SOME INEQUALITIES FOR THE SPECTRAL RADIUS OF NON-NEGATIVE MATRICES AND APPLICATIONS

by

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1. Introduction. The purpose of this work is to establish useful lower and upper estimates for the spectral radius of certain classes of positive matrices which apart from their independent interest are pertinent to the study of a number of mathematical models of population genetics and also apply to the solution of some cases of inverse eigenvalue problems.

In the stability analysis of certain equilibria states of physical and biological systems, it is relevant to determine useful conditions indicating when the largest eigenvalue \( \rho \) for a matrix of the type \( MD \) (composed from a general non-negative and positive diagonal matrix) exceeds or is smaller than 1. In the physical setting, \( M = ||m_{ij}|| \) is commonly an \( n \times n \) matrix of non-negative elements corresponding to a Green's function for a vibrating coupled mechanical system of \( n \) mass points, while \( D \) is a diagonal matrix with positive diagonal entries \( \{d_1, d_2, \cdots, d_n\} \) such that \( d_i, \ i = 1, 2, \cdots, n \), relates to the mass at position \( i \).

In the genetics context, a population is distributed in \( n \) demes (habitats, \( \{\phi_1, \phi_2, \cdots, \phi_n\} \) subject to local natural selection forces and inter-deme migration pressures. The changes in the population composition of a trait expressed by two possible types (genes) labeled \( A \) and \( a \) are observed over

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successive generations. The transformation of gene frequency accountable to
the local selection forces in deme \( \varphi_i \), is characterized by a non linear relation
\[
\xi' = f_i(\xi)
\]
such that if \( \xi \) is the A-frequency in \( \varphi \), at the start of a generation then after
the action of mating and natural selection the resulting A-frequency prior
to migration is \( \xi' \). Generally \( f_i(\xi) \) defined for \( 0 \leq \xi \leq 1 \) is continuously dif-
ferrentiable and monotone increasing obeying the boundary conditions
\[
f_i(0) = 0, \quad f_i(1) = 1
\]
signifying that selection operates to maintain a pure population composition.
(Thus, in this formulation events are ignored, i.e., new mutant forms
arising in the time frame under consideration cannot be established.)

The dispersal (= migration) pattern is described by the matrix \( M = [\mathbf{m}_{i,j}] \)
where \( m_{ij} \) is the proportion of the population in \( \varphi_j \) immigrating from deme
\( \varphi_i \). Thus, from the interpretation, \( M \) is a stochastic matrix. Let \( x_i \) denote
the proportion of type A in deme \( i \) at the start of a generation and \( x_i' \) the
frequency for the next generation. The global transformation equations
connecting \( x = (x_1, \ldots, x_n) \) to \( x' = (x_1', \ldots, x_n') \) over two successive generations
takes the form
\[
x_i' = \sum_{j=1}^{n} m_{ij} f_j(x_j), \quad i = 1, 2, \ldots, n.
\]

If the migration and selection forces operate in reverse order then the trans-
formation equations become
\[
x_i' = f_i\left(\sum_{j=1}^{n} m_{ij} x_j\right), \quad i = 1, 2, \ldots, n.
\]

We abbreviate the transformation (1.2) by \( \mathbf{x}' = T\mathbf{x} \). A tacit assumption
underlying the models (1.2) and (1.3) is that the deme sizes are large and
approximately of equal magnitude. (For variations and other facets of the
general model concerned with population subdivision and selection migration
interaction, we refer to Karlin [9].)

Owing to (1.1) we find that the frequency states \( \mathbf{0} = (0, 0, \ldots, 0) \) and
\( \mathbf{1} = (1, \ldots, 1) \) are invariant points of \( T \) corresponding to fixation of the
population consisting exclusively of the a-type and A-type, respectively. The gradient
matrix of \( T \) at \( \mathbf{0} \) (that is the local linear approximation valid about \( \mathbf{0} \)) reduces to
\[
T(\mathbf{0}) = MD
\]

where
\[
D = \text{diag}(f_1'(0), f_2'(0), \ldots, f_n'(0)).
\]

Manifestly, \( f_i'(0) \geq 0 \) since \( f_i \) is increasing. We will stipulate henceforth
strict inequality (reflecting the usual phenomenon for selection forces, so that
in the previous notation \( d_i = f_i'(0), i = 1, 2, \ldots, n \). Because \( T \) is a monotone
transformation (as each \( f_i(\cdot) \) is monotone), the local instability of the equilibrium
state \( \mathbf{0} \) is assured if the spectral radius
\[
\rho(MD) = (\text{the spectral radius of } MD) \text{ exceeds } 1.
\]

In the presence of (1.5) for any \( \mathbf{x} \) close to but distinct from \( \mathbf{0} \) the iterates \( T^n\mathbf{x} \)
will depart from the neighborhood of \( \mathbf{0} \). The same considerations show that
\( \mathbf{0} \) is locally stable if \( \rho(MD) < 1 \).

The delineation of natural conditions satisfied by the selection coefficients
(covered in \( D \)) and their coupling to the migration rates embodied in \( M \) that imply \( \rho(MD) > 1 \) are of much interest. In fact where the spectral radius
exceeds 1, then the A-type is “protected”, i.e., this type is maintained in the
population and can never approach extinction.

The gradient matrix at \( \mathbf{0} \) for the model (1.3) is \( MD \). In view of the familiar
property that the eigenvalues of \( MD \) and \( MD \) coincide and are identical to
the eigenvalues of \( D^TMD \), all criteria for “protection” of the A-type is the same
for the models (1.2) and (1.3). Thus the operational order of the selection
and migration forces does not influence the occurrence of “A-protection” or
“A extinction.” (See Karlin [9] for other mathematical developments and
extensive discussions of these models.)

The classical inverse eigenvalue problems have the following formulation.
Let \( L \) be an ordinary differential operator on \( R = \text{real line} \). Find a “potential”
\( g(x) \) such that the operator \( K = (Lu)(x) + g(x)u(x) \) coupled with appropriate
boundary conditions possesses a prescribed spectrum. Another version of
the problem states: Determine a “density” \( p(x) \) such that the operator
\( Ku = (Lu)(x)/p(x) \) with associated boundary conditions involves a prescribed
spectrum. A voluminous literature exists on this subject, e.g., a good survey
is contained in Chap. 1 of Hald [5]. Some discrete matrix analogs of these
problems take the following form.

(i) Let an \( n \times n \) matrix \( M \) be given. Determine a diagonal matrix \( D \) with
the property that the spectrum of \( M + D \) coincides with a prescribed set
\( \Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \). This problem is henceforth referred to as the inverse
additive eigenvalue problem (I.A.E.P.).

(ii) If \( M + D \) is replaced by \( MD \) then we are confronted with the inverse
multiplicative eigenvalue problem (I.M.E.P.).

For some references and elaborations on certain aspects of these problems, see
Friedland [3].

It is known that both inverse matrix problems generally do not admit unique
solutions and if the allowable \( D \) matrices are required, say, to be positive definite
(or real) then cases of no solution arise. In the complex domain (i.e., where
\( D \) is permitted to be a diagonal complex matrix) generally \( n! \) solutions are
available. We will construct in Section 6 a class of special spectra \( \Lambda \) related to
\( M \) for which problems (i) and (ii) admit a unique real solution.
With the motivation well rooted, we next highlight some of the principal results of this paper.

Let \( M = ||m_{i,j}||_s \) be a non-negative matrix of order \( n \). Denote by \( \rho(M) \) the spectral radius of \( M \). The classical Perron-Frobenius theorem tells us that \( \rho(M) \) is an eigenvalue of \( M \) with the property that there exist non-trivial non-negative vectors \( u = (u_1, \ldots, u_n) \) and \( v = (v_1, \ldots, v_n) \) satisfying

\[
Mu = \rho(M)u, \quad vM = \rho(M)v.
\]

Moreover, if \( M \) is also irreducible then \( \rho(M) \) is a positive simple eigenvalue and the associated eigenvectors \( u \) and \( v \) display only positive components.

With \( u \) and \( v \) prescribed positive vectors and normalized to satisfy

\[
\sum_{i=1}^{n} u_i v_i = 1
\]

we denote by \( \Xi(u, v) \) the set of all non-negative matrices fulfilling (1.6), with \( \rho(M) = 1 \).

Obviously \( \Xi(u, v) \) is a convex polyhedral set.

In the special case \( u = v = e = (1/\sqrt{n}), (1, 1, \ldots, 1) \), (We shall employ the same notation \( e = (1, 1, \ldots, 1) \) without the scalar factor \( 1/\sqrt{n} \).

There should arise no confusion of the meaning of \( e \) from the context at hand.) then \( \Xi(e, e) \) comprises precisely the set of all doubly stochastic matrices. It is a familiar fact (commonly referred to as the Birkhoff Theorem) that the permutation matrices are the extreme points of \( \Xi(e, e) \).

The theorem now stated and proved in Section 3 is fundamental to a number of the applications.

**Theorem 3.1.** Let \( M \) be an \( n \times n \) irreducible non-negative matrix in \( \Xi(u, v) \). Then for any positive definite diagonal matrix \( D = \text{diag}(d_1, \ldots, d_n) \), we have

\[
\rho(DM) = \rho(MD) \geq \prod_{i=1}^{n} d_i^{u_i v_i}.
\]

If \( \rho(M) \) is not normalized (equal to 1) then (1.8) is replaced by

\[
\rho(DM) \geq \left( \prod_{i=1}^{n} d_i^{u_i v_i} \right)^{\rho(M)}.
\]

A basic inequality underlying (1.8) is that for any positive vector \( x > 0 \) and \( M \in \Xi(u, v) \), we have

\[
\prod_{i=1}^{n} \left[ \frac{(Mx)_i}{x_i} \right]^{u_i v_i} \geq 1.
\]

Equality holds in (1.9) if \( x = \alpha u \) for any \( \alpha > 0 \). Where \( M \) is strictly positive then the equality in (1.9) entails \( x = \alpha u \) for some \( \alpha > 0 \).

A much simpler case of Theorem 3.1 concerns \( \Xi(e, e) = \text{the class of doubly stochastic matrices} \).

**Theorem 2.1.** Let \( M \in \Xi(e, e) \) and \( D \) a positive definite diagonal matrix. Then

\[
\rho(DM) \geq \left( \prod_{i=1}^{n} d_i \right)^{1/n}.
\]

If \( M \) is positive the inequality is sharp unless \( D = \alpha I \) for some \( \alpha > 0 \). Equality holds in (1.10) for the permutation matrix

\[
P = ||m_{i,j}||, \quad m_{i,i+1} = 1, \quad i = 1, 2, \ldots, n - 1; \quad m_{n,1} = 1.
\]

The proof of Theorem 2.1 set forth in Section 2 is elementary but the facts (1.8) and (1.9) are markedly deeper.

The inequality (1.8) can be improved by restricting further the class of matrices.

**Theorem 4.1.**

(i) \( \Xi(e, e) = \text{of the form } M = E_1KE_2 \) where \( E_1 \) and \( E_2 \) are diagonal positive definite matrices and \( K \) is positive semi-definite. Let \( D \) be as in Theorem 3.1. Then

\[
\rho(DM) \geq \sum_{i=1}^{n} u_i v_i d_i.
\]

Moreover, for any \( x > 0 \) and \( M \) irreducible, we have

\[
\sum_{i=1}^{n} u_i v_i \frac{x_i}{(Mx)_i} \leq \frac{1}{\rho(M)}.
\]

If \( M \in \Xi(u, v) \) is the specific rank one matrix, \( M = u'v = ||u,v|| \) then equality occurs in (1.12).

Note for this latter example that \( M = E_1KE_2 \) where \( E_1 = \text{diag}(u_1, u_2, \ldots, u_n) \),

\[
E_2 = \text{diag}(v_1, v_2, \ldots, v_n), \quad \text{and } K = ||k_{i,j}||,
\]

\[
k_{i,j} = 1 \quad \text{for all} \quad i, j.
\]

The arithmetic geometric mean inequality implies

\[
\sum_{i=1}^{n} u_i v_i d_i \geq \prod_{i=1}^{n} d_i^{u_i v_i}.
\]

((1.7) is needed here) which shows that (1.12) is usually a genuine sharpening of (1.8).

Another class of matrices to which (1.12) applies is described in Theorem 4.2 of Section 4. There is evidence for the conjecture that where \( M \in \Xi(u, v) \) is suitably totally positive, then (1.12) is also correct. On the other hand the inequality (1.12) cannot always apply for matrices of class \( \Xi(u, v) \) as revealed by the example (1.11).
Section 5 is devoted to some extensions of (1.8) and (1.12). If $M$ is appropriately totally positive, then by passing to the higher order comounds of $M$ we extract estimates for the product of the largest $k$ eigenvalues of $MD$, that is for the quantity $\prod_{i=1}^{k} \lambda_i(MD)$ where $\lambda_1(MD) > \lambda_2(MD) > \cdots > \lambda_k(MD)$ are the eigenvalues.

A variety of criteria emanating from (1.8) and (1.12) discerning instability of the expected point $0$ of the transformation (1.2) are discussed in Section 5. Finally, the evaluation of $\rho(MD)$ for some important special examples are set forth.

In Section 5 we apply the results of the earlier sections especially Theorems 3.1 and 4.1 to the study of certain matrix inverse eigenvalue problems. We describe here one of our main findings. Prior to doing this we prepare some background.

Let $A$ be non-negative and irreducible. It can be shown that there exists a unique (up to a constant factor) positive definite diagonal matrix $C$ such that

(1.15) $CAC^{-1}e = e(CAC^{-1})$, $e = (1, \cdots , 1)$ (Lemma 6.1).

With (1.15) established, we readily produce a unique real matrix $C_0$ engendering

(1.16) $(CAC^{-1} + C_0)e = e(CAC^{-1} + C_0) = e$.

Now we state our main result on the inverse additive eigenvalue problem (I.A.E.P.).

**Theorem 6.1.** Let $A$ be non-negative and irreducible. Let $C = \text{diag}(c_1, \cdots , c_n)$ and $C_0$ be determined conforming to (1.15) and (1.16), respectively. Denote by $\Lambda^* = \{\lambda_1^*, \lambda_2^*, \cdots , \lambda_m^*\}$ the spectrum of $A + C_0$. Let $\Lambda = \{\lambda_1, \cdots , \lambda_m\}$ be given and suppose the I.A.E.P. is solvable with respect to $\Lambda$, i.e., there exists at least one real diagonal $D$ such that $A + D$ has spectrum $\Lambda$. Then $\Lambda$ satisfies

(1.17) $(n - 1)\lambda_1 \geq \sum_{i=2}^{n} c_i\lambda_i + \sum_{i=2}^{n} \lambda_i$.

Equality occurs in (1.17) only for the prescription $\Lambda^*(\alpha) = \{\lambda_1^* + \alpha, \lambda_2^* + \alpha, \cdots , \lambda_m^* + \alpha\}$ for any real $\alpha$. Where $\Lambda = \Lambda^*(\alpha)$ then the I.A.E.P. is uniquely solvable.

A version of Theorem 6.1 for the I.M.E.P. is the content of Theorem 6.2 and its corollary.

The final Section 7 is devoted to establishing a number of extensions of the previous matrix results to certain integral operator analogs. A convolution example is studied in detail.

**2. Inequalities for the spectral radius for $MD$ where $M$ is doubly stochastic and $D$ is positive diagonal.**

We adhere to the notation of Section 1.

Throughout this section we focus on the collection of matrices $\mathfrak{M}(e, e)$, $e = (1, 1, \cdots , 1)$, (see after (1.7)), consisting of all doubly stochastic matrices. Recall that $\mathfrak{M}(e, e)$ is a convex set spanned by the extreme points, the latter identified as all permutation matrices.

The next lemma is relevant to a number of contexts.

**Lemma 2.1.** Let $M$ be doubly stochastic. Then

(i)

\[ \min_{x > 0} \left[ \frac{\prod_{i=1}^{n} (Mx)_i}{\prod_{i=1}^{n} x_i} \right] \geq 1 \] (the infimum extended over all positive vectors $x$).

Equality is always achieved independent of $x > 0$ provided $M$ is a permutation matrix or for all $M \in \mathfrak{M}(e, e)$ if $x = \alpha e$, $(\alpha > 0)$. If $M$ is positive, that is, all entries of $M$ are positive, then the equality in (2.1) persists only if $x = \alpha e$.

(ii)

\[ \min_{x > 0} \frac{\sum_{i=1}^{n} (Mx)_i^2}{\sum_{i=1}^{n} x_i} \geq n \]

with equality attained for $x = \alpha e$ or $M = \text{the identity matrix}$. If $M$ is irreducible non-negative then equality holds only if $x = \alpha e$, $\alpha > 0$.

**Remark 2.1.** The function $\phi(\xi_1, \xi_2, \cdots, \xi_n) = \prod_{i=1}^{n} \xi_i$, $(\xi_i > 0)$ is Schur-concave and accordingly (2.1) ensures by an elementary application of the theory of "majorization," (Hardy-Littlewood-Pólya [6], Chaps. 1 and 2). To maintain this writing self contained, we provide a direct proof.

**Proof.**

(i) Since the log function is strictly concave and $M$ is doubly stochastic we find

(2.3) \[
\sum_{i=1}^{n} \log \left( \sum_{j=1}^{n} m_{ij}x_i \right) \geq \sum_{i=1}^{n} \frac{\sum_{j=1}^{n} m_{ij} \log x_i}{x_i} = \sum_{i=1}^{n} \frac{\log x_i \left( \sum_{j=1}^{n} m_{ij} \right)}{x_i} = \sum_{i=1}^{n} \log x_i
\]

and the resulting inequality is clearly synonymous with (2.1). The circumstances of equality can readily be discerned.

(ii) This is an immediate consequence of the arithmetic-geometric mean inequality imposed on the fact of (2.1) plus verification of the cases of equality. Indeed, we have

(2.4) \[
\frac{1}{n} \sum_{i=1}^{n} \frac{(Mx)_i}{x_i} \geq \left( \prod_{i=1}^{n} \frac{(Mx)_i}{x_i} \right)^{1/n} \geq 1
\]

and equality prevails throughout for $x = \alpha e$, $\alpha$ positive. Equality in (2.2) implies $(Mx)_i = x_i$, $i = 1, \cdots , n$. As $M$ is irreducible, the Perron-Frobenius theorem tells us that $x = \alpha e$. 

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Let $A$ be a non-negative matrix satisfying
\begin{equation}
\tag{2.5}
A e = eA \quad \text{(i.e., the respective row and column sums agree)}.
\end{equation}
Then
\begin{equation}
\tag{2.6}
\min_{x>0} \frac{\sum_{i=1}^n (Ax)_i}{x_i} = \frac{\sum_{i=1}^n a_{ii}}{\gamma}.
\end{equation}
Equality is achieved for $x = \alpha e$. Where $A$ is irreducible nonnegative then equality holds only for $x = \alpha e$.

Proof. The condition (2.5) entails the existence of a positive diagonal matrix $E = \text{diag}(e_1, e_2, \ldots, e_n)$ such that $A + E = \gamma M$ where $\gamma > 0$ and $M \in \mathbb{R}(e, e)$ is doubly stochastic. Then for $x > 0$
\begin{equation}
\sum_{i=1}^n (Ax)_i = \gamma \sum_{i=1}^n (Mx)_i - \sum_{i=1}^n e_i.
\end{equation}
Since the minimum on the right is always achieved for $x = \alpha e$ independent of $M \in \mathbb{R}(e, e)$, the conclusion (2.6) is valid.

An easy consequence of Lemma 2.1 is the result (1.10), now restated.

**Theorem 2.1.** Let $M \in \mathbb{R}(e, e)$ and $D$ be a positive definite diagonal matrix, $D = \text{diag}(d_1, d_2, \ldots, d_n)$. Then
\begin{equation}
\tag{2.8}
\rho(DM) = \rho(MD) \geq \left( \prod_{i=1}^n d_i \right)^{1/n}
\end{equation}
(recall $\rho(A)$ denotes the spectral radius of $A$). If $M$ is positive then the inequality is sharp unless $D = \alpha I$ for $\alpha > 0$. Equality holds for the permutation matrix (1.11).

Proof. Since the spectrum of $DM$ and $MD$ each coincides with the spectrum of $D^1M D^1$, we have in particular $\rho(DM) = \rho(MD)$.

Assume first that $M$ is positive (all entries are positive). The Perron-Frobenius theorem guarantees the existence of some $x > 0$ satisfying
\begin{equation}
\tag{2.9}
\rho(DM)x = DMx.
\end{equation}
Multiplying components produces
\begin{equation}
\tag{2.10}
[\rho(DM)]^n \prod_{i=1}^n d_i^{-1} = \prod_{i=1}^n (Mx)_i \geq 1
\end{equation}
where the final inequality results on the basis of (2.1) and thereby (2.8) is proved for $M$ positive. By continuity, we infer that (2.8) is correct for all $M \in \mathbb{R}(e, e)$.

For the special permutation matrix $P = [p_{i,j}]$ with $p_{i,i} = 1$, $p_{i,i+1} = 1$, $i = 1, \ldots, n-1$, a direct calculation yields $\rho(PD) = \left( \prod_{i=1}^{n-1} d_i \right)^{1/n}$ attesting to the fact that (2.8) cannot be sharpened without further restricting $\mathbb{R}(e, e)$.

The proof of Theorem 2.1 is complete.

A modified version of the inequality (2.8) can be extended to the class of all positive matrices as follows. It is known (e.g., see Sinkhorn [17]) that if $A$ is positive (or non-negative and fully indecomposable, concerning the latter concept see Brualdi, Parter and Schneider [2]) and Sinkhorn and Knopp [18]) then there exists unique (up to scalar multiples) positive diagonal matrices $E_1 = \text{diag}(e_1, \ldots, e_n)$ and $E_2 = \text{diag}(e_1, \ldots, e_n)$ such that
\begin{equation}
E_1 A E_2 = M
\end{equation}
is doubly stochastic.

Actually, a practical algorithm is available for constructing $E_1$ and $E_2$ where $A$ is positive. In fact, normalize alternately the rows and columns to sum to 1. More specifically, determine a diagonal positive matrix $C_n$ to satisfy $eA_n C_n = e$ and subsequently a diagonal positive $B_n^*$, $B_n A_n e = e$ where $A_n = A B_n^* C_n$, $A_n = B_{n+1} \ldots C_{n+1}$, $n = 1, 2, \ldots$, $(A_1 = A)$. The convergence of
\begin{equation}
\prod_{i=1}^n C_i \rightarrow E_2, \quad \prod_{i=1}^n B_i \rightarrow E_1,
\end{equation}
leading to (2.11) is proved in Sinkhorn [17].

**Corollary 2.1.** Let $A$ be positive (or non-negative and fully indecomposable) and let $E_1, E_2$ be determined as in (2.11).

Then
\begin{equation}
\rho(DA) \geq \left( \prod_{i=1}^n \frac{d_i}{\epsilon_{i,0}} \right)^{1/n}.
\end{equation}

If $A$ is positive and symmetric then a known procedure for calculating $E$ converting $E A E$ doubly stochastic, is by solving the following variational problem.

Determine a positive vector $e = (e_1, e_2, \ldots, e_n)$ which minimizes $g(e) = (e, A e)$ subject to the constraint $\prod_{i=1}^n e_i = 1$.

3. Developments pertaining to Theorem 3.1 and related critical point theory.

The inequality embodied in (1.8) is more far reaching and deeper than the version in the doubly stochastic case covered in Theorem 2.1. The elementary technique of Lemma 2.1 does not carry over.

Let $A$ be positive (it suffices to have $A$ non-negative and irreducible) and suppose
\begin{equation}
Au = \rho(A)u, \quad vA = \rho(A)v
\end{equation}
and suppose $u$ and $v$ normalized to satisfy
\begin{equation}
\sum_{i=1}^n u_i v_i = 1.
\end{equation}
The main objective of this section is to prove...
\begin{equation}
(3.3) \quad f(x) = f(x_1, \ldots, x_n) = \sum_{i=1}^{n} u_i v_i \log \left( \frac{\langle Ax \rangle_i}{x_i} \right) \geq 0 \quad \text{for } x > 0.
\end{equation}

The analysis is done through a series of lemmas.

**Lemma 3.1.** Let \( \Delta \) be an open bounded connected domain contained in \( \mathbb{R}^* \) (Euclidean n-space) and suppose \( f(y) \) defined for \( y \in \Delta \) is real valued of continuity class \( C^2(\Delta) \). Assume that \( \lim_{t \to \infty} f(y(t)) = +\infty \) whenever \( y(t) \) tends to \( \partial \Delta \), the boundary of \( \Delta \). Assume furthermore that for every critical point \( y^* \) of \( f \) in \( \Delta \), the Hessian matrix \( \{f''(y^*)\} \), \( \{i, i, (y^*) = \{\partial^2 f/\partial y_i \partial y_j\}(y^*) \) \) is strictly positive definite. Then there exists a unique critical point \( y^* \) in \( \Delta \) and \( f \) achieves over \( \Delta \) an absolute minimum at \( y^* \).

Remark 3.1. The content of this lemma is undoubtedly known and would follow directly from the Morse critical point inequalities (the second of the series suffices for our purposes) except for the complication that \( \Delta \) is open. Since \( f \) is infinite on \( \partial \Delta \) the behavior near the boundary should be and indeed is irrelevant.

**Proof.** The stipulations of the lemma assure the existence of an absolute minimum \( y^* \in \Delta \) which, of course, is a critical point of \( f \). We need to establish that \( y^* \) is the only critical point. Consider the differential equation system

\begin{equation}
(3.4) \quad \frac{dy_i(t)}{dt} = -\frac{\partial f}{\partial y_i}(y(t), \ldots, y_n(t)), \quad i = 1, 2, \ldots, n
\end{equation}

with the initial conditions

\begin{equation}
(3.5) \quad y_i(0) = y_i^0, \quad y^0 \in \Delta.
\end{equation}

Direct evaluation produces

\begin{equation}
(3.6) \quad \frac{df(y(t))}{dt} = -\sum_{i=1}^{n} \left[ \frac{\partial f}{\partial y_i}(y(t)) \right]^2
\end{equation}

and from this equation we infer that \( f(y(t)) \) strictly decreases as \( t \) increases, \( t > 0 \), unless \( y^0 \) is a critical point. We claim that \( y(t) \) converges as \( t \to \infty \) to a critical point. Let \( \bar{y} \) be a limit point of \( y(t) \) (i.e., \( \bar{y} = \lim_{t \to \infty} y(t) \)) which necessarily exists in \( \Delta \) since \( f \) is unbounded on \( \partial \Delta \). Certainly, \( f(y(t)) \) decreases to \( f(\bar{y}) \) since \( f(y(t)) \) converges to \( f(\bar{y}) \) and \( f(y(t)) \) is decreasing. The presumption \( \sum_{i=1}^{n} \left[ \frac{\partial f}{\partial y_i}(y(t)) \right]^2 = \alpha > 0 \) implies the inequality \( df/df(y(t)) \leq -\alpha/2 \) for all \( i \) large enough which compels therefore \( f(\bar{y}) = -\infty \), an absurdity. To avert this absurdity we must have that \( \bar{y} \) (a conclusion applying to all such limit points) is a critical point. The hypothesis that \( \|f_i(\bar{y})\| \) is positive definite entails that \( \bar{y} \) is locally stable with respect to the flow (3.4). It follows that \( \lim_{t \to \infty} y(t) = \bar{y} \) as previously claimed.

Consider now the domain of attraction \( \Delta(\bar{y}) \) to \( \bar{y} \) consisting of all \( y \) fulfilling the conditions \( y(0) = \bar{y} \) and \( \lim_{t \to \infty} y(t) = \bar{y} \). Since \( \bar{y} \) is locally stable it is clear that \( \Delta(\bar{y}) \) is open. Suppose \( \Delta(\bar{y}) \) is properly contained in \( \Delta \), then (because \( \Delta \) is connected and open) we infer the existence of \( \bar{z} \) in \( \Delta \), located on the boundary of \( \Delta(\bar{y}) \) and the previous analysis shows that \( \lim_{t \to \infty} z(t) = \bar{z} \) for some critical point \( \bar{z} \neq \bar{y} \). But \( \Delta(\bar{y}) \) is an open domain for the same reason that \( \Delta(\bar{y}) \) is open, and a contradiction emerges as \( \bar{z} \) lies on \( \partial \Delta(\bar{y}) \). To avoid this contradiction, we necessarily have \( \Delta(\bar{y}) = \Delta \) and a unique critical point exists. The proof of Lemma 3.1 is complete.

We now recall the following concept.

**Definition 3.1.** A matrix \( A = \{a_{ij}\} \) is said to be a (strict) \( M \)-matrix (after Minkowski who introduced the pertinent classification) if \( a_{ij} \leq 0 \) \( (< 0) \) for all \( i \neq j \) and there exists a positive vector \( x^0 > 0 \) satisfying

\begin{equation}
(3.7) \quad A x^0 \geq 0.
\end{equation}

Such matrices occur widely in applications to econometric theory and in certain contexts of physical systems, e.g., see Karlin [10, chap. 8].

We review for easy reference some of the main characterizations and properties of \( M \)-matrices. A (strict) \( M \)-matrix can be represented in the form

\begin{equation}
(3.8) \quad \rho e \lambda(A) > \lambda_1, \quad i = 2, 3, \ldots, n.
\end{equation}

If \( A \) is a symmetric strict \( M \)-matrix then (3.6) in conjunction with (3.8) imply that \( A \) is positive semi definite with the eigenvalue \( \lambda_1 \) having simple multiplicity.

We are now prepared for the next lemma which highlights a class of functions \( f \) fulfilling the conditions of Lemma 3.1.

**Lemma 3.2.** Let \( \Delta \) be an open connected homogeneous domain in \( \mathbb{R}^{n} \) (i.e., \( \Delta \subset \alpha \Delta \) for any \( \alpha > 0 \)). Assume that \( f \) is a homogeneous function of degree 0 defined on \( \Delta \) of continuity class \( C^0(\Delta) \). Consider \( f(y) \) restricted to the bounded region \( S = \Delta \cap \Sigma \) where

\begin{equation}
\Sigma = \{y \mid y \in \mathbb{R}^n, \Sigma y_i = 1\}.
\end{equation}

Suppose further that \( f(y) \to +\infty \) whenever \( y \in S \) tends to the boundary \( \partial S \). If

\begin{equation}
(3.9) \quad \frac{\partial f(y)}{\partial y_i}(y) < 0 \quad \text{for } i \neq j \quad \text{and } y \in S,
\end{equation}

then \( f(y) \) on \( S \) admits a unique critical point located at the absolute minimum of \( f \).

**Proof.** Since \( f \) is homogeneous of degree 0, it satisfies the Euler equation

\begin{equation}
(3.10) \quad \sum_{i=1}^{n} y_i \frac{\partial f(y)}{\partial y_i}(y) = 0.
\end{equation}
Differentiating (3.10) in $y_j$ produces
\[ \sum_{j=1}^{n} y_j \frac{\partial f}{\partial y_j}(y) + (\partial f/\partial y_i)(y) = 0 \]
for $j = 1, 2, \ldots, n$ and at a critical point $y^*$ these relations reduce to
\[ y^* F = 0 \quad \text{where} \quad F = F(y^*) = \left| \frac{\partial f}{\partial y_j} (y^*) \right| . \]

The assumption (3.9) in conjunction with (3.11) establish that $F$ is a strict $M$-matrix (Definition 3.1). By virtue of (3.8) we know that the eigenvalues of $F(y^*)$ can be arranged in the order
\[ 0 = \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_n \]
where the unique eigenvector (apart from scalar multiples) for $\lambda_1 = 0$ is $y^*$. We infer accordingly that the second variation (the Hessian) of $f$ at $y^*$ with respect to variations in $S$ is positive definite. Thus, $f$ confined to $S$ satisfies the requirements of Lemma 3.1. Applying the conclusion of that lemma, the desired result is achieved.

Proof of (3.3). The function
\[ f(x) = \sum_{i=1}^{n} u_i v_i \log \left( \frac{(Ax)_i}{x_i} \right) \]
defined on
\[ \Sigma = \{ x = (x_1, \ldots, x_n) > 0, |x| = \Sigma x_i = 1 \} \]
is plainly homogeneous of degree 0. Also $f(x)$ tends to $\infty$ if at least one component of $x \in \Sigma$ goes to zero. Clearly $f \in C^2(\Sigma)$.

A direct calculation produces
\[ \frac{\partial^2 f}{\partial x_j \partial x_k} = -\sum_{i=1}^{n} \frac{u_i v_i a_{ij} a_{ik}}{(Ax)_i} \quad \text{for} \quad j \neq k \]
which is certainly negative since $A$ is a positive matrix.

Straightforward computation relying on the relations (3.1) verifies that the vector $u/|u|$ is a critical point of $f$. On the basis of Lemma 3.2 we know that $x = au$ provides the unique absolute minimum of (3.12) and this value is manifestly 0.

Proof of Theorem 3.1. The inequality (1.9) is obviously equivalent to (3.3). The statements discerning the possibilities of equality in (3.3) are decided on the basis of the uniqueness results affirmed in Lemma 3.2.

We turn to the proof of (1.8). Invoking the Perron–Frobenius theory, we obtain
\[ \rho(DM) x_i = d_i \sum_{j=1}^{n} m_{ij} x_j, \quad i = 1, 2, \ldots, n \quad \text{for some} \quad x > 0. \]

Therefore
\[ \rho(DM) \prod_{i=1}^{n} d_i^{-w_{ii}} = \prod_{i=1}^{n} \left[ \frac{(Mx)_i}{x_i} \right]^{w_{ii}}. \]
Now, determine $E_2$ by the prescriptions

\begin{equation}
\epsilon_i = \frac{\xi_i(M)}{u_i} \quad i = 1, 2, \ldots, n
\end{equation}

\begin{equation}
(E_2 = \text{diag}(\epsilon_1, \ldots, \epsilon_n)) \text{ and } E_1 = \text{diag}(\epsilon_1, \ldots, \epsilon_n) \text{ to satisfy}
\end{equation}

\begin{equation}
E_1ME_2u = u
\end{equation}

It follows from (3.16) that

\begin{equation}
\xi(E_1ME_2) = u
\end{equation}

Also, the relations

\begin{equation}
\frac{\partial}{\partial x_i} f(x, E_1ME_2)|_{u} = 0, \quad i = 1, 2, \ldots, n
\end{equation}

since $u$ is the appropriate critical point, reduce to

\begin{equation}
vE_1ME_2 = v.
\end{equation}

Suppose now that $E_1M E_2 \in \Re(u, v)$. By virtue of (3.16) we have

\begin{equation}
u = \alpha E_2^{-1}x(M) \text{ for some } \alpha > 0.
\end{equation}

Comparing to (3.17) we see that

\begin{equation}
E_2 = \alpha^{-1}E_2 \quad \text{and then } E_1 = \alpha E_1
\end{equation}

is clear. The proof of the Theorem is complete.

The natural generalizations of Corollary 3.1 ensues.

**Corollary 3.2.** Let $M$ be an $n \times n$ matrix as in Theorem 3.2. Let $\omega = (\omega_1, \ldots, \omega_n)$ be a positive vector normalized by the condition

\begin{equation}
\sum_{i=1}^{n} \omega_i = 1.
\end{equation}

Then there exists a diagonal positive definite matrix

\begin{equation}
D(\omega) = \text{diag}(d_1(\omega), \ldots, d_n(\omega))
\end{equation}

such that

\begin{equation}
\rho(DA) \geq \prod_{i=1}^{n} \left( \frac{d_i(\omega)}{d(\omega_i)} \right)^{\omega_i}
\end{equation}

for any diagonal positive definite matrix $D$. The equality sign holds if $D = \alpha D(\omega)$ for some $\alpha > 0$.

**Proof.** Take $D = E_1E_2$, where $E_1$ and $E_2$ are defined in Theorem 3.2 and $u = e$ and $v = \omega$.

**Remark 3.3.** According to Brualdi, Parter and Schneider [2, Lemma 2.3] the conditions of Theorem 3.2 imply that $M$ is fully indecomposable. Thus in case $u = v = e$ Theorem 3.2 goes back to Brualdi, Parter and Schneider and Sinkhorn & Knopp [18]. Finally, in view of this lemma it follows that the results of Theorem 3.2 remain valid if we assume that $M$ is a fully indecomposable matrix.

**4. Derivation of the inequality (1.12) and ramifications.**

We begin with some preliminaries

**Lemma 4.1.** Let $M = E_1KE_2$ where $E_1$ and $E_2$ are diagonal positive definite matrices and $K$ is positive definite. Furthermore, assume $M$ is irreducible and nonnegative. Let $u$ and $v$ be positive vectors satisfying

\begin{equation}
Mu = \rho(M)u \quad vM = \nu(M)
\end{equation}

(for simplicity of writing take $\rho(M) = 1$)

and normalized to satisfy

\begin{equation}
\sum_{i=1}^{n} u_i = 1 \quad \text{and} \quad \sum_{i=1}^{n} u_i^2 = 1.
\end{equation}

Define the positive vector $y^0$ by the equation

\begin{equation}
E_1^{-1}E_2^{-1}y^0 = u.
\end{equation}

Then

\begin{equation}
E_1^{-1}E_2^{-1}y^0 = \alpha v \text{ for some } \alpha > 0.
\end{equation}

**Proof.** We evaluate $(E_1^{-1}E_2^{-1}y^0)M^{-1}$ heavily exploiting the fact that $E_1$ and $E_2$ are positive diagonal matrices. We obtain

\begin{equation}(E_1^{-1}E_2^{-1}y^0)M^{-1} = y^0E_1^{-1}E_2^{-1}(E_2^{-1}M^{-1}E_2^{-1}) = y^0E_1^{-1}E_2^{-1}(E_2^{-1}K^{-1}E_2^{-1})E_1^{-1}E_2^{-1}
\end{equation}

\begin{equation}
= u(E_1^{-1}K^{-1}E_2^{-1})E_1^{-1}E_2^{-1} = u(M)^{-1}E_2^{-1}E_1^{-1} = uE_2^{-1}E_1^{-1}
\end{equation}

\begin{equation}
= y^0E_1^{-1}E_2^{-1}y^0.
\end{equation}

Comparing the outside equations we see that $z = E_1^{-1}E_2^{-1}y^0$ is a left eigenvector of $M^{-1}$ with eigenvalue 1, i.e., $zM^{-1} = z$. We infer by the Perron-Frobenius theorem since $\rho(M) = 1$ is a simple eigenvalue that

\begin{equation}
z = E_1^{-1}E_2^{-1}y^0 = \alpha v \text{ for some positive } \alpha.
\end{equation}

The proof of the lemma is complete.

**Proof of (1.12), consult section 1.** Assume first that $K$ is positive definite and positive. Because $K$ is positive definite and $E_1$, $E_2$ and $D$ are diagonal, the eigenvalues of $DM = DE_1KE_2$ coincide with $E_2DE_1K$ and $K^t(E_1E_2)^t$. $D(E_1E_2)^tK^t$. In particular

\begin{equation}
\rho(DM) = \rho(K^t(E_1E_2)^tD(E_1E_2)^tK^t).
\end{equation}
On the basis of a familiar characterization of the largest eigenvalue of a symmetric matrix and taking account of the fact that all component matrices involved are nonnegative, we find that

\[ \rho(DM) = \sup_{x > 0} \frac{(K(E, E)^{\lambda}D(E, E)^{-1}K^{\lambda}x, x)}{(x, x)} = \sup_{y \geq 0} \frac{(Dy, y)}{(K^{-1}(E, E)^{-1}y, (E, E)^{-1}y)} \]

where \( y^0 \) is determined as in (4.3). Substituting for \( y^0 \) and relying on the fact of (4.4) the final denominator in (6.6) reduces to \( \sum_{i=1}^{n} d_i u_i v_i \), while the numerator becomes \( \alpha u_i v_i \).

The case that \( K \) is non-negative and positive semi-definite is handled by standard perturbation procedures.

The result of Theorem 3.1 applies to give

\[ (\text{BY})_i \leq \frac{(\text{M}x)_i}{y_i}, \]

and therefore (4.11) holds.

Proof of (1.13). Application of the Perron-Frobenius theorem gives

\[ \rho(DM) = d_i(Mx), \quad i = 1, 2, \ldots, n \]

for some \( x > 0 \). Therefore

\[ \rho(DM) = \sum_{i=1}^{n} u_i v_i \frac{x_i}{(Mx)_i} = \sum_{i=1}^{n} d_i u_i v_i. \]

The inequality \( \sum_{i=1}^{n} d_i u_i v_i \leq \rho(DM) \) established as (1.12) together with (4.8) (since \( \rho(DM) > 0 \) as \( M \) is irreducible and nonnegative) obviously imply

\[ \sum_{i=1}^{n} u_i v_i \frac{x_i}{(Mx)_i} \leq 1. \]

The inequality (4.9) is validated in this analysis only for \( x \), an eigenvalue of \( MD \) for \( \rho(MD) \). Now given an arbitrary \( y > 0 \), we determine the vector \( d = (d_1, \ldots, d_n) \) by the relation \( d_i = y_i/(Mx)_i \), and obviously this \( y > 0 \) qualifies as the eigenvector for \( \rho(DM) = 1 \). Thus, (4.9) holds for all \( x > 0 \).

Where \( \rho(M) \neq 1 \) then (4.9) should read as,

\[ \sup_{x > 0} \sum_{i=1}^{n} u_i v_i \frac{x_i}{(Mx)_i} \leq \frac{1}{\rho(M)}. \]

With (1.12) and (1.13) in hand, the proof of Theorem 4.1 is complete.

The next theorem describes another class of matrices for which (1.12) and (1.13) hold.

**Theorem 4.2.** Let \( M \in \mathbb{R}(u, v) \). Assume furthermore that \( M^{-1} \) exists as an \( M \)-matrix (see Definition 3.1). Then

\[ \sup_{x > 0} \sum_{i=1}^{n} u_i v_i \frac{x_i}{(Mx)_i} \leq 1 \quad \text{and} \]

\[ \rho(DM) \geq \frac{1}{n} \frac{\sum_{i=1}^{n} d_i}{\sum_{i=1}^{n} d_{(i)} d_{(i)}}, \]

for any positive diagonal matrix \( D \).

**Proof.** We have available the representation

\[ (\text{BY})_i = \frac{(Mx)_i}{y_i}, \]

(Pertaining to (4.13), see the discussion following Definition 3.1).

\[ (\text{DY})_i = \frac{(Dy)_i}{y_i}, \]

\[ (\text{BY})_i \leq \frac{(\text{M}x)_i}{y_i}, \]

where \( y^0 \) is determined as in (4.3). Substituting for \( y^0 \) and relying on the fact of (4.4) the final denominator in (6.6) reduces to \( \sum_{i=1}^{n} d_i u_i v_i \), while the numerator becomes \( \alpha u_i v_i \).

The case that \( K \) is non-negative and positive semi-definite is handled by standard perturbation procedures. Apply the result to \( K, = (1 - 2\epsilon)K + \epsilon I + \epsilon E^{-1}u_i v_i, E^{-1}u_i v_i \) and let \( \epsilon \to 0^+ \). The proof of (1.12) is complete.

Remark 4.1. If there exists a positive number \( \beta > 0 \) such that \( -M^{-1} + \beta I \) is irreducible nonnegative then the equation sign holds in (1.12) iff \( D = \alpha I \) for some \( \alpha > 0 \). This statement is readily confirmed by a careful scrutiny of the details of the analysis.

Remark 4.2. Consider a matrix \( M \) of the form \( M = E_i K E_2 \) where the component matrices have the properties enunciated in Theorem 4.1 and assume \( M \) is positive. Let \( D_1 \) and \( D_2 \) be diagonal positive matrices such that \( D_1 M D_2 \) is doubly stochastic. We adopt the notation \( D_1 = \text{diag} (d_{(1)}, \ldots, d_{(1)}), D_2 = \text{diag} (d_{(2)}, \ldots, d_{(2)}) \). Obviously

\[ D_1 M D_2 = E_i K E_2, \quad E_i = D_i E_i, \quad i = 1, 2 \]

The inequality (1.12) in terms of this reduction becomes

\[ \rho(DM) \geq \frac{1}{n} \frac{\sum_{i=1}^{n} d_i}{\sum_{i=1}^{n} d_{(i)} d_{(i)}}, \]

and therefore (4.11) holds.

Backtracking over the analysis of Theorem 4.1 we can check that the inequality (4.11) entails (4.12). The proof is complete.
5. Extensions, examples and applications.

I. Bounds for lower order eigenvalues; compound matrices

Let $A$ be an oscillatory matrix meaning that every compound of $A$, $A_{[p]}$ ($p = 1, 2, \cdots, n$) is nonnegative and that $A_{[p]}^{m}$ is strictly positive for some integer $m$, $m$ can depend on $p$. Recall that the elements of $A_{[p]}$ consist of all $p$th order minors where the indices can be identified with all $p$ tuples of integers and these arranged in lexicographic order. (Thus if $A = \|a_{ij}\|_{n}$ then the elements of $A_{[p]}$ in row $\alpha = (i_1, i_2, \cdots, i_p)$ column $\beta = (j_1, \cdots, j_p)$ ($1 \leq i_1 < \cdots < i_p \leq n, 1 \leq j_1 < \cdots < j_p \leq n$) is $A_{[p]}(\alpha, \beta) = \det (\|a_{ij}\|_{[p]})$.

In particular, of $A$ is oscillatory then $A_{[1]}$ is a nonnegative and irreducible matrix of size $\binom{n}{1} \times \binom{n}{1}$. Oscillatory matrices have remarkable properties and we record a number of them for ready reference. (Further discussion and validations can be found in Gantmacher and Krein [4], see also Karlin [11], for extensions and applications).

(i) All the eigenvalues of $A$ are positive and distinct. Arranging them in decreasing order gives

$$\lambda_1(A) > \lambda_2(A) > \cdots > \lambda_n(A) > 0$$

Thus

$$Au^{(i)} = \lambda_i u^{(i)} \quad v^{(i)} A = \lambda_i v^{(i)}$$

The multiplying factors for the eigenvectors can be chosen in a manner that the wedge product vectors

$$u^{(1)} \wedge u^{(2)} \wedge \cdots \wedge u^{(k)} > 0, \quad v^{(1)} \wedge v^{(2)} \wedge \cdots \wedge v^{(k)} > 0$$

are positive ($k = 1, 2, \cdots, n$).

Recall that if $x^{(i)} = (x_1^{(i)}, \cdots, x_n^{(i)})$ then $x^{(1)} \wedge x^{(2)} \wedge \cdots \wedge x^{(k)}$ can be concretely expressed as the vector whose $\alpha = (j_1, j_2, \cdots, j_k)$ coordinate is $\det (\|x_i^{(j')\|_{[k]})$). This arrangement of the coordinates is consistent with the lexicographic ordering used in representing $A_{[k]}$.

The vectors listed in (5.3) are right and left eigenvectors, respectively for the spectral radius of the corresponding compound matrix $A_{[k]}$, so that

$$\rho(A_{[k]}) = \sum_{i=1}^{k} \lambda_i$$

and

$$A_{[k]}(u^{(1)} \wedge \cdots \wedge u^{(k)}) = (\prod_{i=1}^{k} \lambda_i)(u^{(1)} \wedge \cdots \wedge u^{(k)}).$$

We will further scale the eigenvectors in (5.2) to satisfy

$$\sum_{\alpha} (u^{(1)} \wedge u^{(2)} \wedge \cdots \wedge u^{(k)})_\alpha (v^{(1)} \wedge v^{(2)} \wedge \cdots \wedge v^{(k)})_\alpha = 1$$

$k = 1, 2, \cdots, n$.

II. Upper bounds for the spectral radius $\rho(DM)$.

The following classical characterizations of the spectral radius for nonnegative matrices will serve us in Theorem 5.2 below and in the discussion of a number of examples.

Let $A$ be a nonnegative matrix. Then

$$\rho(A) = \sup_{x \neq 0} \frac{\Gamma x}{\Gamma}$$

for some $x \geq 0, x \neq 0$.

(The notation $z \geq w$ signifies that $z - w$ is a nonnegative vector.)

If there exists $y > 0$ (strictly positive) satisfying $Ay \leq \gamma y, \gamma > 0$ then

$$\rho(A) \leq \gamma.$$ 

The characterization (5.8) due initially to Wielandt can be cast in the form of a minimax expression. Proofs of (5.8) and (5.9) can be found in innumerable sources, e.g., see the appendix of Karlin [10].

Theorem 5.2. (i) Let $M$ be nonnegative and irreducible and $D$ a positive definite diagonal matrix, $D = \text{diag}(d_1, \cdots, d_n)$, then
(5.10) \[ \rho(DM) \leq \rho(M) \max_{1 \leq i \leq n} (d_i). \]

(ii) If \( M \) is totally positive of order 2 then \( \rho(MD) \leq \sum_{i=1}^n m_{ii} d_i. \)

Proof. (i). Since \( M \) is irreducible there exists \( u > 0 \) satisfying \(Mu = \rho(M)u.\) Therefore
\[ DMu \leq \rho(M)M d. \]
The result (5.10) now follows by direct appeal to (5.9) as \( u \) is strictly positive.

(ii). The proof of (ii) is found in Karlin [12, Th. 10.11].

In Theorem 4.1 (consult also (4.10)) we uncovered the bound
\[ \sup_{x > 0} \sum_{i=1}^n u, v, (Mx)i \leq \frac{1}{\rho(M)}. \]
For comparison purposes we record the following fact. Let \( A \) be an oscillatory matrix of order 2 (designated \( OS_2 \)), i.e., \( A \) is irreducible nonnegative and the second order compund \( A_{12} \) is also nonnegative and irreducible. Assume \( \alpha_i > 0, i = 1, 2, \ldots, n, \)
\[ \min_{x > 0} \sum_{i=1}^n \alpha_i (Ax)i = \frac{1}{\max_{1 \leq i \leq n} \alpha_i}. \]
This finding is implicit in Karlin [12, Eq. (10.8)]. For completeness we review the setting from which (5.11) emanates. Let \( A \) be nonnegative irreducible and let \( D \) be an irreducible diagonal where the vector \( d \) of the diagonal components satisfies \( d_i \geq 0, \Sigma d_i = |d| = 1. \) We wish to evaluate \( \max_{d \geq 0, \ |d| = 1} \rho(DA) \)
where \( D \) is diagonal. Where \( A \) is of class \( OS_2 \) we obtain
\[ \max_{d \geq 0, \ |d| = 1} \rho(DA) = \max_{1 \leq i \leq n} a_{ii}. \]
Obviously, for \( A > 0 \) and each \( d > 0, |d| = 1 \) there exists \( x > 0 \) satisfying
\[ DAx = \rho(DA)x \]
or equivalently \( [d_i / \rho(DA)] = x_i / \rho(DA), \) and therefore \( [1/\rho(DA)] = \sum_{i=1}^n x_i / \rho(Ax)i. \) Since \( d \) is arbitrary subject to the constraints \( d \geq 0, |d| = 1 \) and \( x \) is determined up to a scalar multiple we deduce that
\[ \inf_{x > 0} \sum_{i=1}^n x_i / \rho(Ax)i = \frac{1}{\max_{d \geq 0, \ |d| = 1} \rho(DA)} = \frac{1}{\max_{1 \leq i \leq n} \alpha_i}. \]
the last equation resulting by virtue of (5.12). By absorbing a diagonal matrix composed of the \( \alpha \) into \( A \) we find that (5.11) agrees with (5.13).

III. Applications of the inequalities (1.8) and (1.12) to local stability analysis.

For the population migration models of (1.2) and (1.3) it is of interest to ascertain conditions on the migration matrix \( M \) and the matrix of selection coefficients \( D \) implying
\[ \rho(DM) > 1. \]
Where the above inequality holds, the connotation is (for the model (1.2)) that the \( A \)-type in the population can never tend to extinction and will always be represented with moderate to high frequency.

Obviously from Theorems 3.1 and 4.1 we secure simple sufficient conditions insuring (5.14). More specifically, if
\[ M \in \mathfrak{H}(u, v) \quad \text{and} \quad M = E_1 KE_2 \]
(\( E_1, E_2 \) positive diagonal, \( K \) positive semi-definite) then (5.14) prevails if
\[ \sum_{i=1}^n u, v, d_i > 1. \]
Generally, already the equation \( \sum_{i=1}^n u, v, d_i = 1 \) entails (5.14). However we pointed out earlier in (1.14) that \( M = ||u, v, || \in M(u, v) \) carries the equation sign
\[ \rho(DM) = \sum_{i=1}^n u, v, d_i. \]
and in this case (5.14) and (5.16) are trivially equivalent. It is elementary to check that if \( M \in \mathfrak{H}(u, v) \) is of type (5.15), then \( M^k \) for each integer \( k \geq 1 \) is also of type (5.15). Therefore, in the presence of (5.16), we have
\[ \rho(DM^k) > 1 \quad \text{for all} \ k. \]
Conversely suppose (5.18) holds for all \( k \) (or even for infinitely many \( k \)) with \( M \) irreducible and \( \notin \mathfrak{H}(u, v). \) We know from a standard ergodic theorem that \( M^k \rightarrow ||u, v, || \) as \( k \rightarrow \infty. \) With this convergence and the stipulation of (5.18) we infer the inequality \( \sum_{i=1}^n u, v, d_i > 1. \)

IV. Discussion of some examples.

(a) Consider a tridiagonal (Jacobi) nonnegative matrix
\[ M = \begin{bmatrix} r_1 & p_1 & 0 & 0 & 0 \\ q_2 & r_2 & p_2 & \vdots & \vdots \\ 0 & q_3 & r_3 & p_3 & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & q_n & r_n \\ \end{bmatrix} \]
(5.19)
with \( p_i > 0, q_i > 0 \) and \( r_i \geq 0. \) Assume also \( q_i + r_i + p_i = 1 \) for all \( i \) so that \( M \) is a stochastic matrix. Choosing \( \epsilon, \) positive to satisfy \( \epsilon / \epsilon_{i=1} = p_i / q_i, \)
\( i = 1, \ldots, n - 1 \) and defining \( E_2 = \text{diag}(\epsilon_1, \epsilon_2, \ldots, \epsilon_n) \) we obtain \( ME_2 = K \)
where \( K \) is a symmetric Jacobi matrix. Where the upper bordered principal determinants of \( M \) are positive we deduce that \( K \) is positive definite. More concretely, if
then $M$ is of type $E_{1}K_{2}$ where $E_{i}$ are positive definite diagonal matrices and $K$ is positive definite.

Moreover, it is easy to verify that $M \in \mathfrak{M}(u, v)$ where $u = (1, 1, \ldots, 1)$ and $v = (v_{1}, v_{2}, \ldots, v_{n})$ has the form

$$(5.21) \quad v_{i} = \frac{\pi_{i}}{\sum_{j=1}^{n} \pi_{j}}, \quad \pi_{1} = 1 \quad \text{and} \quad \pi_{j} = \frac{p_{j-1} - p_{j-2} \cdots p_{1}}{q_{j-1} \cdots q_{2}}, \quad j \geq 2.$$ provided $(5.20)$ holds then application of Theorem 4.2 affirms that

$$(5.22) \quad \rho(DM) > 1 \quad \text{if} \quad \sum_{i=1}^{n} d_{i} \pi_{i} \geq 1.$$ The special doubly stochastic matrix

$$(5.23) \quad M = \begin{bmatrix} 1 - m & m & 0 & \ldots & 0 \\ m & 1 - 2m & m & \ldots & \vdots \\ 0 & m & 1 - 2m & \ldots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & \ldots & \ldots & 0 \\ m & 1 - 2m & m & \ldots & 1 - m \end{bmatrix}$$ is positive definite provided $m \leq \frac{1}{2}$ independent of the size of the matrix. Then conforming to $(5.22)$ we have

$$(5.24) \quad \rho(DM) > 1 \quad \text{if} \quad \frac{\sum_{i=1}^{n} d_{i}}{\sum_{i=1}^{n} \pi_{i}} \geq 1.$$ We examine two cases.

Case (i). Suppose $e_{i}d_{i} \geq 1$ for some index $i_{0}$, $1 \leq i_{0} \leq n$. Let $z^{(i_{0})} = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$ with the component 1 appearing only at the $i_{0}$-th coordinate. Since $MD$ is nonnegative then by virtue of $e_{i}d_{i} \geq 1$ we find

$$(5.25) \quad MDz^{(i_{0})} \geq z^{(i_{0})}$$ and the two vectors in $(5.25)$ are not equal. Invoking the characterization of (5.8) implies

$$(5.26) \quad \rho(DM) > 1.$$ Case (ii). Suppose $e_{i}d_{i} < 1$ for all $i = 1, 2, \ldots, n$. Then for each $\lambda \geq 1$, $(ED - \lambda I)^{-1}$ exists. By the prescription of $(5.26)$, $(MD - \lambda I)^{-1}$ exists if $c(\lambda) = 1 + (v, (ED - \lambda I)^{-1}u) \neq 0$ that is provided

$$(5.27) \quad c(\lambda) = 1 + \sum_{i=1}^{n} \frac{v_{i}d_{i}u_{i}}{e_{i}d_{i} - \lambda} \neq 0.$$ Obviously $c(\infty) = 1$ and since $e_{i}d_{i} < 1$ for all $i \geq 1$ the monotone increasing function $c(\lambda)$ vanishes somewhere on the range $1 \leq \lambda < \infty$ and then $\rho(DM) > 1$ if and only if $0 < c(1) = 1 + \sum_{i=1}^{n} \frac{v_{i}d_{i}u_{i}}{e_{i}d_{i} - 1}$ or equivalently

$$(5.28) \quad \sum_{i=1}^{n} \frac{v_{i}d_{i}u_{i}}{1 - e_{i}d_{i}} > 1$$ To sum up, if $M$ has the form $(5.24)$, then

$$(5.29) \quad \rho(DM) > 1 \quad \text{iff} \quad \text{either} \quad e_{i}d_{i} \geq 1 \quad \text{for some} \quad i \quad \text{or} \quad \sum_{i=1}^{n} \frac{v_{i}d_{i}u_{i}}{1 - e_{i}d_{i}} > 1$$ holds.

Where $e_{i}d_{i} < 1$ for all $i$ and $(5.28)$ prevails then the actual value of $\rho(DM)$ can be calculated as the unique solution $\lambda$ of the equation

so that

$$(5.25) \quad m_{i} = (1 - \alpha)\delta_{i} + \alpha v_{i} \cdot (\delta_{i} = \text{Kronecker delta}, \sum_{i=1}^{n} v_{i} = 1).$$ The determination of $\rho(DM)$ depends on the following known formula for calculating the inverse of $(5.24)$. Consider a matrix $C = B + R$ where $R$ is of rank 1 as in $(5.24)$. If $B^{-1}$ exists then provided $c = 1 + (v, B^{-1}u) \neq 0$, $C^{-1}$ has the form

$$(5.26) \quad C^{-1} = B^{-1} - \left(\frac{1}{c}\right)S$$ where $S$ of rank 1 has the form $IL_{\text{rank} 1}$ and specifically $r = vB^{-1}$, $s = B^{-1}u$ with $M$ as in $(5.24)$. We can also represent $MD - \lambda I$ in the same form, viz

$$(5.27) \quad MD - \lambda I = (ED - \lambda I) + R$$ with $R = ||u, v, d_{i}||$. We examine two cases.

Case (i). Suppose $e_{i}d_{i} \geq 1$ for some index $i_{0}$, $1 \leq i_{0} \leq n$. Let $x^{(i_{0})} = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$ with the component 1 appearing only at the $i_{0}$-th coordinate. Since $MD$ is nonnegative then by virtue of $e_{i}d_{i} \geq 1$ we find

$$(5.28) \quad MDx^{(i_{0})} \geq x^{(i_{0})}$$ and the two vectors in $(5.28)$ are not equal. Invoking the characterization of (5.8) implies

$$(5.29) \quad \rho(DM) > 1.$$ Case (ii). Suppose $e_{i}d_{i} < 1$ for all $i = 1, 2, \ldots, n$. Then for each $\lambda \geq 1$, $(ED - \lambda I)^{-1}$ exists. By the prescription of $(5.26)$, $(MD - \lambda I)^{-1}$ exists if $c(\lambda) = 1 + (v, (ED - \lambda I)^{-1}u) \neq 0$ that is provided

$$(5.27) \quad c(\lambda) = 1 + \sum_{i=1}^{n} \frac{v_{i}d_{i}u_{i}}{e_{i}d_{i} - \lambda} \neq 0.$$ Obviously $c(\infty) = 1$ and since $e_{i}d_{i} < 1$ for all $i \geq 1$ the monotone increasing function $c(\lambda)$ vanishes somewhere on the range $1 \leq \lambda < \infty$ and then $\rho(DM) > 1$ if and only if $0 < c(1) = 1 + \sum_{i=1}^{n} \frac{v_{i}d_{i}u_{i}}{e_{i}d_{i} - 1}$ or equivalently

$$(5.28) \quad \sum_{i=1}^{n} \frac{v_{i}d_{i}u_{i}}{1 - e_{i}d_{i}} > 1$$ To sum up, if $M$ has the form $(5.24)$, then

$$(5.29) \quad \rho(DM) > 1 \quad \text{iff} \quad \text{either} \quad e_{i}d_{i} \geq 1 \quad \text{for some} \quad i \quad \text{or} \quad \sum_{i=1}^{n} \frac{v_{i}d_{i}u_{i}}{1 - e_{i}d_{i}} > 1$$ holds.
\[ \sum \frac{d_i u_i v_i}{\lambda - e_d_i} = 1 \]

located in the interval \( \max_{1 \leq i \leq n} e_d_i < \lambda < \infty \). The preceding discussion interpreted for the special example (5.25) leads to the following characterization: \( \rho(DM) > 1 \) iff either \( (1 - \alpha)d_i > 1 \) for some \( i \) or

\[ \alpha \sum \frac{v_i}{d_i - 1 + \alpha} > 1 \]

holds.

(c) A hybrid version of the previous example combined with a permutation matrix is as follows. Consider \( M \) of the form

\[ M = E + R \]

where \( R \) decomposes into block matrices explicit structure

\[ R = \begin{bmatrix} 0 & R_{12} & 0 & \cdots & 0 \\ 0 & 0 & R_{23} & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & R_{n-1,n} \end{bmatrix} \]

where \( R_{k,k+1} = \|u_i^{(k)} v_j^{(k+1)}\| \) is a rank one matrix of order \( n_k \times n_{k+1} \), \( k = 1, 2, \cdots, p \) (interpret \( p + 1 \) as 1) and \( \sum_{k=1}^{p} n_k = n, E = \text{diag}(e_1, e_2, \cdots, e_n) \) is a positive diagonal matrix. It is convenient to view \( e \) as a juxtaposition of \( k \) vectors \( e^{(1)}, e^{(2)}, \cdots, e^{(k)} \) where \( e^{(i)} \) consist of the first \( n_i \) components of \( e \) and the next \( n_k \) components, etc.

The complete result of instability corresponding to the matrix \( DM \) (\( D = \text{diag}(d_1, \cdots, d_n) \)) is a positive diagonal matrix as usual) supplementing (5.29) is as follows: (We decompose the vector \( d \) analogously to \( e \).)

\[ \rho(DM) > 1 \] iff one of the following two conditions hold: either \( e, d_i \geq 1 \) for some \( i \) or

\[ (-1)^p \prod_{k=1}^{p} \left( \sum_{i=1}^{n_k} \frac{u_i^{(k)} v_j^{(k)} d_i^{(k)}}{e_i^{(k)} d_i^{(k)}} - 1 \right) > 1 \]

6. Some classes of inverse eigenvalue problems.

We will apply the relations (2.6) and (2.8) to set forth general necessary conditions for the solution of certain inverse eigenvalue problems. The uniqueness characterizations underlying (1.9) will serve to secure unique solutions for special classes of inverse eigenvalue problems. At this point it would be helpful to consult the background material of the introductory section. We start our deliberations with the inverse additive eigenvalue problem (I.A.E.P.) whose formulation is as follows

(i) Let \( \Lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_n\} \) consist of \( n \) values in \( R^1 \) and let a matrix \( A \) be prescribed. The problem is to ascertain criteria for the existence and determination of at least one real valued diagonal matrix \( D \) with the property that the spectrum of \( A + D \) coincides with \( \Lambda \). In our context, the I.A.E.P. is said to be solvable for the prescriptions \( A \) and \( \Lambda \) if there exists at least one real diagonal matrix \( D \) such that \( A + D \) has spectrum \( \Lambda \).

Our main finding on problem (i) is summarized in Theorem 6.1 highlighted in the introductory section. We proceed to the proof. The following lemma will be needed.

**Lemma 6.1.** Let \( A \) be an irreducible nonnegative matrix. There exists a unique (up to a constant factor) diagonal matrix \( C = \text{diag}(c_1, c_2, \cdots, c_n) \) such that \( \Lambda \) is solvable such that the equation

\[ (6.1) \quad C AC^{-1} e = e \quad e = (1, 1, \cdots, 1) \]

Proof. Consider the function

\[ f(x) = \sum_{i=1}^{n} \frac{(Ax_i)}{x_i} \quad \text{defined for } x > 0, \quad |x| = \sum x_i = 1 \]

Straightforward adaptation of the reasoning of Lemma 3.2 establishes the existence of a unique critical point \( \xi \) for \( f(x) \). Thus \( \xi = (\xi_1, \cdots, \xi_n) \) satisfies the equations

\[ (6.3) \quad \frac{-(A \xi_i)}{\xi_i} + \sum_{j=1}^{n} a_{ij} = 0 \quad j = 1, 2, \cdots, n \]

Defining \( C = \text{diag}(1/\xi_1, 1/\xi_2, \cdots, 1/\xi_n) \) the equations (6.3) are synonymous with (6.1). Conversely if \( c^* = \text{diag}(c_1^*, c_2^*, \cdots, c_n^*) \) qualifies for (6.3), with \( c^* \) normalized by \( \sum c_j^* = 1 \) then we can backtrack and verify that the vector \( \xi^* = (1/c_1^*, \cdots, 1/c_n^*) \) is a critical point of \( f(x) \). The uniqueness assertion of the lemma ensues since a single critical point exists. The proof of lemma 6.1 is complete.

With \( C \) constructed conforming to the relations (6.1), we next determine \( C_0 \) as the unique diagonal matrix satisfying

\[ (6.4) \quad (CAC^{-1} + C_0) e = e(CAC^{-1} + C_0) e = e \]

We are now prepared to deal with Theorem 6.1 which for convenience is restated.

**Theorem 6.1.** Let \( A \) be nonnegative and irreducible. Let \( C \) and \( C_0 \) be determined fulfilling (6.1) and (6.4). Denote by \( \Lambda^0 = [\lambda_1^0, \lambda_2^0, \cdots, \lambda_n^0] \) the spectrum of \( A + C_0 \) (since \( A \geq 0 \) is irreducible and \( C_0 \) real it follows from the Perron Frobenius theorem that \( \lambda_i^0 \) is certainly real and \( \lambda_i^0 > \text{Re} \lambda_i^0, i = 2, 3, \cdots, n \)). Suppose the I.A.E.P. with respect to \( A \) and the prescribed spectrum \( \Lambda = [\lambda_1, \lambda_2, \cdots, \lambda_n] \) having \( \lambda_i > \text{Re} \lambda_i^0, i = 2, 3, \cdots, n \), is solvable such that the spectrum of \( A + D = \Lambda \) with \( D \) real diagonal. Then \( \Lambda \) satisfies

\[ (6.5) \quad (n - 1)\lambda_i \geq \sum_{j=1}^{n} c_j a_{ij} \frac{1}{c_i} + \sum_{j=2}^{n} \lambda_j \]
The equality sign holds only for the spectral set \( \{ \lambda_1^0 + \beta, \lambda_2^0 + \beta, \ldots, \lambda_n^0 + \beta \} \) for any real \( \beta \).

Proof. Suppose the spectrum of \( A + D \) is \( \Lambda \). Because \( C \) and \( D \) are diagonal, the spectrum of \( CAC^{-1} + D \) is likewise \( \Lambda \). On the basis of the Perron Frobenius theorem we obtain

\[
(CAC^{-1} + D)x = \lambda x \quad \text{for} \quad x > 0.
\]

Therefore,

\[
n\lambda_1 = \sum_{i=1}^{n} \frac{(CAC^{-1} + D)x_i}{x_i} = \sum_{i=1}^{n} d_i + \sum_{i=1}^{n} \frac{(CAC^{-1}x)_i}{x_i}
\]

Invoking Lemma 2.2 we find that the last sum is bounded below by

\[
\sum_{i=1}^{n} c_i a_{ii} \frac{1}{e_i}
\]

and equality is attained only if \( x = \alpha e \). The equation \( \text{Trace} (A + D) = \sum_{i=1}^{n} \lambda_i \), written out is

\[
\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} d_i \quad \text{or} \quad \sum_{i=1}^{n} d_i = \sum_{i=1}^{n} \lambda_i - \sum_{i=1}^{n} a_{ii}.
\]

Substituting for \( \sum d \), the inequality (6.5) is achieved with the equation sign possible only if

\[
(CAC^{-1} + D)e = e(CAC^{-1} + D) = \alpha e \quad \text{prevails.}
\]

But the fulfillment of (6.7) in view of the results of Lemma 6.1 and the nature of the determination of \( C_0 \) imply \( D = D_0 + \beta I \) for some appropriate real \( \beta \). For \( D_0 = C_0 + \beta I \) the spectrum of \( A + D_0 \) is precisely \( \{ \lambda_1^0 + \beta, \ldots, \lambda_n^0 + \beta \} \). The proof of Theorem 6.1 is complete.

Corollary 6.1. Let the hypothesis and notation of Theorem 6.1 prevail. Consider the I.A.E.P. for \( A \) with prescribed spectrum \( \Lambda = \{ \lambda_1^0 + \alpha, \ldots, \lambda_n^0 + \alpha \} \), \( \alpha \) real and fixed. This inverse additive eigenvalue problem has a unique real solution given by \( D_0 = C_0 + \alpha I \).

We develop next the analog of Theorem 6.1 for the inverse multiplicative eigenvalue problem (I.M.E.P.)

Theorem 6.2. Let \( A \) be a positive nonsingular matrix. Let \( D_1 \) and \( D_2 \) be diagonal positive matrices rendering \( D_1 AD_2 \) doubly stochastic (in this connection see the discussion culminating section 2). The matrices \( D_1 \) and \( D_2 \) are unique up to constant multipliers. Define \( D_0 = D_1 D_2 \) and denote the spectrum of \( AD_0 \) by \( \{ \lambda_1^0, \lambda_2^0, \ldots, \lambda_n^0 \} \), \( \lambda_1^0 = \rho(AD_0) \). If the I.M.E.P. is solvable for \( \Lambda \) with \( \lambda_1 = \rho(AD) \), then we have

\[
\prod_{i=1}^{n} \lambda_i^0 \geq \prod_{i=1}^{n} \lambda_i
\]

Equality holds only if \( \lambda_i = \alpha \lambda_j^0, i = 1, 2, \ldots, n, \) for some \( \alpha \).

Proof. Consider that \( \lambda_i(AD) = \lambda_i \) with \( \lambda_i(AD) = \rho(AD) \). If \( d_i = 0 \) for some \( i \) then \( \Pi_{i=1}^{n} \lambda_i = 0 \) and (6.8) is trivially true. Suppose now that \( d_i > 0, i = 1, \ldots, n \). Invoking the inequality (2.8) (since \( D_2 AD_1 \) is doubly stochastic) yields

\[
\lambda_i^0 = [\rho(AD)]^0 = [\rho(AD_0 D_0^{-1})]^0 \geq \left[ \prod_{i=1}^{n} d_i \left( \prod_{i=1}^{n} \lambda_i \right) \right]
\]

Plainly

\[
\prod_{i=1}^{n} d_i = \left( \prod_{i=1}^{n} \lambda_i \right) (\text{det } A)^{-1}, \quad \prod_{i=1}^{n} d_i = \left( \prod_{i=1}^{n} \lambda_i \right) (\text{det } A)^{-1}.
\]

Of course, \( \lambda_i^0 = 1 \) since \( D_0 AD_1 \) is doubly stochastic. Combining (6.10) into (6.9) produces (6.8).

The equation sign in (6.8), as \( \Lambda \) is strictly positive compels the case of equality in Lemma 2.2 and then necessarily \( D_0 D_1 AD_2^{-1} = \alpha I \). The proof is finished.

Corollary 6.2. Let \( A \) satisfy the assumptions and notation of Theorem 6.2. We prescribe \( \Lambda = \{ \beta_1 \lambda_1^0, \beta_2 \lambda_2^0, \ldots, \beta_n \lambda_n^0 \} \) for \( \beta > 0 \). Then the I.M.E.P. has a unique solution \( D = \beta D_0 \).

Another result supplementary to Theorem 6.2 relies on (1.12) instead of (2.8). Accordingly we obtain

Theorem 6.3. Let \( A \) be a doubly stochastic matrix such that \( a_{ii} = a > 0 \) (with constant diagonal.) Assume furthermore that either \( A^{-1} \) is an M-matrix (see Definition 3.1) or \( A = KEK_2 \) where \( K \), \( E_1 \), and \( E_2 \); the latter two diagonal, are all positive definite. If for some positive diagonal \( D \) the spectrum of \( DA \) is the set \( \Lambda = \{ \lambda_1, \lambda_2, \ldots, \lambda_n \} \) with \( \lambda_i > \text{Re} \lambda_i, i = 2, \ldots, n \), then

\[
\lambda_1 \geq \frac{1}{na} \sum_{i=1}^{n} \lambda_i.
\]

The equality sign is maintained if and only if \( D = \alpha I \) for some \( \alpha > 0 \).

Remark 6.1. The uniqueness feature of the inverse problems of Corollaries 6.1 and 6.2 are akin or perhaps discrete analogs to the uniqueness result demonstrated for the system of eigenvalues occurring in the case of a vibrating membrane, see Kac [7] and Kac and Van Moerbeke [8].

7. Inequalities for the spectral radius of some integral operators. By a standard discretization akin to the analysis of Fredholm kernel operators,
approximating an integral operator by a sequence of matrix operators we can secure extensions of (1.8) and (1.12) to certain classes of integral operators. This procedure will be described succinctly. Let \( \Omega \) be a compact set in \( \mathbb{R}^n \). As usual, denote by \( C(\Omega) \) the collection of all real valued continuous functions on \( \Omega \). The subset of nonnegative functions of \( C(\Omega) \) is designated by \( C_+(\Omega) \) and by \( C_+^*(\Omega) \), the positive function. Throughout, let

\[
K(x, y) \text{ be a continuous nonnegative kernel on } \Omega \times \Omega.
\]

Let \( \sigma \) be a prescribed nonnegative regular finite measure on \( \Omega \) so that \( \sigma(\Omega) < \infty \). Consider the linear operator

\[
K_* : C(\Omega) \to C(\Omega)
\]

defined explicitly by the integral kernel mapping

\[
K_*(f)(x) = \int_\Omega K(x, y)f(y) \, d\sigma(y) \quad f \in C(\Omega).
\]

The operator \( K_* \) is obviously compact, even transforming a bounded set of measurable functions into an equi-continuous family of functions. The assumption (7.1) entails that \( K_* \) preserves the cone \( C_+(\Omega) \). The theory of operators leaving invariant a cone (dubbed positive operators) is well developed, e.g., see Krein and Rutman [15], Karlin [13], Sawashima [16], Krasnoselskii [14, Chap. 2], Anselone and Lee [1] among others. Most of the Frobenius theory of nonnegative matrices carries over to the case of compact nonnegative operators. More specifically,

(i) The spectral radius \( \rho(K_*) > 0 \) and \( K_* \) is a simple eigenvalue of \( K_* \).
(ii) Suppose \( \Omega = \Omega^0 \) (\( \Omega^0 \) is the closure of its interior), then there exists a unique (up to scalar multipliers) functions \( u(x) \) and \( v(x) \in C_+^*(\Omega) \) with \( u(x) \) and \( v(x) \) strictly positive on \( \Omega^0 \) satisfying

\[
\rho(K_*)u(x) = \int_\Omega K(x, y)u(y) \, d\sigma(y) \quad \text{indicating that } u \text{ is an eigenvector for } K_* \text{ corresponding to } \lambda_0 = \rho(K_*)
\]

\[
\rho(K_*)v(x) = \int_\Omega v(y)K(x, y) \, d\sigma(x) \quad \text{signifying } v \text{ is an eigenvector for } K_*^*; \text{ the adjoint to } K_*(i) \text{ if } \Omega^0 \text{ is empty, then we stipulate that } u \text{ and } v \in C_+^*(\Omega) \text{. In the presence of (ii) and (ii') we normalize } u \text{ and } v \text{ to satisfy}
\]

\[
\int_\Omega u(y) \, d\sigma(y) = 1 \quad \text{and} \quad \int_\Omega u(y)v(y) \, d\sigma(y) = 1
\]

We call an operator endowed with the properties (i) and (ii) irreducible.

A number of useful sufficient conditions implying that \( K_* \) is irreducible are subsumed in the following two cases. Thus, \( K_* \) is nonnegative irreducible if either of the following two hypotheses holds:

(a) \( K(x, x) > 0 \) for \( x \in \Omega \) (K is strictly positive on the diagonal), Sawashima [16];

(b) \( K(x, y) = r(x)G(x, y)s(y) \) where \( r \) and \( s \in C_+^*(\Omega) \) and \( G \) is symmetric such that \( G \) induces a positive definite operator.

We now state the first result which extends the inequality (1.8).

**Theorem 7.1.** Let \( K_*^1 \) be defined as in (7.3) satisfying (7.1) and either (7.6a) or (7.6b). (Thus \( K_*^1 \) is irreducible.) Let \( p(x) \in C_+^*(\Omega) \). Denote by \( (Kp)_* \), the integral operator (7.3) induced by the kernel \( K_*(x, y)p(y) \).

\[
\rho((Kp)_*) \geq \rho(K_*) \exp \left[ \int_\Omega u(x)v(x)(\log p(x)) \, d\sigma(x) \right]
\]

**Proof.** We will be brief. Let \( X \subseteq \{x_1\}^* \) be a dense set of points in \( \Omega \). The span of a convex set by its extreme point (Krein-Milman Theorem) provides the existence of a sequence of discrete measures \( \sigma^{(k)} \) concentrating at a finite sub-set of \( X \) say \( X^k = \{x_i\}^k \) with \( \sigma^{(k)} \) converging in the weak * topology to \( \sigma \), that is

\[
\lim_{k \to \infty} \int_\Omega f \, d\sigma^{(k)} = \int_\Omega f \, d\sigma \quad \text{for all } f \in C(\Omega)
\]

The integral operators \( K_*^k \) and \( (Kp)_*^k \) can be approximated in the weak operator topology by a sequence of operators with finite range induced by the matrix mappings.

\[
\int K(x_i, y)f(y) \, d\sigma \quad \text{and} \quad \int K(x_i, y)p(y)f(y) \, d\sigma
\]

\[
i = 1, 2, \ldots, k.
\]

It can be checked that each of the operators (7.8) is nonnegative and irreducible. The inequality (1.8) is in force involving the eigenvectors \( u_k = \{u_k(x_i)\}^k \) and \( v_k = \{v_k(x_i)\}^k \) appropriately normalized. A routine compactness argument, involving the Fredholm determinants, validates that we can extract an appropriate subsequence of \( u_k \) converging uniformly to \( u(x) \) on \( X \) and similarly, \( v_k \to v(x) \). The inequality (7.7) ensues by proceeding to the limit in \( k \) applying at each stage the corresponding matrix inequality. The proof is complete.

By completely similar means, we deduce

**Theorem 7.2.** Let the assumptions of theorem 7.1 hold. Assume also that

\[
K(x, y) = r(x)G(x, y)s(y) \quad \text{where } r \text{ and } s \in C_+^*(\Omega) \text{ and } G \text{ is a positive definite symmetric kernel, i.e.,}
\]

\[
\int \int G(x, y)f(x)(y) \, d\sigma(x) \, d\sigma(y) > 0 \quad \text{for } f \in C(\Omega) \text{ and } f \neq 0.
\]

Let \( p(x) \in C_+^*(\Omega) \). Then

\[
\rho((Kp)_*) \geq \rho(K_*) \int_\Omega u(x)v(x)p(x) \, d\sigma(x)
\]
Remark 7.1. A more delicate analysis shows that equality holds in (7.10) if and only if \( p(x) = \alpha > 0 \) on \( \Omega \). For this purpose, we rely on the minimax characterization of the spectral radius for symmetrizable operators.

An application
Consider a translation kernel \( K(x, y) = k(x - y) \). Take \( \Omega = (-\pi, \pi) \).
Assume \( k(x) \) is periodic, symmetric with respect to 0 (an even function) and \( \in C(\Omega) \).
Accordingly, the Fourier expression of \( k \) has the form

\[
k(x) \sim \sum_{n=0}^{\infty} a_n \cos nx
\]

Assume further that the matrix

\[
||\{(x_n - x_j)\}|^r \text{ is nonnegative definite for every collection } -\pi < x_1 < x_2 < \cdots < x_s < \pi, p \text{ arbitrary.}
\]

The celebrated Bochner theorem applied to \( k(x) \) guarantees

\[
a_n \geq 0, \quad n = 0, 1, 2, \ldots, \sum_{n=0}^{\infty} a_n < \infty
\]
and the fact of (7.13) entails that

\[
k(x) = \sum_{n=0}^{\infty} a_n \cos nx \text{ with absolute and uniform convergence present}
\]

We also assume henceforth

\[
k(x) > 0 \text{ for } -\pi < x < \pi \text{ and normalized such that } k(0) = 1 = \sum_{n=0}^{\infty} a_n.
\]

Let \( K \) be the operator

\[
K(f) = \int_{-\pi}^{\pi} k(x - y)f(y)dy \text{ defined for } f \in C(\Omega) \text{ or even for } f \in L_1(\Omega).
\]

The facts of (7.13) and (7.14) imply that \( K \) on \( L_1(\Omega) \) is a trace (nuclear) operator. Obviously, the eigenvalues of \( K \) are \( \{a_n\}_{n=1}^{\infty} \). In particular,

\[
\rho(K) = 2\pi a_0 = \int_{-\pi}^{\pi} k(x)dx
\]
and the associated eigenfunctions are the constant functions

\[
u(x) = v(x) = \frac{1}{\sqrt{2\pi}}
\]

Let \( p(x) \in C_{\ast}^{\ast}(\Omega) \) and consider the operator on \( L_1(\Omega) \) or \( C(\Omega) \)

\[
Kp(f) = \int_{-\pi}^{\pi} k(x - y)p(y)f(y)dy
\]

Denote the spectrum of \( Kp \) by \( \{\lambda_1(p)\}_{n=1}^{\infty} \) arranged so that \( \lambda_1(p) = \rho(Kp) \).
Standard theory tells us that \( Kp \) is also a nuclear (trace class 1) operator and therefore \( \sum_{n=1}^{\infty} |\lambda_i(p)| < \infty \). Since \( Kp \) is a trace operator we find recalling the normalization \( k(0) = 1 \) that

\[
\int_{-\pi}^{\pi} k(x - y)p(x)dx = \int_{-\pi}^{\pi} p(x)dx = \sum_{n=1}^{\infty} \lambda_n(p)
\]

Application of theorem 7.2 gives

**Theorem 7.3.** Let \( k(x) \) satisfy the assumptions stated prior to (7.11), and also (7.12) and (7.15). Let \( p(x) \in C_{\ast}^{\ast}(\Omega) \). Then

\[
\rho(Kp) \geq \left( \frac{\int_{-\pi}^{\pi} k(x)dx}{2\pi} \right)^{1/2} \int_{-\pi}^{\pi} p(x)dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} k(x)dx \sum_{n=1}^{\infty} \lambda_n(p)
\]
(notice that \( 1/2\pi \int_{-\pi}^{\pi} p(x)dx = f_{-\pi}^{\pi} p(x)u(x)v(x)dx \) and \( \int_{-\pi}^{\pi} k(x)dx = \rho(K) \) by (7.17).)
Equality sign holds iff \( p(x) = \alpha > 0 \).

From (7.15) and (7.13) we have that \( 1/2\pi \int_{-\pi}^{\pi} k(x)dx = a_0 < 1 \), so that (7.20) is equivalent to

\[
\rho(1 - \frac{a_0}{2\pi} Kp) \geq \frac{a_0}{2\pi} \sum_{n=0}^{\infty} \lambda_n(p).
\]

Since the spectrum of the integral operator with kernel \( k(x - y)p(y) \) coincides with the operator having kernel \( \sqrt{p(x)}k(x - y)\sqrt{p(y)} \) we deduce that \( \lambda_i(p) \geq 0 \), \( i = 1, 2, \ldots \). As a consequence we have

**Theorem 7.4.** Consider the multiplicative inverse eigenvalue problem to determine \( p \) belonging to \( C_{\ast}^{\ast}(\Omega) \) with the prescribed spectrum \( \Lambda = \{\lambda_i\}_{n=1}^{\infty} \). Assume that the spectrum is \( \alpha \Lambda^\alpha \) where \( \Lambda^\alpha = \{a_n\}_{n=1}^{\infty}, \alpha > 0 \).
Then \( p(x) = \alpha \).

**References**


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