i} \frac{d(x_i, y_i)}{\rho^{i-1}},$$
$$d(\{x_i\}_{i \in \mathbf{Z}}, \{y_i\}_{i \in \mathbf{Z}}) = \sup_{i \in \mathbf{Z}} \frac{d(x_i, y_i)}{\rho^{|i-1|}}.$$

Here $\rho > 1$ to be fixed later. Since X is compact it follows that $X^k, X_+^\infty, X^\infty$ are compact metric spaces where the infinite products have the Tychonoff topology. Let

$$\sigma : X_+^\infty \to X_+^\infty, \sigma((x_i)_1^\infty) = (x_{i+1})_1^\infty,$$
$$\sigma : X^\infty \to X^\infty, \sigma((x_i)_{i \in \mathbf{Z}}) = (x_{i+1})_{i \in \mathbf{Z}}$$

be the one sided shift and two sided shift respectively. We refer to Walters [**Wal**] for the definitions and properties of dynamical systems used here. Note that $\Gamma_+^\infty, \Gamma^\infty$ are invariant subsets of one sided and two sided shifts, i.e.

$$\sigma : \Gamma_+^\infty \to \Gamma_+^\infty, \sigma : \Gamma^\infty \to \Gamma^\infty.$$

We call the above restrictons of σ as the dynamics (maps) induced by Γ. As Γ was assumed to be closed it follows that $\Gamma_+^\infty, \Gamma^\infty$ are closed too. Hence, we can define the topological entropies $h(\sigma|\Gamma_+^\infty), h(\sigma|\Gamma^\infty)$ of the corresponding restrictions. We shall show that these two entropies are equal. The above entropy is $h(\Gamma)$.

Denote by $C(X)$ the Banach space of all continuous functions $f : X \to \mathbf{R}$. For $f \in C(X)$ it is possible to define the topological pressure $P(\Gamma, f)$ as follows. First observe that f induces the following continuous functions

$$f_1 : \Gamma_+^\infty \to \mathbf{R}, f_1((x_i)_1^\infty) = f(x_1),$$
$$f_2 : \Gamma^\infty \to \mathbf{R}, f_2((x_i)_{i\in\mathbf{Z}}) = f(x_1).$$

Let $P(\sigma, f_1), P(\sigma, f_2)$ be the topological pressures of f_1, f_2 with respect to the map σ acting on $\Gamma_+^\infty, \Gamma^\infty$ respectively. We shall show that the above topological pressures coincide. We then let $P(\Gamma, f) = P(\sigma, f_1) = P(\sigma, f_2)$.

Let $T : X \to X$ be a continuous map. Set $\Gamma = \Gamma(T) = \{(x, y) : x \in X, y = T(x)\}$ be the graph of T. Denote by $h(T)$ the topological entropy of T. It then follows that $h(T) = h(\Gamma)$. Indeed, observe that $x \mapsto orb_T(x) = (T^{i-1}(x))_1^\infty$ induces a homeomorphism $\phi : X \to \Gamma(T)_+^\infty$ such that $T = \phi^{-1} \circ \sigma \circ \phi$ and the equality $h(T) = h(\sigma\big|\Gamma_+^\infty)$ follows. Similarly, for $f \in C(X)$ we have the equality $P(T, f) = P(\sigma, f_1) = P(\Gamma(T), f)$.

Let $\Gamma_\alpha, \alpha \in \mathcal{A}$ be a family of closed graphs in $X \times X$. Set

$$\vee_{\alpha\in\mathcal{A}}\Gamma_\alpha = Closure(\cup_{\alpha\in\mathcal{A}}\Gamma_\alpha).$$

Note that if $\mathcal{A}$ is finite then $\vee\Gamma_\alpha = \cup\Gamma_\alpha$. The dynamics of $\Gamma = \vee\Gamma_\alpha$ is called the product dynamics induced by $\Gamma_\alpha, \alpha \in \mathcal{A}$. Let $T_\alpha : X \to X, \alpha \in \mathcal{A}$ be a set of continuous maps. Set

$$\mathcal{T} = \cup_{\alpha\in\mathcal{A}}T_\alpha, \Gamma(\mathcal{T}) = Closure(\cup_{\alpha\in\mathcal{A}}\Gamma(T_\alpha)).$$

Then the dynamics of $\Gamma(\mathcal{T})$ is the dynamics of a semigroup $\mathcal{S}(\mathcal{T})$ generated by $\mathcal{T}$. If each $T_\alpha, \alpha \in \mathcal{A}$ is a homeomorphism and $\mathcal{T}^{-1} = \mathcal{T}$ then the dynamics of $\Gamma(\mathcal{T})$ is the dynamics of a group $\mathcal{G}(\mathcal{T})$ generated by $\mathcal{T}$. Note that for a fixed $x \in X$ the orbit of x is given by the formula

$$orb_\mathcal{T}(x) = \{(x_i)_1^\infty, x_1 = x, x_i \in Closure(T_{\alpha_{i-1}} \circ \cdots \circ T_{\alpha_1}(x)), \alpha_1, ..., \alpha_{i-1} \in \mathcal{A}, i = 2, ..., \}.$$

If $\mathcal{A}$ is finite then we can drop the closure in the above definition.

Let $\mathcal{T}$ be a set of continuous transformations of X as above. We then define

$$h(\mathcal{S}(\mathcal{T})) = h(\Gamma(\mathcal{T})), \ \ P(\mathcal{S}(\mathcal{T}), f) = P(\Gamma(\mathcal{T}), f), f \in C(X)$$

to be the entropy of $\mathcal{S}(\mathcal{T})$ and the topological pressure of f with respect to the set of generators $\mathcal{T}$. In order to ensure that the above quantities are finite we shall assume that $\mathcal{T}$ is a finite set. Given a finitely generated semigroup $\mathcal{S}$ of $T : X \to X$ let

$$h(\mathcal{S}) = \inf_{\mathcal{T}, \mathcal{S}=\mathcal{S}(\mathcal{T})} h(\mathcal{S}(\mathcal{T})), \ P(\mathcal{S}, f) = \inf_{\mathcal{T}, S=S(\mathcal{T})} P(S(\mathcal{T}), f), f \in C(X).$$

Here, the infimum is taken over all finite generators of $\mathcal{S}$.

§2. Entropy of graphs on finite spaces

Let X be a finite space. We assume that $X = \{1, ..., n\}$. Then each $\Gamma \subset X \times X$ is in one to one correspondence with a $n \times n$ $0-1$ matrix $A = (a_{ij})_1^n$. That is $(i,j) \in \Gamma \iff a_{ij} = 1$. As usual we let $M_n(\{0-1\})$ be the set of $0-1$ $n \times n$ matrices. For $\Gamma \subset X \times X$ we let $A(\Gamma) \in M_n(\{0-1\})$ to be the matrix induced by Γ and for $A \in M_n(\{0-1\})$ we let $\Gamma(A)$ to be the graph induced by A. The assumption that $\Gamma^k \neq \emptyset, k = 1, 2, ...,$ is equivalent to $\rho(A(\Gamma)) > 0 \iff \rho(A(\Gamma)) \geq 1$. Here, for any A in the set of $n \times n$ complex valued matrices $M_n(\mathbf{C})$ we let $\rho(A)$ to be the spectral radius of A. For $\Gamma \subset X \times X$ consider the sets $X_l = \pi_{l,l}(\Gamma^l), l = 2, ...,$. It easily follows that $X_2 \supset X_3 \supset \cdots X_n = X_{n+1} = \cdots = X'$. Then $\Gamma^l \neq \emptyset, l = 2, ...,$ iff $X' \neq \emptyset$. Set $\Gamma' = \Gamma \cap X' \times X'$. It then follows that $\Gamma^\infty = \Gamma'^\infty$. Moreover,

$$\pi_{1,\infty}(\Gamma^\infty) = \pi_{1,\infty}(\Gamma'^\infty) = \Gamma'^\infty_+ \subset \Gamma^\infty_+.$$

Here the containment is strict iff $X' \neq X$. It is well known fact in symbolic dynamics that if $X' \neq \emptyset$ then

$$h(\sigma|\Gamma^\infty_+) = h(\sigma|\Gamma^\infty) = \log \rho(A(\Gamma)) = \log \rho(A(\Gamma')) = h(\sigma|\Gamma'^\infty) = h(\sigma|\Gamma'^\infty_+).$$

See for example [**Wal**]. We thus let $h(\Gamma)$ - the entropy of the graph Γ to be any of the above numbers. In fact, X' can be viewed as a limit set of the "transformation" induced by Γ on X'. If $\rho(A(\Gamma)) = 0$, i.e. $X' = \emptyset$ we then let $h(\Gamma) = \log^+ \rho(A(\Gamma))$. Here, $\log^+ x = \log \max(x, 1)$.

Let $\Gamma_\alpha \subset X \times X, \alpha \in \mathcal{A}$ be a family of graphs. Set $A_\alpha = (a_{ij}^{(\alpha)})_1^n = A(\Gamma_\alpha), \alpha \in \mathcal{A}$. It then follows that

$$\vee_{\alpha \in \mathcal{A}} A_\alpha \stackrel{\text{def}}{=} (\max_{\alpha \in \mathcal{A}} a_{ij}^{(\alpha)})_1^n = A(\vee_{\alpha \in \mathcal{A}} \Gamma_\alpha).$$

The Perron-Frobenius theory of nonnegative matrices yields straightforward that $\rho(A_\alpha) \leq \rho(\vee A_\beta)$. This is equivalent to the obvious inequality $h(\Gamma_\alpha) \leq h(\vee \Gamma_\beta)$. We now point out that we can not obtain an upper bound on $h(\vee \Gamma_\alpha)$ as a function of $h(\Gamma_\alpha), \alpha \in \mathcal{A}$. It suffices to pass to the corresponding matrices and their spectral radii. Let $A = (a_{ij})_1^n \in M_n(\{0-1\})$ matrix such that $a_{ij} = 1 \iff i \leq j$. Assume that $B = A^T$. Then $\rho(A) = \rho(B) = 1, \rho(A \vee B) = n$.

Let $\|\cdot\| : \mathbf{C}^n \to \mathbf{R}_+$ be a norm on $\mathbf{C}^n$. Denote by $\|\cdot\| : M_n(\mathbf{C}) \to \mathbf{R}_+$ the induced operator norm. Clearly, $\rho(A) \leq \|A\|$. Hence

$$\rho(\vee_{\alpha \in \mathcal{A}} A_\alpha) \leq \rho(\sum_{\alpha \in \mathcal{A}} A_\alpha) \leq \sum_{\alpha \in \mathcal{A}} \|A_\alpha\|.$$

Thus

$$h(\vee_{\alpha \in \mathcal{A}} \Gamma_\alpha) \leq \log^+ \sum_{\alpha \in \mathcal{A}} \|A_\alpha\|. \tag{2.1}$$

In the next section we shall consider analogs of $\|A(\Gamma)\|$ for which we have the inequality (2.1) for any set $\mathcal{A}$. For a graph $\Gamma \subset X \times X$ let $\Gamma^T = \{(x, y) : (y, x) \in \Gamma\}$. That is, $A(\Gamma^T) = A^T(\Gamma)$. A graph Γ is symmetric if $\Gamma^T = \Gamma$. Assume that Γ is symmetric.

It then follows that $\rho(A(\Gamma)) = \|A(\Gamma)\|$ where $\|\cdot\|$ is the spectral norm on $M_n(\mathbf{C})$, i.e. $\|A\| = \rho(AA^*)^{\frac{1}{2}}$. Thus, for a family $\Gamma_\alpha, \alpha \in \mathcal{A}$ of symmetric graphs we have the inequalites

$$h(\vee_{\alpha\in\mathcal{A}}\Gamma_\alpha) \le \log \sum_{\alpha\in\mathcal{A}} e^{h(\Gamma_\alpha)}. \tag{2.2}$$

More generally, for any family of graphs we have the inequalites

$$h(\vee_{\alpha\in\mathcal{A}}\Gamma_\alpha) \le h(\vee_{\alpha\in\mathcal{A}}(\Gamma_\alpha \vee \Gamma_\alpha^T)) \le \log \sum_{\alpha\in\mathcal{A}} e^{h(\Gamma_\alpha\vee\Gamma_\alpha^T)}. \tag{2.3}$$

Let $T : X \to X$ be a transformation. Then $A(T) = A(\Gamma(T))$ is a $0-1$ stochastic matrix, i.e. each row of $A(T)$ contains exactly one 1. Vice versa, if $A \in M_n(\{0-1\})$ is a stochastic matrix then $A = A(T)$ for some transformation $T : X \to X$. Furthermore, $T : X \to X$ is a homeomorphism iff $A(T)$ is a permutation matrix. For $\mathcal{T} = \{T_1, ..., T_k\}$ $\mathcal{S}(\mathcal{T})$ is a group iff each T_i is a homeomorphism, i.e. $A(T_i)$ is a permutation matrix for $i = 1, ..., k$. Clearly, any group of homeomorphisms $\mathcal{S}$ of X is a subgroup of the symmetric group $S_n, n = Card(X)$.

(2.4) Theorem. *Let X be a finite space and assume that $T_i : X \to X, i = 1, ..., k,$ be a set of transformation. Set*

$$\mathcal{T} = \{T_1, ..., T_k\}, \Gamma = \Gamma(\mathcal{T}) = \cup_1^k \Gamma(T_i), A = A(\Gamma).$$

Then $h(\mathcal{S}(\mathcal{T})) \le \log k$. Furthermore, $h(\mathcal{S}(\mathcal{T})) = 0$ iff $A(\Gamma')$ is a permutation matrix. Assume that $k \ge 2$. Then $h(\mathcal{S}(\mathcal{T})) = \log k$ iff there exists an irreducible component $\hat{X} \subset X'$ on which $\mathcal{S}(\mathcal{T})$ acts transitively such that $A(\Gamma \cap \hat{X} \times \hat{X})$ is $0-1$ matrix with k ones in each row. In particular, $h(\mathcal{S}(\{T, T^{-1}\})) = \log 2$ for $T^2 \ne Id$. Assume finally that $\mathcal{S}(\mathcal{T})$ is a commutative group. Then $h(\mathcal{S}(\mathcal{T})) = \log k'$ for some integer $1 \le k' \le k$.

Proof. Recall that $h(\mathcal{S}(\mathcal{T})) = \log \rho(A)$. As $A(T_i)$ is a stochasic matrix it follows that $\rho(A(T_i)) = 1, i = 1, ..., k$. Since $A \ge A(T_i)$ we deduce that $\rho(A) \ge 1$. Thus, $X' \ne \emptyset$. Then $X' = \cup_1^m X_i, X_i \cap X_j = \emptyset, 1 \le i < j \le m$. Here, A acts transitively on each X_i. Set $\Gamma_i = \Gamma \cap X_i \times X_i, A_i = A(\Gamma_i), i = 1, ..., m$. Note that each A_i is an irreducible matrix. It then follow that $h(\Gamma) = \max \log \rho(A_i)$. Set $u_i : X_i \to \{1\}$. Then $A_i u_i \le k u_i$. The *minmax* characterization of Wielandt for an irreducible A_i yields that $\rho(A_i) \le k$. The equality holds iff each row of A_i has exactly k ones. Thus, $h(\Gamma) = \log k, k > 1$ iff each row of some A_i has k ones.

Assume next that T is a homeomorphism such that $T^2 \ne Id$. Set $\Gamma = \Gamma(T) \cup \Gamma(T^{-1})$. Then $X' = X = \cup_1^m X_i$ and least one X_i contains more then one point. Clearly, this A_i has two ones in each row and column. Hence, $h(\Gamma) = \log 2$.

Assume now that $\mathcal{G} = \mathcal{S}(\mathcal{T})$ is a commutative group. Then $X = X' = \cup_1^m X_l$. We claim that the following dichotomy holds for each pair $T_i, T_j, i \ne j$. Either $T_i(x) \ne T_j(x) \forall x \in X_l$ or $T_i(x) = T_j(x) \forall x \in X_l$. Indeed, assume that $T_i(x) = T_j(x)$ for some $x \in X_l$. As $\mathcal{G}$ acts transitively on X_l and is commutative we deduce that $T_i(x) = T_j(x) \forall x \in T_l$.

Thus $\Gamma(T_i) \cap X_l \times X_l, i = 1, ..., k$, consists of k_l distinct permutation matrices which do not have any 1 in common. That is $\Gamma_l = \Gamma \cap X_l \times X_l$ is a matrix with k_l ones in each row and column. Hence,

$$h(\Gamma_l) = \log k_l, l = 1, ..., m, h(\Gamma) = \log \max_{1 \leq l \leq m} k_l.$$

$\diamond$

(2.5) Theorem. *Let X be a finite space of n points. If $\mathcal{G}$ is commutative then $h(\mathcal{G}) = \log k$ for some integer k which is not greater then the number of the minimal generators of $\mathcal{G}$. If $\mathcal{G}$ acts transitively on X or the restriciton of $\mathcal{G}$ to one of the irreducible (transitive) components is faithful then k is the minimal number of generators of $\mathcal{G}$. In particular, for any $\mathcal{G}$ $h(\mathcal{G}) = 0$ iff $\mathcal{G}$ is cyclic. For each $n \geq 3$ there exists a group $\mathcal{G}$ which acts transitively on X so that $0 < h(\mathcal{G}) < \log 2$.*

Proof. Assume first that $\mathcal{G}$ is commutative. Let $\mathcal{T} = \{T_1, ..., T_p\}$ be a set of generators. Theorem 2.4 yields that $h(\mathcal{G}(\mathcal{T})) = \log k(\mathcal{T}), k(\mathcal{T}) \leq p$. Choose a minimal subset of generators $\mathcal{T}' \subset \mathcal{T}$. Clearly, $h(\mathcal{G}(\mathcal{T}')) \leq h(\mathcal{G}(\mathcal{T}))$. Thus, to compute $h(\mathcal{G})$ it is enough to assume that $\mathcal{T}$ consists of a minimal set of generators of $\mathcal{G}$. Hence, $h(\mathcal{G}) = \log k$ and k is at most the number of the minimal generators of $\mathcal{G}$.

Assume now that $\mathcal{G}$ acts transitively on X. The arguments of the proof of Theorem 2.4 yield that $x \in X, T_i(x) \neq T_j(x)$ for $i \neq j$. Therefore, $h(\mathcal{G}(\mathcal{T})) = \log p$. In particular, $h(\mathcal{G}) = \log k$ where k is the minimal number of generators for $\mathcal{G}$. Suppose now that X is reducible under the action of $\mathcal{G}$ and the restriction of $\mathcal{G}$ to one of its irreducible components is faithful. Then the above results yield that $h(\mathcal{G}) = \log k$ where k is the minimal number of generators of $\mathcal{G}$.

Assume now that $h(\mathcal{G}) = 0$. Let $h(\mathcal{G}) = h(\mathcal{G}(\mathcal{T}))$. Assume first that $\mathcal{G}$ acts irreducibly on X. If $\mathcal{T}$ consists of one element T we are done. Assume to the contrary that $\mathcal{T} = \{T_1, ..., T_q\}, q > 1$. Then $A(\Gamma) \geq A(T_1)$. Since $A(\Gamma)$ is irreducible as $\mathcal{G}$ acts transitively, and $A(\Gamma) \neq A(T_1)$ we deduce that $\rho(A(\Gamma)) > 1$. See for example [**Gan**]. This contradicts our assumption that $h(\mathcal{G}) = 0$. Hence, $\mathcal{G}$ is generated by one element, i.e. $\mathcal{G}$ is cyclic. Assume now that $X = \cup_1^m X_i$ is the decompostion of X to its irreducible components. According to the above arguments $\Gamma(\mathcal{T}) \cap X_i \times X_i$ is a permutation matrix. Hence $\Gamma(\mathcal{T})$ is a permutation matrix corresponding to the homeomorphism T. Thus $\mathcal{G}$ is generated by T.

Assume that $Card(X) = n \geq 3$. Let $T : X \to X$ be a homeomorphism that acts transitively on X, i.e. $T^n = Id, T^{n-1} \neq Id$. Let $Q : X \to X, Q \neq T$ be another homeomorphism so that $Q(x) = T(x)$ for some $x \in X$. Set $\mathcal{G} = \mathcal{G}(\{T, Q\})$. According to Theorem 2.4 $h(\mathcal{G}(\{T, Q\})) < \log 2$. Hence, $h(\mathcal{G}) < \log 2$. As $\mathcal{G}$ is not cyclic it follows that $h(\mathcal{G}) > 0$. $\diamond$

It is an interesting problem to determine the entropy of a commutative group in the general case.

§3. Entropy of graphs on compact spaces

Let X be a compact metric space and $\Gamma \subset X \times X$ be a closed graph. As in the previous section set $X_l = \pi_{l,l}(\Gamma^l), l = 2, ...,$. Then $\{X_l\}_2^\infty$ is a sequence of decreasing closed spaces. Let $X' = \cap_2^\infty X_l, \Gamma' = \Gamma \cap X' \times X'$. Clearly,

$$\Gamma^\infty = \Gamma'^\infty, \pi_{1,\infty}(\Gamma^\infty) = \pi_{1,\infty}(\Gamma'^\infty) = \Gamma'^\infty_+ \subset \Gamma^\infty_+.$$

(3.1) Theorem. *Let X be a compact metric space and $\Gamma \subset X \times X$ be a closed set. Then*

$$h(\sigma\big|\Gamma^\infty_+) = h(\sigma\big|\Gamma'^\infty_+) = h(\sigma\big|\Gamma^\infty),$$
$$P(\Gamma^\infty_+, f) = P(\Gamma'^\infty_+, f) = P(\Gamma^\infty, f), f \in C(X).$$

Proof. The equality $h(\sigma\big|\Gamma^\infty_+) = h(\sigma\big|\Gamma'^\infty_+)$ follows from the observation that $\Gamma'^\infty_+ = \cap_0^\infty \sigma^l(\Gamma^\infty_+)$. See [**Wal**, Cor. 8.6.1.]. We now prove the equality $h(\sigma\big|\Gamma'^\infty_+) = h(\sigma\big|\Gamma^\infty)$ It is enough to assume that $X' = X$. Set $X_1 = \Gamma^\infty_+, X_2 = \Gamma^\infty$. Let $\pi : X_2 \to X_1$ be the projection $\pi_{1,\infty}$. It then follows that $\pi(X_2) = X_1, \pi \circ \sigma_2 = \sigma_1 \circ \pi$. Denote by σ_i the restriction of σ to X_i and let $h_i = h(\sigma_i)$ be the topological entropy of σ_i. As σ_1 is a factor of σ_2 one deduces $h_1 \le h_2$.

We now prove the reversed inequality $h_1 \ge h_2$. Let Y be a compact metric space and assume that $T : Y \to Y$ is a continuous transformation. Denote by $\Pi(Y)$ the set of all probability measures on the Borel σ-algebra generated by all open sets of Y. Let $\mathcal{M}(T) \subset \Pi(Y)$ be the set of all T-invariant probability measures. Assume that $\mu \in \mathcal{M}(T)$. Then one defines the Kolmogorov-Sinai entropy $h_\mu(T)$. The variational principle states that

$$h(T) = \sup_{\mu \in \mathcal{M}(T)} h_\mu(T),\ P(T, f) = \sup_{\mu \in \mathcal{M}(T)} \big(h_\mu(T) + \int f d\mu\big), f \in C(X).$$

Let $\mathcal{B}_2$ be the σ-algebra generated by open sets in X_2. An open set $A \subset X_2$ is called cylindrical if there exist $p \le q$ with the following property. Let $y \in \pi_{i,i}(A)$. Then for $i \le p$ we have the property $\pi^2_{1,1}((\pi^2_{2,2})^{-1}(y)) \subset \pi_{i-1,i-1}(A)$. For $i \ge q$ we have the property $\pi^2_{2,2}((\pi^2_{1,1})^{-1}(y)) \subset \pi_{i+1,i+1}(A)$. Let $\mathcal{C} \subset \mathcal{B}_2$ be the finite Borel subalgebra generated by open cylindrical sets. Note that each set in $\mathcal{C}$ is cylindrical. Since σ_2 is a homeomorphism it follows that for any $\mu \in \mathcal{M}(\sigma_2)$ $\mathcal{B}(\mathcal{C}) \overset{o}{=} \mathcal{B}_2$. That is up a set of zero μ-measure every set in $\mathcal{B}_2$ can be presented as a set in σ-Borel algebra generated by $\mathcal{C}$. Let $\alpha \subset \mathcal{C}$ be a finite partition of X_2. One then can define the entropy $h(\sigma_2, \alpha)$ with respect to the measure μ [**Wal**, Ch.4]. Since σ_2 is a homeomorphism and μ is σ_2 invariant it follows that $h(\sigma_2, \alpha) = h(\sigma_2, \sigma_2^m(\alpha))$ for any $m \in \mathbf{Z}$. The assumption that $\mathcal{B}(\mathcal{C}) \overset{o}{=} \mathcal{B}_2$ implies that $\sup_{\alpha \in \mathcal{C}} h(\sigma_2, \alpha) = h_\mu(\sigma_2)$. Taking m big enough in the previous equality we deduce that it is enough to consider all finite partitions $\alpha \subset \mathcal{C}$ with the following property. For each $A \in \alpha$ and each $i \le 1, y \in \pi_{i,i}(A)$ we have the condition $\pi^2_{1,1}((\pi^2_{2,2})^{-1}(y)) \subset \pi_{i-1,i-1}(A)$. It then follows that μ projects on $\mu' \in \mathcal{M}(\sigma_1)$ and $h_\mu(\sigma_2) = h_{\mu'}(\sigma_1)$. The variational principle yields $h_2 \le h_1$ and the equalities of all three entropies are established.

To prove the three equalities on the topological pressure we use the analogous arguments for the topological pressure. $\diamond$

Let $h(\Gamma)$ to be one of the entropies in Theorem 3.1. We call $h(\Gamma)$ the entropy of Γ. For $f \in C(X)$ we denote by $P(\Gamma, f)$ to be one of the topological in Theorem 3.1. Let X be a complete metric space with a metric d. Denote by $B(x,r)$ the open ball of radius r centered in x. Let $\bar{B}(x,r) = Closure(B(x,r))$. We say that X is semi-Riemannian of Hausdorff dimension $n \geq 0$ if for every open ball $B(x,r), 0 < r < \delta$ the Hausdorff dimension of $\bar{B}(x,r)$ is n and its Hausdorf volume $vol(\bar{B}(x,r))$ satisfies the inequality

$$\alpha r^n \leq vol(\bar{B}(x,r))$$

for some $0 < \alpha$. Recall that if the Hausdorff dimension of a compact set $Y \subset X$ is m then its Hausdorff volume is defined as follows.

$$vol(Y) = \lim_{\epsilon \to 0} \liminf_{x_i, 0 < \epsilon_i \leq \epsilon, i=1,...,k, \cup B(x_i,\epsilon_i) \supset Y} \sum_{1}^{k} \epsilon_i^m.$$

The following lemma is a straightforward generalization of Bowen's inequality [**Bow**], [**Wal**, Thm. 7.15].

(3.2) Lemma. *Let X be a semi-Riemannian compact metric space of Hausdorff dimension n. Assume that $T : X \to X$ is Lipschitzian - $d(T(x), T(y)) \leq \lambda d(x,y)$ for all $x, y \in X$ and some $\lambda \geq 1$. Suppose furthermore that X has a finite n dimensional Hausdorff volume. Then $h(T) \leq \log \lambda^n$.*

Proof. As X is compact and semi-Riemannian it follows that X has the Hausdorff dimension n. Let $N(k,\epsilon)$ be the cardinality of the maximal (k,ϵ) separated set. Assume that $\{x_1, ..., x_{N(k,\epsilon)}\}$ is a maximal (k,ϵ) separated set. That is for $i \neq j$

$$\max_{0 \leq l \leq k-1} d(T^l(x_i), T^l(x_j)) > \epsilon.$$

We claim that

$$\bar{B}(x_i, \epsilon_k) \cap \bar{B}(x_j, \epsilon_k) = \emptyset, i \neq j, \epsilon_k = \frac{\epsilon}{3\lambda^{k-1}}.$$

This is immediate from the inequality $d(T^l(x), T^l(y)) \leq \lambda^l d(x,y)$ and the (k,ϵ) separability of $\{x_1, ..., x_{N(k,\epsilon)}\}$. We thus deduce the obvious inequality

$$\sum_{l=1}^{N(k,\epsilon)} vol(\bar{B}(x_l, \epsilon_k)) \leq vol(X).$$

In the above inequality assume that $\epsilon \leq \delta$. Then the lower bound on $vol(\bar{B}(x_l, \epsilon_k))$ yields

$$N(k,\epsilon) \leq \frac{vol(X) 3^n \lambda^{n(k-1)}}{\alpha \epsilon^n}.$$

Thus

$$h(T) = \lim_{\epsilon \to 0} \limsup_{k \to \infty} \frac{\log N(k, \epsilon)}{k} \leq n \log \lambda$$

and the proof of the lemma is completed. $\diamond$

The above estimate can be improved as follows. Let X be a compact metric space and $T : X \to X$. Set

$$L(T) = \sup_{x \neq y \in X} \frac{d(T(x), T(y))}{d(x, y)}, \ L_+(T) = \max(L(T), 1).$$

Thus T is Lipschitzian iff $L(T) < \infty$. Let

$$l(T) = \liminf_{k \to \infty} L_+^{\frac{1}{k}}(T^k).$$

Note that T^k is Lipschtzian for some $k \geq 1$ iff $l(T) < \infty$. $l(T)$ can be considered as a generalization of the maximal Lyapunov exponent for the mapping T. As $h(T^k) = kh(T), k \geq 0$ from Lemma 3.2 we obtain.

(3.3) Theorem. *Let X be a semi-Riemannian compact metric space of Hausdorff dimension n. Assume that $T : X \to X$ is a continuous map. Suppose furthermore that X has a finite n dimensional Hausdorff measure. Then $h(T) \leq n \log l(T)$.*

We have in mind the following application. Let $T : \mathbf{CP}^1 \to \mathbf{CP}^1$ be a rational map of the Riemann sphere $\mathbf{CP}^1$. Let $X = J(T)$ be its Julia set. It is plausible to assume that $\log l(T)$ on X is the Lyapunov exponent corresponding to T and the maximal T-invariant measure on X. Suppose that the Hausdorff dimension of X is n and X has a finite Hausdorff volume. Assume furthermore that X is semi-Riemannian of Hausdorff dimension n. We then can apply Theorem 3.3. As $h(T) = \log deg(T)$ we have the inequality $deg(f) \leq l(f)^n$.

(3.4) Theorem. *Let X be a semi-Riemannian compact metric space of Hausdorff dimension n. Assume that $T_i : X \to X, i = 1, ..., m$, are continuous maps. Let $\Gamma(T_i)$ be the graph of $T_i = 1, ..., m$. Set $\Gamma = \cup_1^m \Gamma(T_i)$. Suppose furthermore that X has a finite n dimensional Hausdorff volume. Then*

$$h(\Gamma) \leq \log \sum_1^m L_+(T_i)^n.$$

.

Proof. It is enough to consider the nontrivial case where each T_i is Lipschitzian. In the definitions of the metrics on $\Gamma^k, \Gamma_+^\infty$ set

$$\rho > \max_{1 \leq i \leq m} L_+(T_i).$$

Let $M=\{1,...,m\}$. Then for $\omega=(\omega_1,...,\omega_{k-1})\in M^{k-1}$ we let

$$\Gamma(\omega)=\{(x_i)_1^k:\ x_1\in X, x_i=T_{\omega_{i-1}}\circ\cdots\circ T_{\omega_1}(x_1), i=2,...,k\}\subset\Gamma^k, \omega\in M^{k-1}.$$

Clearly, each $\Gamma(\omega)$ is isometric to X. Hence, the Hausdorff dimension of $\Gamma(\omega)$ is n and $vol(\Gamma(\omega))=vol(X)$. Furthermore, $\cup_{\omega\in M^{k-1}}\Gamma(\omega)=\Gamma^k$. It then follows that each Γ^k has Hausdorff dimension n, has finite Hausdorff volume not exceeding $m^{k-1}vol(X)$ and is semi-Riemannian compact metric space of Hausdorff dimension n. Moreover, the volume of any closed ball $\bar{B}(y,r)\subset\Gamma^k$ is at least αr^n where α is the constant for X. Let $Y=\Gamma_+^\infty$ and consider a maximal (k,ϵ) separated set in Y of cardinality $N(k,\epsilon)$ - $y^j\in Y, j=1,...,N(k,\epsilon)$. That is

$$y^j=(x_i^j)_{i=1}^\infty, (x_i^j,x_{i+1}^j)\in\Gamma, i=1,..., j=1,...,N(k,\epsilon),$$
$$\max_{1\le i}\frac{d(x_i^j,x_i^l)}{\rho^{(i-k)^+}}>\epsilon, 1\le j\ne l\le N(k,\epsilon).$$

Here, $a^+=\max(a,0), a\in\mathbf{R}$. Fix $\epsilon, 0<\epsilon<\delta$. Assume that D is the diameter of X and let $K(\epsilon)=\lceil log_\rho D-log_\rho\epsilon\rceil$. It then follows that

$$\max_{1\le i\le k+K(\epsilon)} d(x_i^j,x_i^l)>\epsilon, 1\le j\ne l\le N(k,\epsilon). \tag{3.5}$$

Set $z^j=(x_i^j)_{i=1}^{k+K(\epsilon)}\subset\Gamma^{k+K(\epsilon)}, j=1,...,N(k,\epsilon)$. Clearly,

$$\{z^j\}_1^{N(k+K(\epsilon))}=\cup_{\omega\in M^{k+K(\epsilon)-1}}(\{z^j\}_1^{N(k,\epsilon)}\cap\Gamma(\omega))\Rightarrow$$
$$N(k,\epsilon)\le\sum_{\omega\in M^{k+K(\epsilon)-1}}Card(\{z^j\}_1^{N(k,\epsilon)}\cap\Gamma(\omega)).$$

We now estimate $Card(\{z^j\}_1^{N(k,\epsilon)}\cap\Gamma(\omega))$ for a fixed $\omega=(\omega_1,...,\omega_{k+K(\epsilon)-1})\in M^{k+K(\epsilon)-1}$. For each $z^j=(x_i^j)_{i=1}^{k+K(\epsilon)}\in\Gamma(\omega)$ consider the closed set ball

$$\bar{B}(z^j,\epsilon(\omega))\subset\Gamma(\omega), \epsilon(\omega)=\frac{\epsilon}{3\prod_1^{k+K(\epsilon)-1}L_+(T_{\omega_i})}.$$

(We restrict here our discussion to the compact metric space $\Gamma(\omega)$ with the metric induced from $\Gamma^{k+K(\epsilon)}$.) Let $z^j\ne z^l\in\Gamma(\omega)$. The condition (3.5) yields that $\bar{B}(z^j,\epsilon(\omega))\cap\bar{B}(z^l,\epsilon(\omega))=\emptyset$. As $\Gamma(\omega)$ is isometric to X we deduce that

$$Card(\{z^j\}_1^{N(k,\epsilon)}\cap\Gamma(\omega))\le\frac{vol(X)3^n\prod_{i=1}^{k+K(\epsilon)-1}L_+(T_{\omega_i})^n}{\alpha\epsilon^n}.$$

Hence,

$$N(k,\epsilon)\le\sum_{\omega\in M^{k+K(\epsilon)-1}}\frac{vol(X)3^n\prod_{i=1}^{k+K(\epsilon)-1}L_+(T_{\omega_i})^n}{\alpha\epsilon^n}=$$
$$\frac{vol(X)3^n\big(\sum_{i=1}^m L_+(T_i)^n\big)^{k+K(\epsilon)-1}}{\alpha\epsilon^n}.$$

Thus

$$h(\Gamma) = \lim_{\epsilon\to 0} \limsup_{k\to\infty} \frac{logN(k,\epsilon)}{k} \le \log \sum_{i=1}^{n} L_+(T_i)^n$$

and the theorem is proved. $\diamond$

We remark that the inequality of Theorem 3.4 holds if we replace the assumption that X has a finite n-Hausdorff volume by the following one: the number of points of every $r-$ *separated* set in X does not exceed Cr^{-n} for some positive constant C.

Let X satisfies the assumptions of Theorem 3.4. It then follows that for the Lipschitzian maps $f : X \to X$ the quantity $L_+(T)^n$ is the "norm" of the graph $\Gamma(f)$ discussed in §2.

(3.6) Lemma. *Let X be a compact metric space and $T : X \to X$ be a noninvolutive homeomorphism ($T^2 \ne Id$). Then $\log 2 \le h(\Gamma(T) \cup \Gamma(T^{-1}))$. If $T, T^{-1} : X \to X$ are noninvolutive isometries then $h(\Gamma(T) \cup \Gamma(T^{-1})) = \log 2$.*

Proof. Assume first that T has a periodic orbit $Y = \{y_1, ..., y_p\}$ of period $p > 2$. Restrict T, T^{-1} to this orbit. Theorem 2.4 yields the desired inequality. Assume now that we have an infinite orbit $y_i = T^i(y), i = 1, 2, ...,$. Fix $n \ge 3$. Let $Y_n = \{y_1, ..., y_n\}$. Denote by $\Gamma_n \subset Y_n \times Y_n$ the graph corresponding to the undirected linear graph on the vertices $y_1, ..., y_n$. That $(i,j) \in \Gamma_n \iff |i-j| = 1$. Clearly

$$\Gamma_n^\infty \subset \Gamma^\infty, \ \Gamma = \Gamma(T) \cup \Gamma(T^{-1}).$$

Hence $h(\Gamma_n) \le h(\Gamma)$. Obviuosly, $h(\Gamma_n) = log\rho(A(\Gamma_n))$. It is well known that $\rho(A(\Gamma_n)) = 2cos\frac{\pi}{n+1}$. (The eigenvalues of $A(\Gamma_n)$ are the roots of the Chebycheff polynomial.) Let $n \to \infty$ and deduce $h(\Gamma) \ge \log 2$. Assume now that T and T^{-1} are noninvolutive isometries. Then Theorem 3.4 and the above inequality implies that $h(\Gamma(T) \cup \Gamma(T^{-1})) = \log 2$. $\diamond$

Thus, Theorem 3.4 is sharp for $m = 2$. Similar examples using isometries and Theorem 2.4 show that Theorem 3.4 is sharp in general.

Let X be a compact metric space and $T_i : X \to X, i = 1, ..., m$, be a set of continuous transformations. Let $\mathcal{T} = \{T_1, ..., T_m\}$. Then $h(\mathcal{S}(\mathcal{T}))$ was defined to be the entropy of the graph $\Gamma = \cup_1^m \Gamma(T_i)$. As in the case of $m = 1$ this entropy can be defined in terms of "(k,ϵ)" separated (spanning) sets as follows. Set

$$d_{k+1}(x,y) = \max(\max_{1\le i_1, j_1, ..., i_k, j_k \le m,} d(T_{i_1} \dots T_{i_k}(x), T_{j_1} \dots T_{j_k}(y)), d(x,y)), k = 1, 2, ..., .$$

Let $M(k,\epsilon)$ be the maximal cardinality the ϵ separated set in the metric d_k.

(3.7) Lemma. *Let X be a compact metric space and assume that $T_i : X \to X, i = 1, ..., m$, are continuous transformations. Then*

$$h(\mathcal{S}(\{T_1, ..., T_m\}) = \lim_{\epsilon\to 0} \limsup_{k\to\infty} \frac{\log M(k,\epsilon)}{k}.$$

Proof. From the definiton of the (k, ϵ) separated set for Γ_+^∞ it immediately follows that

$$M(k, \epsilon) \leq N(k, \epsilon).$$

The arguments in the proof of Theorem 3.4 yield that

$$N(k, \epsilon) \leq M(k + K(\epsilon))$$

and the lemma follows. $\diamond$

§4. Approximating entropy of graphs by entropy of subshifts of finite type

Let X be a set. $\mathcal{U} = \{U_1, ..., U_m\} \subset 2^X$ is called a finite cover of X if $X = \cup_1^m U_i$. The cover $\mathcal{U}$ is called minimal if any strict subset of $\mathcal{U}$ is not a cover of X. Let $\Gamma \subset X \times X$ be any subset. Introduce the following graph and its corresponding matrix on the space $< m >= \{1, ..., m\}$:

$$\mathcal{U} = \{U_1, ..., U_m\}, \ \Gamma(\mathcal{U}) = \{(i, j) : \Gamma \cap U_i \times U_j \neq \emptyset\} \subset < m > \times < m >,$$
$$A(\Gamma(\mathcal{U})) = (a_{ij})_1^m \in M_m(\{0 - 1\}), a_{ij} = 1 \iff (i, j) \in \Gamma(\mathcal{U})\}.$$

Note that $\Gamma(\mathcal{U})$ induces a subshift of a finite type on $< m >$. Thus, $\log^+ \rho(\Gamma(\mathcal{U}))$ is the entropy of Γ induced by the cover $\mathcal{U}$. Let $\mathcal{V}$ be also a finite cover of X. Then $\mathcal{V}$ is called a refinement of $\mathcal{U}$, written $\mathcal{U} < \mathcal{V}$, if every member of $\mathcal{V}$ is a subset of a member of $\mathcal{U}$. Assume that $\mathcal{V} = \{V_1, ..., V_m\}$ is a refinement of $\mathcal{U}$ such that $V_i \subset U_i, i = 1, ..., m$. It then follows that $A(\Gamma(\mathcal{U})) \geq A(\Gamma(\mathcal{V}))$ for any $\Gamma \subset X \times X$. Hence, $\rho(A(\Gamma(\mathcal{U}))) \geq \rho(A(\Gamma(\mathcal{V})))$. If $U_i \cap U_j = \emptyset, 1 \leq i < j \leq m$, then $\mathcal{U}$ is called a finite partition of X. Given a finite minimal cover $\mathcal{U} = \{U_1, ..., U_m\}$ there always exist a partition $\mathcal{V} = \{V_1, ..., V_m\}$ such that $V_i \subset U_i, i = 1, ..., m$. Indeed, consider a partition $\mathcal{U}'$ corresponding to the subalgebra generated by $\mathcal{U}$. This partition is a refinement of $\mathcal{U}$. Then each U_i is union of some sets in $\mathcal{U}'$. Set $V_1 = U_1$. Let $V_2 \subset U_2$ be the union of sets of $\mathcal{U}'$ which are subsets of $U_2 \backslash U_1$. Continue this process to construct $\mathcal{V}$. In particular, $\rho(A(T, \mathcal{U})) \geq \rho(A(T, \mathcal{V}))$.

Let $\mathcal{U} < \mathcal{V}$ be finite partitions of X. Assume that $\Gamma \subset X \times X$. In general, there is no relation between $\rho(A(\Gamma(\mathcal{U})))$ and $\rho(A(\Gamma(\mathcal{V})))$. Indeed, if $A(\Gamma(\mathcal{V}))$ is a matrix whose all entries are equal to 1 then $A(\Gamma(\mathcal{U}))$ is also a matrix whose all entries are equal to 1. Hence

$$\rho(A(\Gamma(\mathcal{V}))) = Card(\mathcal{V}) > \rho(A(\Gamma(\mathcal{U}))) = Card(\mathcal{U}) \iff \mathcal{U} \neq \mathcal{V}.$$

Assume now that $Card(\mathcal{V}) = n, A(\Gamma(\mathcal{V})) = (\delta_{(i+1)j})_1^n, n + 1 \equiv 1$ be the matrix corresponding to a cyclic graph on $< n >$. Suppose furthermore that $n \geq 3$ and let $U_1 = V_1 \cup V_2, U_i = V_{i+1}, i = 2, ..., n-1$. It then follows that $\rho(A(\Gamma(\mathcal{U}))) > \rho(A(\Gamma(\mathcal{V}))) = 1$.

Let $\mathcal{F}_\epsilon, 0 < \epsilon < 1$ be a family of finite covers of X increasing in ϵ. That is, $\mathcal{F}_\delta \subset \mathcal{F}_\epsilon, 0 < \delta \leq \epsilon < 1$. Assume that $\Gamma \subset X \times X$ be any set. We then set

$$e(\Gamma, \mathcal{F}) = \lim_{\epsilon \to 0^+} \inf_{\mathcal{U} \in \mathcal{F}_\epsilon} \log^+ \rho(A(\Gamma(\mathcal{U}))).$$

Thus, $e(\Gamma,\mathcal{F})$ can be considered as the entropy of Γ induced by the family $\mathcal{F}_\epsilon$. Its definition is reminiscent of the definition of the Hausdorff dimension of a metric space X. Let $\mathcal{U}$ be a finite cover of X. Clearly, $A(\Gamma^T(\mathcal{U})) = A^T(\Gamma(\mathcal{U}))$. Hence, $\rho(A(\Gamma(\mathcal{U}))) = \rho(A(\Gamma^T(\mathcal{U})))$ and $e(\Gamma,\mathcal{F}) = e(\Gamma^T,\mathcal{F})$.

(4.1) Lemma. *Let $\mathcal{U} = \{U_1,...,U_m\}$ be a finite cover of compact metric space X. Assume that $\mathrm{diam}(\mathcal{U}) \stackrel{\mathrm{def}}{=} \max \mathrm{diam}(U_i) \le \frac{\delta}{2}$. Let $\Gamma \subset X \times X$ be a closed set. Assume that $N_k(\delta)$ is the maximal cardinality of (k,δ) separated set for $\sigma : \Gamma_+^\infty \to \Gamma_+^\infty$. Then*

$$\limsup_{k\to\infty} \frac{\log N_k(\delta)}{k} \le \log \rho(A(\Gamma(\mathcal{U}))).$$

Proof. Set

$$A^k(\Gamma(\mathcal{U})) = (a_{ij}^{(k)})_1^m, \nu_k(\mathcal{U}) = \sum_1^m a_{ij}^{(k-1)}.$$

Then $\nu_k(\mathcal{U})$ is counting the number of distinct point

$$(y_i)_1^k \in < m >^k, (y_i, y_{i+1}) \in \Gamma(\mathcal{U}), i = 1,...,k-1.$$

Let $K(\delta)$ be defined as in the proof of Theorem 3.4. We claim that $N(k,\delta) \le \nu_{k+K(\delta)}(\mathcal{U})$. Indeed, assume that $x^i = (x_j^i)_{j=1}^\infty, i = 1,...,N(k,\delta)$, is a (k,δ) separated set. Then each x^i generates at least one point $y^i = (y_1^i,...,y_p^i) \in < m >^p$ as follows: $x_j^i \in U_{y_j^i}, j = 1,...,p$. From (3.5) and the assumption that $\mathrm{diam}(\mathcal{U}) < \frac{\delta}{2}$ we deduce that for $p = k + K(\delta)$ $i \ne l \Rightarrow y^i \ne y^l$. Hence $N(k,\delta) \le \nu_{k+K(\delta)}(\mathcal{U})$. As a point x^i may generate more then one point y^i in general we have strict inequality. Since $A(T,\mathcal{U})$ is a nonnegative matrix it is well known that

$$K_1 \rho(A)^k \le \nu_k \le K_2 k^{m-1} \rho(A(T,\mathcal{U}))^k, k = 1,...,.$$

See for example [**F-S**]. The above inequalities yield the lemma. $\diamond$

Let $\{\mathcal{U}_i\}_1^\infty$ be sequence of finite open covers such $\mathrm{diam}(\mathcal{U}_i) \to 0$. Assume that $\Gamma \subset X \times X$ is closed. Then $\{\mathcal{U}_i\}_1^\infty$ is called an approximation cover sequence for Γ if

$$lim_{i\to\infty} \log^+ \rho(A(\Gamma(\mathcal{U}_i))) = h(\Gamma).$$

Note as $\rho(A^T) = \rho(A), \forall A \in M_n(\mathbf{C})$ and $h(\Gamma) = h(\Gamma^T)$ we deduce that $\{\mathcal{U}_i\}_1^\infty$ is also an approximation cover for Γ^T. Use Lemma 4.1 and (2.2) for finite graphs to obtain sufficient conditions for the validity of the inequality (2.2) for infinite graphs.

(4.2) Corollary. *Let X be a compact metric space and $\Gamma_j^T = \Gamma_j \subset X \times X, j = 1,...,m$ be closed sets. Assume that there exist a sequence of open finite covers*

$$\{\mathcal{U}_i\}_1^\infty, \lim_{i\to\infty} \mathrm{diam}(\mathcal{U}_i) = 0,$$

which is an approximation cover for $\Gamma_1, ..., \Gamma_m$*. Then*

$$h(\cup_1^m \Gamma_j) \leq log \sum_1^m e^{h(\Gamma_j)}.$$

Let Z be a compact metric space and $T : Z \to Z$ is a homeomorphism. Then T is called expansive if there exists $\delta > 0$ such that

$$\sup_{n \in \mathbf{Z}} d(T^n(x), T^n(y)) > \delta, \forall x, y \in Z, x \neq y.$$

A finite open cover $\mathcal{U}$ of Z is called a generator for homeomorphism T if for every bisequence $\{U_n\}_{-\infty}^{\infty}$ of members of $\mathcal{U}$ the set $\cap_{n=-\infty}^{\infty} T^{-n}\bar{U}_n$ contains at most one point of X. If this condition is replaced by $\cap_{n=-\infty}^{\infty} U_n$ then $\mathcal{U}$ is called a weak generator. A basic result due to Keynes and Robertson [**K-R**] and Reddy [**Red**] claims that T is expansive iff T has a generator iff T has a weak generator. See [**Wal**, §5.6]. Moreover, T is a factor of the restriction of a shift S on a finite number of symbols to a closed $S-$invariant set Δ [**Wal**, Thm 5.24]. If Δ is a subshift of a finite type then T is called FP. See [**Fr**] for the theory of FP maps. In particular, for any expansive T, $h(T) < \infty$.

Let $\Gamma \subset X \times X$ be a closed set such that $\Gamma^\infty \neq \emptyset$. Then Γ is called expansive if

$$\sup_{n \in \mathbf{Z}} d(\sigma^n(x), \sigma^n(y)) > \delta, \forall x, y \in \Gamma^\infty, x \neq y$$

for some $\delta > 0$. A finite open cover $\mathcal{U}$ of X is called a generator for Γ if for every bisequence $\{U_n\}_{-\infty}^{\infty}$ of members of $\mathcal{U}$ the set

$$x = (x_n)_{-\infty}^{\infty} \in \Gamma^\infty, x_n \in \bar{U}_n, n \in \mathbf{Z}$$

contains at most one point of Γ^∞. If this condition is replaced by $x_n \in U_n$ then $\mathcal{U}$ is called a weak generator. We claim that Γ is expansive iff Γ has a generator iff Γ has a weak generator. Indeed, observe first that the condition that Γ is expansive is equivalent to the assumption that σ is expansive on Γ^∞. Let $V_i = \pi_{1,1}^{-1}(U_i) \subset X^\infty, i = 1, ..., m$. That is, V_i is an open cylindrical set in X^∞ whose projection on the first coordinate is U_i while on all other coordinates is X. Set $W_i = V_i \cap \Gamma^\infty, i = 1, ..., m$. It now follows that $W_1, ..., W_m$ is a standard set of generators for the map $\sigma : \Gamma^\infty \to \Gamma^\infty$.

Assume that $T : X \to X$ is expansive with the expansive constant δ. It is known [**Wal**, Thm. 7.11] that

$$h(T) = \limsup_{k \to \infty} \frac{\log N(k, \delta_0)}{k}, \delta_0 < \frac{\delta}{4}.$$

Thus, according to Lemma 4.1 $h(\Gamma) \leq \log \rho(A(\Gamma(\mathcal{U})))$ if Γ is expansive with an expansive constant δ and $\mathrm{diam}(\mathcal{U}) < \frac{\delta}{8}$. Assume that $T_i : X \to X, i = 1, ..., m$, are expansive maps. We claim that for $m > 1$ it can happen that $h(\cup_1^m \Gamma(T_i))$ is infinite. Let T_1 be Anosov map on the 2-torus X in the standard coordinates. Now change the coordinates in X by a homeomorphism and let T_2 be Anosov with respect to the new coordinates. It is possible to choose a homeomorphism (which is not diffeo!) so that that $T_2 \circ T_1$ contains horseshoes of arbitrary many folds. Hence $h(\Gamma(T_1) \cup \Gamma(T_2) \geq h(T_2 \circ T_1) = \infty$.

§5. Entropy of semigroups of Möbius transformations

Let $X \subset \mathbf{CP}^n$ be an irreducible smooth projective variety of complex dimension n. Assume that $\Gamma \subset X \times X$ be a projective variety such that the projections $\pi_{i,i} : \Gamma \to X, i = 1, 2$ are onto and finite to one. Then Γ can be viewed as a graph of an algebraic function. In algebraic geometry such a graph is called a correspondence. Furthermore, Γ induces a linear operator

$$\Gamma^* : H_{*,a}(X) \to H_{*,a}(X), \;\; H_{*,a}(X) = \sum_{j=0}^{n} H_{2j,a}(X),$$
$$\Gamma^* : H_{2j,a}(X) \to H_{2j,a}(X), j = 0, ..., n.$$

Here, $H_{2j,a}(X)$ is the homology generated by the algebraic cycles of X of complex dimension j over the rationals $\mathbf{Q}$. Indeed, if $Y \subset X$ is an irreducible projective variety then $\Gamma^*([Y]) = [\pi^2_{2,2}((\pi^2_{1,1})^{-1}(Y))]$. Let $\rho(\Gamma^*)$ be the spectral radius of Γ^*. Assume that first that Γ is irreducible. In [**Fri3**] we showed that $h(\Gamma) \leq \log \rho(\Gamma^*)$. However our arguments apply also to the case Γ is reducible. We also conjectured in [**Fri3**] that in the case that Γ is irreducible we have the equality $h(\Gamma) = \log \rho(\Gamma^*)$. We now doubt the validity of this conjecture. We will show that in the reducible case we can have a strict inequality $h(\Gamma) < \log \rho(\Gamma^*)$. Let $\Gamma_i \subset X \times X, i = 1, ..., m$, be algebraic correspondences as above. Set $\Gamma = \cup_1^m \Gamma_i$. Then

$$\Gamma^* = \sum_1^m \Gamma_i^*, \;\; h(\Gamma) \leq log\rho(\sum_1^m \Gamma_i^*).$$

Thus, there is a close analogy between the entropy of algebraic (finite to one) correspondences and entropy of shifts of finite types. Consider the simplest case of the above situation. Let $X = \mathbf{CP}^1$ be the Riemann sphere and Γ be an algebraic curve given by a polynomial $p(x, y) = 0$ on some chart $\mathbf{C}^2 \subset \mathbf{CP}^1 \times \mathbf{CP}^1$. Let $d_1 = deg_y(p), d_2 = deg_x(p), d_1 \geq 1, d_2 \geq 1$. It then follows that $\rho(\Gamma^*) = \max(d_1, d_2)$. Note that $\rho(\Gamma^*) = 1$ iff Γ is the graph of a Möbius transformation. Observe next that if $f_i : \mathbf{CP}^1 \to \mathbf{CP}^1, i = 1, ..., m$, are nonconstant rational maps then the correspondance given by $p(x, y) = \prod_1^m (y - f_i(x))$ is induced by $\Gamma = \cup_1^m \Gamma(f_i)$. In particular,

$$h(\Gamma) \leq \log \sum_1^m deg(f_i). \tag{5.1}$$

Here, by $deg(f_i)$ we denote the topological degree of the map f_i. Combine the above inequality with Lemma 3.6 to deduce that for any noninvolutive Möbius transformation f we have the equality $h(\Gamma(f) \cup \Gamma(f^{-1})) = \log 2$.

(5.2) Lemma. *Let $f, g : \mathbf{CP}^1 \to \mathbf{CP}^1$ be two Möbius transformations such that x as a common fixed attracting point of f and g and y is a common repelling point of f and g, Then $h(\Gamma(f) \cup \Gamma(g)) = 0$.*

Proof. We may assume that

$$f = az, g = bz, 0 < |a|, |b| < 1.$$

Set $\Gamma = \Gamma(f) \cup \Gamma(g)$. It the follows that for any point $\zeta = (z_i)_1^\infty \neq \eta = (\infty)_1^\infty$ $\sigma^l(z)$ converges to the fixed point $\xi = (0)_1^\infty$. That is, the nonwondering set of σ is the set $\{\xi, \eta\}$ on which σ acts trivially. Hence $h(\Gamma) = 0$. $\diamond$

(5.3) Lemma. *Let $f, g : \mathbf{CP}^1 \to \mathbf{CP}^1$ be two parabolic Möbius transformation with the same fixed point -∞, i.e. $f = z + a, g = z + b$. If either a, b are linearly independent over $\mathbf{R}$ or $b = \alpha a, \alpha \geq 0$ then $h(\Gamma(f) \cup \Gamma(g)) = 0$.*

Proof. Let $\Gamma = \Gamma(f) \cup \Gamma(g), \eta = (\infty)_1^\infty$. If ether a, b are linearly independent over $\mathbf{R}$ or $b = \alpha a, \alpha > 0, a \neq 0$ then for any point $\zeta \in \Gamma_+^\infty$ $\sigma^l(\zeta)$ converges to the fixed point η. Hence $h(\Gamma) = 0$. Suppose next that $a = b = 0$. Then σ is the identity map on Γ_+^∞ and $h(\Gamma) = 0$. Assume finally that $b = 0, a \neq 0$. Then Ω limit set of σ consists of all points $\zeta = (z_i)_1^\infty, z_i = z_1, i = 2, ...,$. So $\sigma\big|\Omega$ is identity and $h(\Gamma) = 0$. $\diamond$

(5.4) Theorem. *Let $T = z + a, Q = z + b, ab \neq 0$ be two Möbius transformations of $\mathbf{CP}^1$. Assume that there $\frac{b}{a}$ is a negative rational number. Then*

$$h(\Gamma) = -\frac{|a|}{|a| + |b|} \log \frac{|a|}{|a| + |b|} - \frac{|b|}{|a| + |b|} \log \frac{|b|}{|a| + |b|}.$$

We first state an approximation lemma which will be used later.

(5.5) Lemma. *Let X be compact metric space and $T : X \to X$ be a continuous transformation. Assume that we have a sequence of closed subsets $X_i \subset X, i = 1, ...,$ which are $T-$invariant, i.e. $T(X_i) \subset X_i, i = 1, 2, ...,$. Suppose furthermore that $\forall \delta > 0 \exists M(\delta)$ with the following property. $\forall x \in X \backslash X_i \exists y = y(x, i) \in X_i, \sup_{n \geq 0} d(T^n(x), T^n(y)) \leq \delta$ for each $i > M(\delta)$. Then $\lim_{i \to \infty} h(T\big|X_i) = h(T)$.*

Proof. Observe first that $h(T) \geq h(T\big|X_i)$. Thus it is left to show

$$\liminf_{i \to \infty} h(T\big|X_i) \geq h(T).$$

Let $N(k, \epsilon), N_i(k, \epsilon)$ be the cardinality of maximal (k, ϵ) separating set of X and X_i respectively. Clearly, $N_i(k, \epsilon) \leq N(k, \epsilon)$. Let $x_1, ..., x_{N(k,\epsilon)}$ be a (k, ϵ) separating set of X. Then

$$\forall i > M(\frac{\epsilon}{4}),\ \forall x_j \exists y_{j,i} \in X_i, \sup_{n \geq 0} d(T^n(x_j), T^n(y_{j,i})) \leq \frac{\epsilon}{4}.$$

Hence, $y_{j,i}, j = 1, ..., N(k, \epsilon)$, is $\frac{\epsilon}{2}$ separated set in X_i. In particular, $N(k, \epsilon) \le N_i(k, \frac{\epsilon}{2}), i > M(\frac{\epsilon}{4})$. Thus

$$\limsup_{k \to \infty} \frac{\log N(k, \epsilon)}{k} \le \limsup_{k \to \infty} \frac{\log N_i(k, \frac{\epsilon}{2})}{k} \le h(T \big| X_i), i > M(\frac{\epsilon}{4}).$$

The characterization of $h(T)$ yields the lemma. $\diamond$

Proof of Theorem 5.4. W.l.o.g. (without loss of generality) we may assume that $a = p, b = -q$ where p, q are two positive coprime integers. First note that $\mathbf{CP}^1$ is foliated by the invariant lines $\Im z = Const.$ Hence, the maximal characterization of $h(\sigma)$ as the supremum over all measure entropy $h_\mu(\sigma)$ for all extremal σ invariant measures yields that it enough to restrict ourselves to the action of T, Q on (closure of) the real line. Using the same argument again it is enough to consider the action on the lattice $\mathbf{Z} \subset \mathbf{R}$ plus the point at ∞. We may view $Y = \mathbf{Z} \cup \{\infty\}$ as a compact subspace of $S^1 = \{z : |z| = 1\}$.

$$0 \mapsto 1, \infty \mapsto -1, j \mapsto e^{\frac{\pi\sqrt{-1}(1+2j)}{2j}}, 0 \neq j \in \mathbf{Z}.$$

For a positive integer i let $Y_i = \{-ipq, -ipq + 1, ..., ipq - 1, ipq\}$. Set

$$\Gamma = \Gamma(T) \cup \Gamma(Q) \subset Y \times Y, X = \Gamma_+^\infty, \Gamma_i = \Gamma \cap Y_i \times Y_i, X_i = (\Gamma_i)_+^\infty, i = 1, ...,.$$

We will view a point $x = (x_j)_1^\infty \in X$ a path of a particle who starts at time 1 at x_1 and jumps from the place x_i at time i to the place x_{i+1} at time $i+1$. At each point of the lattice $\mathbf{Z}$ a particle is allowed to jump p steps forward and q backwards. The point $\xi = (\infty)_1^\infty$ is the fixed point of our random walk. Observe next that Γ_i is a subshift of a finite type on $2ipq + 1$ points corresponding to the random walk in which a particle stays in the space Y_i. Note that $A_i = A(\Gamma_i)$ is a matrix whose almost each row (column) sums to two, except the first and the last $\max(p, q) - 1$ rows (columns). Moreover, $h(\sigma \big| X_i) = \log \rho(A_i)$. We claim that $X, X_i = 1, ...,$ satisfy the assumption of Lemma 5.5. That is any point $x = (x_j)_1^\infty \in X$ can be approximated up to an arbitrary $\epsilon > 0$ by $y_i = (y_{j,i})_{j=1}^\infty \in X_i$ for $i > M(\epsilon)$. We assume that $i > L$ some fixed big L. Suppose first that $x_j > ipq, j = 1, ...,.$ That is the path described by the vector x never enters X_i. Then consider the following path $y_i = (y_{j,i})_{j=1}^\infty \in X_i$. It starts at the point ipq, i.e. $y_{1,i} = ipq$. Then it jumps p times to the left to the point $(i-1)pq$. Then it the particle jumps q time to the right back to the the point ipq and so on. Clearly, $\sup_{n \ge 0} d(\sigma^n(x), \sigma^n(y_i)) \le d((i-1)pq, \infty)$. Hence for i big enough the above distance is less than ϵ. Same arguments apply to the case $x_j < -ipq, j = 1, ...,.$ Consider next a path $x = (x_j)_1^\infty$ which starts outside X_i and then enters X_i at some time. If the particle enters to X_i and then stays for a short time, e.g. $\le pq$, every time it enters X_i then we can approximate this path by a path looping around the vertex ipq or $-ipq$ in X_i as above. Now suppose that we have a path which enters to X_i at least one time for a longer period of time. We then approximate this path by a path $(y_{i,j})_{j=1}^\infty \in X_i$ such that this path coincide with x for all time when x is in X_i except the short period when x leaves X_i. One can show that such path exists. (Start with the simple example $p = 1, q = 2$.) It then follows that $\sup_{n \ge 0} d(\sigma^n(x), \sigma^n(y_i)) \le d((i-K)pq, \infty)$ for some $K = K(p,q)$. If i is big enough then we have the desired approximation. Lemma 5.5 yields

$$h(\Gamma) = \lim_{i\to\infty} \log \rho(A_i).$$

We now estimate $\log \rho(A_i)$ from above and from below. Recall the well known formula for the spectral radius of a nonnegative $n \times n$ matrix A:

$$\rho = \limsup_{m\to\infty} \big(trace(A^m)\big)^{\frac{1}{m}} = \limsup_{m\to\infty} \big(\max_{1\le j\le n} a_{jj}^{(m)}\big)^{\frac{1}{m}}, A^m = (a_{ij}^{(m)})_1^n.$$

Let $A = A_i$. We now estimate $a_{jj}^{(m)}$. Obviously, $a_{jj}^{(m)}$ is positive if $m = (p+q)k$ as we have to move kq times to the right and kp times to the left. Assume that $m = (p+q)k$. To estimate $a_{jj}^{(m)}$ we assume that we have an uncostrained motion on $\mathbf{Z}$. Then the number of all possible moves on $\mathbf{Z}$ bringing us back to the original point is equal to

$$\frac{((p+q)k)!}{(qk)!(pk)!} \le K\sqrt{p+q}\frac{(p+q)^{(p+q)k}}{q^{qk}p^{pk}}.$$

The last part of inequality follows from the Stirling formula for some suitable K. The characterization of $\rho(A)$ gives the inequality

$$\log \rho(A_i) \le \log \alpha = \log(p+q) - \frac{p}{p+q}\log p - \frac{q}{p+q}\log q.$$

We thus deduce the upper bound on $h(\Gamma) \le \log\alpha$. Let $0 < \delta < \alpha$. The Stirling formula yields that for $k > M(\delta)$

$$\frac{((p+q)k)!}{(qk)!(pk)!} \ge (\alpha-\delta)^{(p+q)k}.$$

Fix $k > M(\delta)$ and let $i > k$. Then for $m = (p+q)k$

$$a_{00}^{(m)} = \frac{((p+q)k)!}{(qk)!(pk)!}.$$

Clearly,

$$\rho(A)^m = \rho(A^m) \ge a_{00}^{(m)}.$$

Thus, $h(\Gamma) \ge \log\rho(A_i) \ge \log(\alpha-\delta)$. Let $\delta \to 0$ and deduce the theorem. $\diamond$

Note that $h(\Gamma)$ is the entropy of the Bernoulli shift on two symbols with the distribution $(\frac{p}{p+q}, \frac{q}{p+q})$. This can be explained by the fact that to have a closed orbit of length $k(p+q)$ we need move to the right kq times and to the left kp. That is, the frequency of the right motion is $\frac{q}{p+q}$ and the left motion is $\frac{p}{p+q}$. It seems that Theorem 5.4 remains valid as long as $\frac{a}{b}$ is a real negative number.

(5.6) Theorem. *Let $f, g : \mathbf{CP}^1 \to \mathbf{CP}^1$ be two parabolic Möbius transformations with the same fixed point - ∞, i.e. $f = z + a, g = z + b$ where a, b are linearly independent over $\mathbf{R}$. Let $\Gamma = \Gamma(f) \cup \Gamma(f^{-1}) \cup \Gamma(g) \cup \Gamma(g^{-1})$. Then $h(\Gamma) = \log 4$.*

Proof. The orbit of any fixed point $z \in \mathbf{C}$ under the action of the group generated by f, g is a lattice in $\mathbf{C}$ which has one accumulation point $\infty \in \mathbf{CP}^1$. Let Y is defined in the proof of Theorem 5.4. Consider the dynamics of $\sigma \times \sigma$ on $Y_j \times Y_j$, for $j = 1, ...,$ as in the proof Theorem 5.4. It then follows that $h(\Gamma) = 2h(\sigma \big| X) = 2 \log 2$. $\diamond$

Let $\mathcal{T} = \{f_1, ..., f_k\}$ be a set of k - Möbius transformations. Set $\Gamma = \cup_1^k \Gamma(f_i)$. Then (5.1) yields $h(\Gamma) \le \log k$. Our examples show that we may have a strict inequality even for the case $k = 2$. Let Γ be the correspondence of the Gauss arithmetic-geometric mean $y^2 = \frac{(x+1)^2}{4x}$ [**Bul2**]. Our inequality in [**Fri3**] yield that $h(\Gamma) \le \log 2$. According to Bullet [**Bul2**] it is possible to view the dynamics of Γ as a factor of the dynamics of $\tilde{\Gamma} = \Gamma(f_1) \cup \Gamma(f_2)$ for some two Möbius transformations f_1, f_2. Hence, $h(\Gamma) \le h(\tilde{\Gamma})$. If $h(\tilde{\Gamma}) < \log 2$ we will have a counterexample to our conjecture that $h(\Gamma) = \log 2$. Even if $h(\tilde{\Gamma}) = \log 2$ we can still have the inequality $h(\Gamma) < \log 2$ as the dynamics of Γ is a subfactor of the dynamics of $h(\tilde{\Gamma})$. Thus, it would be very interesting to compute $h(\Gamma)$.

Assume that $\mathcal{T}$ generates nonelementary Kleinian group. Theorem 2.5 suggests that $e^{h(\Gamma)}$ may have a noninteger value. It would be very interesting to find such a Kleinian group.

We now state an open problem which is inspired by Furstenberg's conjecture [**Fur**]. Assume that $1 < p < q$ are two co-prime integers. (More generally $p^m = q^n \Rightarrow m = n = 0$.) Let

$$f, g : \mathbf{CP}^1 \to \mathbf{CP}^1, T_1(z) = z^p, T_2(z) = z^q, z \in \mathbf{C}^1, f(\infty) = g(\infty). = \infty$$

Note that for f and g $0, \infty$ are two attractive points with the interior and the exterior of the unit disk as basins of attraction respectively. Thus, the nontrivial dynamics takes place on the unit circle S^1. Note that $f \circ g = g \circ f$. Hence f and g have common invariant probability measures. Let $\mathcal{M}$ be the convex set of all probability measures invariant under f, g. Denote by $\mathcal{E} \subset \mathcal{M}$ the set of the extreme points of $\mathcal{M}$ in the standard w^* topology. Then $\mathcal{E}$ is the set of ergodic measures with respect to f, g. (For a recent discussion on the common invariant measure of a semigroup of commuting transformation see [Fri4]). Furstenberg's conjecture (for $p = 2, q = 3$) is that any ergodic measure $\mu \in \mathcal{E}$ is either supported on a finite number of points or is the Lebesgue (Haar) measure on S^1. See [**Rud**] and [**K-S**] for the recent results on this conjecture. Let $\mathcal{G}$ be the semigroup generated by $\mathcal{T} = \{f, g\}$. Then (0.2) for $X = S^1$ or the results of [**Fri3**] yield the inequality $h(\mathcal{G}(\mathcal{T})) \le \log(p + q)$. What is the value of $h(\mathcal{G}(\mathcal{T}))$? It is plausible to conjecture equality in this inequality.

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Acknowledgement. I would like to thank to M. Boyle, M. Gromov and F. Przytycki for useful remarks.