A Proof of the Set-theoretic Version of a Salmon Conjecture

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Summary

1. Statement of the problem
2. Known results
3. New conditions
4. Outline of the complete solution

Basic notions

scalar $a \in \mathbb{F}$, vector $\mathbf{x} = (x_1, \ldots, x_n)^\top \in \mathbb{F}^n$,

matrix $A = [a_{ij}] \in \mathbb{F}^{m \times n}$,

3-tensor $\mathcal{T} = [t_{i,j,k}] \in \mathbb{F}^{m \times n \times l}$,

Rank one tensor $t_{i,j,k} = x_i y_j z_k$, $(i,j,k) = (1,1,1), \ldots, (m,n,l)$

or decomposable tensor $\mathbf{x} \otimes \mathbf{y} \otimes \mathbf{z}$

rank $\mathcal{T}$ minimal r:

$\mathcal{T} = \sum_{i=1}^{r} \mathbf{x}_i \otimes \mathbf{y}_i \otimes \mathbf{z}_i$,

Border rank of $\mathcal{T}$ the minimum k s.t. $\mathcal{T}$ is a limit of $\mathcal{T}_j, j \in \mathbb{N}$, rank $T_j = k$.

Notations

$\mathbb{C}^{m\times n\times l} := \mathbb{C}^m \otimes \mathbb{C}^n \otimes \mathbb{C}^l$ consists of $\mathcal{T} = [t_{i,j,k}]_{i=j=k=1}^{m,n,l}$

$V_r(m,n,l) \subset \mathbb{C}^{m\times n\times l}$ the closure of 3-mode tensors of rank r at most

$\mathbb{P}V_r(m,n,l) = \mathrm{Sec}_r(\mathbb{P}^{m-1} \times \mathbb{P}^{m-1} \times \mathbb{P}^{l-1})$.

$I_r(m,n,l) \subset \mathbb{C}[\mathbb{C}^{m\times n\times l}]$ the ideal defining $V_r(m,n,l)$.

$\mathbf{T}_3(\mathcal{T}) \subset \mathbb{C}^{m\times n}$ subspace spanned by l frontal sections $[t_{i,j,k}]_{i=j=1}^{m,l}, k = 1, \ldots, l$.

Similarly $\mathbf{T}_1(\mathcal{T}) \subset \mathbb{C}^{n\times l}$, $\mathbf{T}_2(\mathcal{T}) \subset \mathbb{C}^{m\times l}$

$\mathrm{S}_n(\mathbb{C})$ - symmetric $n \times n$ matrices

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In Pachter-Sturmfels book [2, Conjecture 3.24] states $I_4(4, 4, 4)$ is generated by polynomials of degree 5 and 9. The degree 5 are coming from Strassen's commutative conditions [3, 1], degree 9 from Strassen's result: $V_4(3, 3, 3)$ is a hypersurface of degree 9.

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In view of degree 6 polynomials in $I_4(4,4,4)$ found by Landsberg and Manivel [5] Sturmfels revised the Salmon conjecture:

$I_4(4,4,4)$ is generated by polynomials of degree $5,6,9$ [4, §2].

Tensors of rank m in $\mathbb{C}^{m\times m\times l}$ Strassen's commutative conditions

$\mathcal{T} \in \mathbb{C}^{m\times m\times l}$, rank $\mathcal{T} = m$, $\mathbf{W} = \mathrm{span}(T_{1,3}, \ldots, T_{l,3}) \in \mathbb{C}^{m\times m}$ spanned by $\mathbf{u}_1\mathbf{v}_1^\top, \ldots, \mathbf{u}_m\mathbf{v}_m^\top$.

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generic case: $\exists P, Q \in \mathbf{GL}(m, \mathbb{C})$ $P\mathbf{W}Q$
subspace of commuting of diagonal matrices.
If $\mathbf{W}$ contains invertible Z then
$(PXQ)(PZQ)^{-1}(PYQ) = (PYQ)(PZQ)^{-1}(PXQ) \Rightarrow$
$X(\mathrm{adj}Z)Y = Y(\mathrm{adj}Z)X$
for all $X, Y \in \mathbf{W}$
equations of degree 5 for $m = 4$

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similarly $\mathrm{C}_r(X)\widetilde{\mathrm{C}_{m-r}(Z)}\mathrm{C}_r(Y) = \mathrm{C}_r(Y)\widetilde{\mathrm{C}_{m-r}(Z)}\mathrm{C}_r(X)$
equations of degree $m + r$ for $r = 1, \ldots, \lfloor \frac{m}{2} \rfloor$.

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For $m = 4, r = 2$ polynomials of degree 6 but no new info.

Strassen and Manivel-Landsberg conditions

Strassen 1983 $V_4(3,3,3)$ is a hypersurface of degree 9

$$\frac{1}{\det Z} \det \left(X(\mathrm{adj}\, Z)Y - Y(\mathrm{adj}\, Z)X\right) = 0$$

X, Y, Z are three sections of $\mathcal{T} = [t_{i,j,k}] \in \mathbb{C}^{3\times 3\times 3}$

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Landsberg-Manivel 2004: $I_4(3,3,4)$ contains polynomials of degree 6.
Study the action of $\mathbf{GL}(3,\mathbb{C}) \times \mathbf{GL}(3,\mathbb{C}) \times \mathbf{GL}(4,\mathbb{C})$ on $\mathbb{C}[\mathbb{C}^{3\times 3\times 4}]$
use Schur duality and symbolic computations to conclude existence of polynomials of degree 6.

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Bates and Oeding[4] constructed explicitly using symbolic computations 10 polynomials of degree 6 in $I_4(3,3,4)$.

Symmetrization conditions for $V_{m+1}(m, m, l)$ [1]

For a generic $\mathcal{T} = [x_{i,j,k}] \in \mathbb{C}^{m\times m\times l}$, $X_k = [t_{i,j,k}]_{i=j=1}^{mm}$ of rank $m+1$
$\mathbf{T}_3(\mathcal{T}) \in \mathbb{C}^{m\times m}$ generated by $\mathbf{u}_1\mathbf{v}_1^\top, \ldots, \mathbf{u}_{m+1}\mathbf{v}_{m+1}^\top$,
any m vectors out of $\mathbf{u}_1, \ldots, \mathbf{u}_{m+1}$ or $\mathbf{v}_1, \ldots, \mathbf{v}_{m+1}$ linearly independent

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$\exists P, Q \in \mathbf{GL}(m, \mathbb{C}) \Rightarrow P\mathbf{u}_i\mathbf{v}_i^\top Q^\top = \mathbf{e}_i\mathbf{e}_i^\top$ for $i = 1, \ldots, m$
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$\exists L, R \in \mathbf{GL}(m, \mathbb{C})$ such that $L\mathbf{T}_3(\mathcal{T}), \mathbf{T}_3(\mathcal{T})R \in \mathrm{S}_n(\mathbb{C})$ (Symcon)
$LX_i - (LX_i)^\top = 0, i = 1, \ldots, l$ (Lsymcon): $(\frac{l(m(m-1)}{2})$ linear equation in entries of L
$X_iR - (X_iR)^\top = 0, i = 1, \ldots, l$ (Rsymcon): $(\frac{l(m(m-1)}{2})$ linear equation in entries of R
and $LR^\top = R^\top L = \frac{1}{n}\mathrm{tr}(LR^\top)I_n$ - (LRcond)

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and $LR^\top = R^\top L = \frac{1}{n}\mathrm{tr}(LR^\top)I_n$ - (LRcond)
Existence of nonzero L, R: entries of $\mathcal{T}$ satisfy polynomial equations of degree m^2
(LRcond) yield polynomial equations of degree $2(m^2-1)$ when $\frac{l(m(m-1)}{2} \geq m^2$.

Characterization of $V_4(3,3,4)$

Generic subspace $\mathbf{W} \subset \mathrm{S}_m(\mathbb{C}), \dim \mathbf{W} = \frac{m(m-1)}{2} + 1$ intersects variety of symmetric matrices of rank 1 at least at $\frac{m(m-1)}{2} + 1$ lin. ind. mat.

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(LRcond) (degree 16) yield $LR^\top = R^\top L = 0 \Rightarrow \mathcal{T} \in V_4(3,3,4)$

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$$R = \mathbf{e}_3\mathbf{e}_3^\top \Rightarrow X_k = \begin{bmatrix} x_{1,1,k} & x_{1,2,k} & 0 \\ x_{2,1,k} & x_{2,2,k} & 0 \\ 0 & 0 & x_{3,3,k} \end{bmatrix}, \quad k = 1, 2, 3, 4,$$

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10 polynomials [4] are $x_{3,3,k}x_{3,3,l}f(\mathcal{X}) = 0, 1 \leq k \leq l \leq 4$

$$f(\mathcal{X}) := \det \begin{bmatrix} x_{1,1,1} & x_{1,2,1} & x_{2,1,1} & x_{2,2,1} \\ x_{1,1,2} & x_{1,2,2} & x_{2,1,2} & x_{2,2,2} \\ x_{1,1,3} & x_{1,2,3} & x_{2,1,3} & x_{2,2,3} \\ x_{1,1,4} & x_{1,2,4} & x_{2,1,4} & x_{2,2,4} \end{bmatrix}$$

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In first 3 cases for R $\mathcal{T} \in V_4(3,3,4)$

$$R = \mathbf{e}_3\mathbf{e}_3^\top \Rightarrow X_k = \begin{bmatrix} x_{1,1,k} & x_{1,2,k} & 0 \\ x_{2,1,k} & x_{2,2,k} & 0 \\ 0 & 0 & x_{3,3,k} \end{bmatrix}, \quad k = 1,2,3,4,$$

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So $\mathcal{T} \in V_4(3,3,4)$ since for $\mathcal{X} \in \mathbb{C}^{2\times 2\times 4}$: $\mathrm{rank}\ \mathcal{X} \le 4$
and $\mathrm{rank}\ \mathcal{X} \le 3$ if $\dim \mathbf{T}_3(\mathcal{X}) \le 3$.

From $V_4(3,3,4)$ to $V_4(4,4,4)$

Manivel-Landsberg[1]: Cor. 5.6: Let $\mathcal{T} \in \mathbb{C}^{4\times4\times4}$ satisfies Strassen's commutative conditions of degree 5. Then either $\mathcal{T} \in V_4(4,4,4)$ or there exists $p \in \{1,2,3\}, \mathbf{u}, \mathbf{v} \in \mathbb{C}^4 \setminus \{\mathbf{0}\}$ such that
$\mathbf{u}^\top \mathbf{T}_p(\mathcal{T}) = \mathbf{0}^\top, \mathbf{T}_p(\mathcal{T})\mathbf{v} = \mathbf{0}$.
I.e. after change of bases and permuting the factors of $\mathbb{C}^4$ $\mathcal{T} \in \mathbb{C}^{3\times3\times4}$

From $V_4(3,3,4)$ to $V_4(4,4,4)$

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[1] characterizes subspace $\mathbf{U} \subset \mathbb{C}^{m\times m}$ where most of the matrices are of rank $m-1$ and satisfy Strassen's commutative condition.

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