Some open problems in matchings in graphs

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Overview

- Matchings in graphs
- Number of $k$-matchings in bipartite graphs as permanents and haffnians
- Upper bounds on permanents and haffnians: results and conjectures.
- Lower bounds on permanents and haffnians: results and conjectures.
- Probabilistic methods
Matchings

- $G = (V, E)$ undirected graph with vertices $V$, edges $E$.
- matching in $G$: $M \subseteq E$
  - no two edges in $M$ share a common endpoint.
- $e = (u, v) \in M$ is dimer
- $v$ not covered by $M$ is monomer.
- $M$ called monomer-dimer cover of $G$.
- $M$ is perfect matching $\iff$ no monomers.
- $M$ is $k$-matching $\iff$ $\#M = k$. 

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Generating matching polynomial

- $\phi(k, G)$ number of $k$-matchings in $G$, $\phi(0, G) := 1$
- $\Phi_G(x) := \sum_k \phi(k, G)x^k$ matching generating polyn.
- roots of $\Phi_G(x)$ are real nonpositive Heilmann-Lieb 1972.
  Newton inequalities hold
- $\Phi_{G_1 \cup G_2}(x) = \Phi_{G_1}(x)\Phi_{G_2}(x)$

Examples:

$\Phi_{K_{2r}}(x) = \sum_{k=0}^{r} \binom{2r}{2k} \frac{\prod_{j=0}^{k-1} \binom{2k-2j}{2}}{k!} x^k = \sum_{k=0}^{r} \frac{(2r)!}{(2r-2k)!2^k k!} x^k$

$\Phi_{K_{r, r}}(x) = \sum_{k=0}^{r} \binom{r}{k} 2k! x^k$

$G(r, 2n) \supset GB(r, 2n)$ set of $r$-regular and regular bipartite graphs on $2n$ vertices, respectively

$qK_{r, r} \in GB(r, 2rq)$ a union of $q$ copies of $K_{r, r}$.

$\Phi_{qK_{r, r}} = \Phi_{K_{r, r}}^q$
Formulas for $k$-matchings in bipartite graphs

$G = (V, E)$ bipartite $V = V_1 \cup V_2, E \subset V_1 \times V_2$, represented by bipartite adjacency matrix $B(G) = B = [b_{ij}]_{i,j=1}^{m \times n} \in \{0, 1\}^{m \times n}, \#V_1 = m, V_2 = n$.

Example: Any subgraph of $\mathbb{Z}^d$ is bipartite

CLAIM: $\phi(k, G) = \text{perm}_k(B(G))$.

Prf: Suppose $n = \#V_1 = \#V_2$.
Then permutation $\sigma : \langle n \rangle \rightarrow \langle n \rangle$ is a perfect match iff $\prod_{i=1}^{n} b_{i\sigma(i)} = 1$.
The number of perfect matchings in $G$ is $\phi(n, G) = \text{perm} B(G)$.

Computing $\phi(n, G)$ is $\#P$-complete problem Valiant 1979

For $G = (\langle 2n \rangle, E)$ bipartite $G \in GB(r, 2n) \iff \frac{1}{r}B(G) \in \Omega_n \iff G$ is a disjoint (edge) union of $r$ perfect matchings.
Matching on nonbipartite graphs

\[ G = (V, E), |V| = 2n, \]
\[ A(G) = [a_{ij}] \in S_0(2n, \{0, 1\}) \text{ - adjacency matrix of } G \]

\[ \phi(n, G) = \text{haf}(A(G)) = \sum_{M \in \mathcal{M}(K_{2n})} \prod_{(i,j) \in M} a_{ij} \]
\[ \mathcal{M}(K_{2n}) \text{ the set of perfect matchings in } K_{2n} \]

\[ \phi(k, G) = \text{haf}_k(A(G)) = \sum_{M \in \mathcal{M}_k(K_{2n})} \prod_{(i,j) \in M} a_{ij} \]
\[ \mathcal{M}_k(K_{2n}) \text{ the set of } k \text{ matchings in } K_{2n} \]

Claim \( \text{perm}(A(G)) \geq \text{haf}(A(G))^2 \). Equality holds if \( G \) is bipartite.
Main problems

Find good estimates on

\[ s_n(k, r) := \min_{G \in G(r, 2n)} \phi(k, G) \leq t_n(k, r) := \min_{G \in G_B(r, 2n)} \phi(k, G) \]

\[ S_n(k, r) := \max_{G \in G(r, 2n)} \phi(k, G) \geq T_n(k, r) := \max_{G \in G_B(r, 2n)} \phi(k, G) \]

Completely solved case \( r = 2 \) [7]

\[ S_n(k, 2) = T_n(k, 2) \text{ achieved only for } G = nK_{2,2} \text{ or } G = mK_{2,2} \cup C_6. \]

\[ t_n(k, 2) \text{ achieved only for } C_{2n} \]

\[ s_n(k, 2) \text{ achieved only for } mC_3, mC_3 \cup C_4 \text{ or } mC_3 \cup C_5. \]
The upper bound conjecture

\[ S_{qr}(k, r) = T_{qr}(k, r) = \phi(k, qK_r, r) \]

\( k = qr \) Follows from Bregman’s inequality (see also [3])

\[ \text{perm } A \leq \prod_{i=1}^{n} (r_i!)^{\frac{1}{r_i}} \]

\[ A = [a_{ij}] \in \{0, 1\}^{n \times n} \quad r_i = \sum_{j=1}^{n}, i = 1, \ldots, n \]

Egorichev-Alon-Friedland for \( G = (V, E), |V| = 2n \)
\[ \phi(n, G) \leq \prod_{v \in V} (\deg(v)!)^{\frac{1}{2\deg(v)}} \]

Equality holds iff \( G \) a union of complete bipartite graphs

\[ S_n(k, r) \leq \binom{2n}{2k} (r!)^{\frac{k}{r}} \]
\[ T_n(k, r) \leq \min\left( \binom{n}{k}^2 (r!)^{\frac{k}{r}}, \binom{n}{k} r^k \right) \]

Asymptotic versions

\[ Sa(p, r) = \limsup_{n_j \to \infty, \frac{k_j}{n_j} \to p \in [0, 1]} \frac{\log S_{n_j}(k_j, r)}{2n_j} \]

\[ Ta(p, r) = \limsup_{n_j \to \infty, \frac{k_j}{n_j} \to p \in [0, 1]} \frac{\log T_{n_j}(k_j, r)}{2n_j} \]

\[ sa(p, r) = \liminf_{n_j \to \infty, \frac{k_j}{n_j} \to p \in [0, 1]} \frac{\log s_{n_j}(k_j, r)}{2n_j} \]

\[ ta(p, r) = \liminf_{n_j \to \infty, \frac{k_j}{n_j} \to p \in [0, 1]} \frac{\log t_{n_j}(k_j, r)}{2n_j} \]

Next slide gives the graphs of AUMC and the upper bounds for \( Ta(p, 4) \).
$r = 4$ upper bounds

Figure: $h_{K(4)}$-green, $\text{upp}_{4,1}$-blue, $\text{upp}_{4,2}$-orange
The lower bounds: Bipartite case

\[ r^n \min_{C \in \Omega_n} \perm_k C \leq \phi(k, G) \text{ for any } G \in \mathcal{GB}(r, 2n) \]

\[ J_n = B(K_{n,n}) = [1] \text{ the incidence matrix of the complete bipartite graph } K_{n,n} \text{ on } 2n \text{ vertices} \]

van der Waerden permanent conjecture 1926:

\[ \min_{C \in \Omega_n} \perm C = \perm \frac{1}{n} J_n \left( = \frac{n!}{n^n} \approx \sqrt{2\pi n} e^{-n} \right) \]

Tverberg permanent conjecture 1963:

\[ \min_{C \in \Omega_n} \perm_k C = \perm_k \frac{1}{n} J_n \left( = \binom{n}{k}^2 \frac{k!}{n^k} \right) \]

for all \( k = 1, \ldots, n \).
History

- In 1979, Friedland showed the lower bound \( \text{perm } C \geq e^{-n} \) for any \( C \in \Omega_n \) following T. Bang’s announcement 1976. This settled the conjecture of Erdös-Rényi on the exponential growth of the number of perfect matchings in \( d \geq 3 \)-regular bipartite graphs 1968, Voorhoeve 1979.

- van der Waerden permanent conjecture was proved by Egorichev and Falikman 1981.

- Tverberg conjecture was proved by Friedland 1982.

- 1979 proof is tour de force according to Bang.

- 1981 proofs involve directly (Egorichev) and indirectly (Falikman) use of Alexandroff mixed volume inequalities with the conditions for the extremal matrix.


- There are new simple proofs using nonnegative hyperbolic polynomials e.g. Friedland-Gurvits 2008.
Lower matching bounds for bipartite graphs

Voorhoeve-1979 \((r = 3)\) Schrijver-1998

\[
\phi(n, G) \geq \left( \frac{(r-1)^{r-1}}{r^{r-2}} \right)^n \quad \text{for} \quad G \in GB(r, 2n)
\]

Gurvits 2006: \(A \in \Omega_n\), each column has at most \(r\) nonzero entries:

\[
\text{perm} A \geq \frac{r!}{r^r}\left( \frac{r}{r-1} \right)^{r(r-1)} \left( \frac{r-1}{r} \right)^{(r-1)n}.
\]

Cor: \(\phi(n, G) \geq \frac{r!}{r^r}\left( \frac{r}{r-1} \right)^{r(r-1)} \left( \frac{r-1}{r^{r-2}} \right)^n\)

Con FKM 2006: \(\phi(k, G) \geq {n \choose k}^2 \left( \frac{nr-k}{nr} \right)^{nr-k} \left( \frac{kr}{n} \right)^k, \ G \in GB(r, 2n)\)

F-G 2008 showed weaker inequalities
Thm: $r \geq 3, s \geq 1$ integers, $B_n \in \Omega_n, n = 1, 2, \ldots$ each column of $B_n$ has at most $r$-nonzero entries. $k_n \in [0, n] \cap \mathbb{N}, n = 1, 2, \ldots$, \(\lim_{n \to \infty} \frac{k_n}{n} = p \in (0, 1)\) then

$$\liminf_{n \to \infty} \frac{\log \text{perm}_{k_n} B_n}{2n} \geq \frac{1}{2} \left( -p \log p - 2(1 - p) \log(1 - p) \right) + \frac{1}{2}(r + s - 1) \log(1 - \frac{1}{r + s}) - \frac{1}{2}(s - 1 + p) \log(1 - \frac{1 - p}{s})$$

Prf combines properties positive hyperbolic polynomials, capacity and the measure on $\mathcal{G}(r, 2n)$

- Cor: $r$-ALMC holds for $p_s = \frac{r}{r+s}, s = 0, 1, \ldots$,
- Con: under Thm assumptions

$$\liminf_{n \to \infty} \frac{\log \text{perm}_{k_n} B_n}{2n} \geq f(r, p) - \frac{p}{2} \log r$$

- For $p_s = \frac{r}{r+s}, s = 0, 1, \ldots$, conjecture holds
Lower bounds for matchings in regular non-bipartite graphs

Petersen’s THM: A bridgeless cubic graph has a perfect match

Problem: Find the minimum of the biggest match in \( G(r, 2n) \) for \( r > 2 \).

Does every \( G \in G(r, 2n) \) has a match of size \( \left\lfloor \frac{2n}{3} \right\rfloor \)? (True for \( r = 2 \).)

Esperet-Kardos-King-Král-Norine:
Every cubic bridgeless graph has at least \( 2^{\frac{|V|}{3656}} \) perfect matchings
An analog the van der Waerden conjecture

THM Edmonds 1965: A symmetric doubly stochastic matrix with zero diagonal of even order $A = [a_{ij}]_{i,j=1}^{2n}$ is a convex combination of symmetric permutation matrices with zero diagonal if and only if

$$\sum_{i,j \in S} a_{ij} \leq |S| - 1$$

for any odd subset $S \subset \{1, \ldots, 2n\}$ (*)

Denote by $\Psi_{2n}$ the subset of all symmetric doubly stochastic matrices of the above form

Problem: Find $\min \text{haf}(A), A \in \Psi_{2n}$

CONJECTURE: The minimum is achieved only for the matrix

$$\frac{1}{2n-1} A(K_{2n})$$

$$\text{haf}(\frac{1}{2n-1} A(K_{2n})) \approx e^{-n} \sqrt{2e} < \text{haf}(\frac{1}{n} A(K_{n,n})) \approx e^{-n} \sqrt{2\pi n}$$
An analog the Tverberg conjecture

$\min_{A \in \Psi_{2n}} haf_k(A) = haf_k \left( \frac{1}{2n-1} A(K_{2n}) \right) =$

$$\binom{2n}{2k} \frac{1}{(2n-1)^k} \frac{(2k)!}{2^k k!}$$

Note $\frac{1}{3} A(G) \in \Psi_{2n}$, $G \in G(3, 2n)$ iff there are at least 3 edges coming out of any the set of odd number of vertices (Improves significantly the lower bound [5] if gen. v.d. Waerden conj. true)
**Hyperbolic polynomials**

**THM:** Good lower bounds hold for $\text{haf}_k(A)$ if $A \in \Psi_{2n}$ if $n - 1$ eigenvalues of $A$ are nonpositive.

**Outline of proof:** Fact $x^\top Ax$ is a hyperbolic polynomial for a nonnegative symmetric matrix iff $A$ has all but one nonpositive eigenvalues [4]

$$\text{haf}_k A = (2^k k!)^{-1} \sum_{1 \leq i_1 < \ldots < i_{2k} \leq 2n} \frac{\partial^{2k}}{\partial x_{i_1} \ldots \partial x_{i_{2k}}} (x^\top A x)^k$$

Use the arguments of [1] to show

$$\text{haf} A \geq \left(\frac{n-1}{n}\right)^{(n-1)n} \geq e^{-n}$$
\( A = [a_{ij}] \in \mathbb{R}_+^{n \times n}, X(A) := [\sqrt{a_{ij}}x_{ij}], \)

\( x_{ij} \) independent random variables \( E(x_{ij}) = 0, E(x_{ij}^2) = 1 \)

\( E((\det X(A))^2) = \perm A. \) Godsil-Gutman 1981

**Concentration results**

**A. Barvinok 1999 -**

1. \( x_{ij} \) real Gaussian \( \Rightarrow \det X(A)^2 \) with high probability

\( \in [c^n \perm A, \perm A] \ c \approx 0.28 \)

2. \( x_{ij} \) complex Gaussian \( E(|x_{ij}|^2) = 1 \Rightarrow |\det X(A)|^2 \) with high probability

\( \in [c^n \perm A, \perm A] \ c \approx 0.56 \)

3. \( x_{ij} \) quaternion Gaussian \( E(|x_{ij}|^2) = 1 \Rightarrow |\det X(A)|^2 \) with high probability

\( \in [c^n \perm A, \perm A] \ c \approx 0.76 \)

**Friedland-Rider-Zeitouni 2004:**

\( 0 < a \leq a_{ij} \leq b, x_{ij} \) real Gaussian \( \Rightarrow \det X(A)^2 \) with high probability

\( \in [(1 - \varepsilon_n) \perm A, \perm A] \ \varepsilon_n \to 0 \)
FRZ results use concentration for $\log_\varepsilon \det Z(A) = \text{tr} f(Z(A))$, 
$Z(A) = X(A)^\top X(A) \succeq 0, f = \log_\varepsilon x = \log \max(x, \varepsilon)$.

or $\log_\varepsilon \det Y(A)$, $Y(A) = \begin{bmatrix} 0 & X(A) \\ -X(A)^\top & 0 \end{bmatrix}$

$E(\det(\sqrt{t}I + Y(A))) = \Phi_{G_w}(t), t \geq 0$
matching polynomial for weighted graph induced by $A$

**Thm:** Concentration of $\log \det(\sqrt{t}I + Y(A))$ around expected value
$\log \tilde{\Phi}_{G_w}(t), t > 0$ which less $\log \Phi_{G_w}(t)$

$\frac{1}{n} \log \tilde{\Phi}(t, G_\omega) \leq \frac{1}{n} \log \Phi(t, G_\omega) \leq \frac{1}{n} \log \tilde{\Phi}(t, G_\omega) + \min(\frac{\max_{i,j} |a_{ij}|}{2t}, 1.271)$

Meaning of $\tilde{\Phi}_{G_w}(t)$?
Make each undirected edge \((i, j)\) with weight \(a_{ij} = a_{ji} \geq 0\) to two opposite directed edges with weights \(\pm a_{ij}\) to obtain a skew symmetric matrix

\[
B = [b_{ij}] \in \mathbb{R}^{(2n) \times (2n)}, \quad b_{ii} = 0
\]

\[
Y(B) = [\text{sign}(b_{ij}) \sqrt{|b_{ij}|} x_{ij}], \quad x_{ij} = x_{ji}, \quad x_{12}, \ldots, x_{(2n-1)(2n)} \quad \text{i.r.v}
\]

\[
E(x_{ij}) = 0, \quad E(x_{ij}^2) = 1
\]

\[
E(\det Y(B)) = \text{haf}A - \text{total weight of weighted matchings in induced graph by } A
\]

\[
E(\det(\sqrt{tI} + Y(B))) = \Phi_{G_w}(t) - \text{the weighted matching polynomial of } G(A).
\]

All the results for bipartite graphs carry over to nonbipartite graphs.
Jerrum-Sinclair-Vigoda 2004: fully polynomial randomized approximation scheme (fpras) to compute $\text{perm } A$
A variation of MCMC method using rapidly mixed Markov chains converging to equilibrium point

The proofs do not carry over for nonbipartite graphs

Any $\#P$ complete problem has fpras?
Expected values of $k$-matchings for bipartite graphs

- Permutation $\sigma: \langle nr \rangle \rightarrow \langle nr \rangle$ induces $G(\sigma) \in \mathcal{GB}_{\text{mult}}(r, 2n)$ and vice versa.
  
  $$G(\sigma) = \{(i, \lceil \frac{\sigma((i-1)r+j)}{r} \rceil), \ j = 1, \ldots, r, \ i = 1, \ldots, n\} \subset \langle n \rangle \times \langle n \rangle$$

- $\mu$ probability measure on $\mathcal{GB}_{\text{mult}}(r, 2n)$:
  
  $$\mu(G(\sigma)) = (((nr)!)^{-1}$$

- FKM 06:
  
  $$E(k, n, r) := E(\phi(k, G)) = \binom{n}{k}^2 r^{2k} k! (nr - k!) (nr!)^{-1}, \quad k = 1, \ldots, n$$

- $1 \leq k_l \leq n_l, \ l = 1, \ldots$, increasing sequences of integers s.t.
  
  $$\lim_{l \to \infty} \frac{k_l}{n_l} = p \in [0, 1].$$

  Then

  $$\lim_{l \to \infty} \frac{\log E(k_l, n_l, r)}{2nk} = f(p, r)$$

  $$f(p, r) := \frac{1}{2} (p \log r - p \log p - 2(1 - p) \log(1 - p) + (r - p) \log(1 - \frac{p}{r}))$$
Similar results for non-bipartite graphs?
References


References


References


