Tensors

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In the past ten years, tensors again became a hot topic of research in pure and applied mathematics. In applied mathematics it is driven by data which has a few parameters. In pure math. it is quantum information theory, and multilinear algebra. There are many interesting numerical and theoretical problems that need to be resolved. Tensors are related to matrices one one hand and on the other hand are related to polynomial maps.
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To paraphrase Max Noether:
Matrices were created by God and tensors by Devil.
Overview

Ranks of 3-tensors
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1 Basic facts.
Overview

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1. Basic facts.
2. Complexity.
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3. Matrix multiplication
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4. Results and conjectures
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Approximations of tensors
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1 Rank one approximation.
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1. Rank one approximation.
2. Perron-Frobenius theorem
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3. Rank \((R_1, R_2, R_3)\) approximations
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4. CUR approximations
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Diagonal scaling of nonnegative tensors to tensors with given rows, columns and depth sums
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Diagonal scaling of nonnegative tensors to tensors with given rows, columns and depth sums

Characterization of tensor in \(\mathbb{C}^{4 \times 4 \times 4}\) of border rank 4
Basic notions

scalar $a \in \mathbb{F}$, vector $\mathbf{x} = (x_1, \ldots, x_n)^\top \in \mathbb{F}^n$, matrix $A = [a_{ij}] \in \mathbb{F}^{m \times n}$,
3-tensor $T = [t_{i,j,k}] \in \mathbb{F}^{m \times n \times l}$, p-tensor $T = [t_{i_1, \ldots, i_p}] \in \mathbb{F}^{n_1 \times \ldots \times n_p}$
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Abstractly \( \mathbb{U} := \mathbb{U}_1 \otimes \mathbb{U}_2 \otimes \mathbb{U}_3 \) \( \dim \mathbb{U}_i = m_i, i = 1, 2, 3 \), \( \dim \mathbb{U} = m_1 m_2 m_3 \)
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Tensor \( \tau \in U_1 \otimes U_2 \otimes U_3 \)
Basic notions

**Scalar:** \( a \in F \), **vector:** \( x = (x_1, \ldots, x_n)^\top \in F^n \), **matrix:** \( A = [a_{ij}] \in F^{m \times n} \),

**3-tensor:** \( T = [t_{i,j,k}] \in F^{m \times n \times l} \), **p-tensor:** \( T = [t_{i_1,\ldots,i_p}] \in F^{n_1 \times \ldots \times n_p} \)

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**HISTORY:** Tensors-as now W. Voigt 1898

Basic notions

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Tensor \( \tau \in \mathbb{U}_1 \otimes \mathbb{U}_2 \otimes \mathbb{U}_3 \)

HISTORY: Tensors—as now W. Voigt 1898
Tensor calculus 1890 G. Ricci-Curbastro: absolute differential calculus,
T. Levi-Civita: 1900, A. Einstein: General relativity 1915

Rank one tensor \( t_{i,j,k} = x_i y_j z_k, (i, j, k) = (1, 1, 1), \ldots, (m_1, m_2, m_3) \)
or decomposable tensor \( \mathbf{x} \otimes \mathbf{y} \otimes \mathbf{z} \)
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basis of \( \mathbb{U}_j \): \( [\mathbf{u}_{1,j}, \ldots, \mathbf{u}_{m_j,j}] j = 1, 2, 3 \)
basis of \( \mathbb{U} \): \( \mathbf{u}_{i_1,1} \otimes \mathbf{u}_{i_2,2} \otimes \mathbf{u}_{i_3,3}, i_j = 1, \ldots, m_j, j = 1, 2, 3, \)
Basic notions

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Abstractly $\mathbb{U} := \mathbb{U}_1 \otimes \mathbb{U}_2 \otimes \mathbb{U}_3 \text{ dim } \mathbb{U}_i = m_i, i = 1, 2, 3, \text{ dim } \mathbb{U} = m_1 m_2 m_3$

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basis of $\mathbb{U}_j$: $[u_{1,j}, \ldots, u_{m_j,j}] j = 1, 2, 3$
basis of $\mathbb{U}$: $u_{i_1,1} \otimes u_{i_2,2} \otimes u_{i_3,3}, i_j = 1, \ldots, m_j, j = 1, 2, 3,$
$\tau = \sum_{i_1=i_2=i_3=1}^{m_1,m_2,m_3} t_{i_1,i_2,i_3} u_{i_1,1} \otimes u_{i_2,2} \otimes u_{i_3,3}$
Ranks of tensors

Unfolding tensor: in direction 1:
\[ T = [t_{ij,k}] \]
view as a matrix
\[ A_1 = [t_{i_1,j_1,k_1}] \in \mathbb{F}^{m_1 \times (m_2 \cdot m_3)} \]
\[ R_1 := \text{rank } A_1 : \text{dimension of row or column subspace spanned in direction 1} \]

\[ T_{i_1} = [t_{i_1,j_1,k_1}]_{m_2 \times m_3} \quad \text{for } i_1 = 1, \ldots, m_1 \]
\[ T = \sum_{i_1=1}^{m_1} T_{i_1} e_{i_1} \quad \text{(convenient notation)} \]
\[ R_1 := \text{dim span} (T_{i_1}, \ldots, T_{m_1}) \]

Similarly, unfolding in directions 2, 3
\[ \text{rank } T \text{ minimal } r : \]
\[ T = f(x_1, y_1, z_1, \ldots, x_r, y_r, z_r) := \sum_{i=1}^{r} x_i \otimes y_i \otimes z_i , \quad \text{(CANDEC, PARFAC)} \]
Ranks of tensors

Unfolding tensor: in direction 1:

$\mathcal{T} = [t_{i,j,k}]$ view as a matrix $A_1 = [t_{i,(j,k)}] \in \mathbb{F}^{m_1 \times (m_2 \cdot m_3)}$
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\[ T_{i,1} := [t_{i,j,k}]^{m_2,m_3}_{j,k=1} \in \mathbb{F}^{m_2 \times m_3}, \quad i = 1, \ldots, m_1 \]
Ranks of tensors

Unfolding tensor: in direction $1$:
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$R_1 := \text{rank } A_1$:
dimension of row or column subspace spanned in direction $1$

$T_{i,1} := [t_{i,j,k}]_{j,k=1}^{m_2,m_3} \in \mathbb{F}^{m_2 \times m_3}, i = 1, \ldots, m_1$
$\mathcal{T} = \sum_{i=1}^{m_1} T_{i,1} e_{i,1}$ (convenient notation)
Ranks of tensors

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Similarly, unfolding in directions 2, 3

rank \( T \) minimal \( r \):

\[ T = f_r(x_1, y_1, z_1, \ldots, x_r, y_r, z_r) := \sum_{i=1}^{r} x_i \otimes y_i \otimes z_i, \]
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(CANDEC, PARFAC)
Basic facts

FACT I: \[ T \geq \max (R_1, R_2, R_3) \]

Reason: \[ U_2 \otimes U_3 \sim F_{m_2 \times m_3} \equiv F_{m_2 m_3} \]

Note: \( R_1, R_2, R_3 \) are easily computable.

It is possible that \( R_1 \neq R_2 \neq R_3 \).

FACT II: For \( \tau = T = [t_i, j, k] \), let \( T_k, 3 = [t_i, j, k] \). Then \[ \text{rank } T = \text{minimal dimension of subspace } L \subset F_{m_1 \times m_2} \text{spanned by rank one matrices containing } T_1, 3, \ldots, T_{m_3}, 3. \]

COR: \[ \text{rank } T \leq \min (m_1 n_l, m_2 l, n_3 l) \]
Basic facts

**FACT I:** \( \text{rank } T \geq \max(R_1, R_2, R_3) \)

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\( T_{k, 3} = [t_i, j, k] \)

\( m_1, m_2 i, j = 1 \in F_{m_1 \times m_2}, k = 1, \ldots, m_3 \).

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Basic facts

**FACT I:** $\text{rank } \mathcal{T} \geq \max(R_1, R_2, R_3)$

**Reason** $U_2 \otimes U_3 \sim \mathbb{F}^{m_2 \times m_3} \equiv \mathbb{F}^{m_2m_3}$

Note: $R_1, R_2, R_3$ are easily computable.

It is possible that $R_1 \neq R_2 \neq R_3$.

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**COR** $\text{rank } T \leq \min(mn, ml, nl)$.
Basic facts

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Complexity of rank of 3-tensor

Hastad 1990: Tensor rank is NP-complete for any finite field and NP-hard for rational numbers.

PRF: 3-sat with \( n \) variables, \( m \) clauses is satisfiable iff \( \text{rank } T = 4n + 2m \), \( T \in \mathbb{F}(2n+3m) \times (3n) \times (3n+m) \), otherwise rank is larger.
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PRF: 3-sat with $n$ variables $m$ clauses
satisfiable iff $\text{rank } T = 4n + 2m$, $T \in \mathbb{F}^{(2n+3m) \times (3n) \times (3n+m)}$
otherwise rank is larger
Generic and typical ranks

Let $R_r(m, n, l) \subset F^{m \times n \times l}$ be the set of all tensors of rank $\leq r$. For $r \geq 2$, $R_r(m, n, l)$ is not a closed variety.

The border rank of a tensor $T$ is defined as the minimum $k$ such that $T$ is a limit of tensors of rank $k$. The generic rank $grank(m, n, l)$ is the rank of a random tensor in $F^{m \times n \times l}$. The typical rank $trank(m, n, l)$ takes all values from $grank(m, n, l)$ to $mtrank(m, n, l)$.

In all the examples we know, $mtrank(m, n, l) \leq grank(m, n, l) + 1$. 
Generic and typical ranks

\[ \mathcal{R}_r(m, n, l) \subset \mathbb{F}^{m \times n \times l} : \text{ all tensors of rank } \leq r \]
Generic and typical ranks

\[ \mathcal{R}_r(m, n, l) \subset \mathbb{F}^{m \times n \times l} : \text{ all tensors of rank } \leq r \]

\[ \mathcal{R}_r(m, n, l) \text{ not closed variety for } r \geq 2 \]
Generic and typical ranks

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Border rank of \( \mathcal{T} \) the minimum \( k \) s.t. \( \mathcal{T} \) is a limit of \( \mathcal{T}_j, j \in \mathbb{N} \), rank \( T_j = k \).
Generic and typical ranks

\[ \mathcal{R}_r(m, n, l) \subset \mathbb{F}^{m \times n \times l} : \text{ all tensors of rank } \leq r \]

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Border rank of \( \mathcal{T} \) the minimum \( k \) s.t. \( \mathcal{T} \) is a limit of \( \mathcal{T}_j, j \in \mathbb{N} \), rank \( \mathcal{T}_j = k \).

generic rank is the rank of a random tensor \( \mathcal{T} \in \mathbb{C}^{m \times n \times l} : \text{grank}(m, n, l) \)
Generic and typical ranks

$\mathcal{R}_r(m, n, l) \subset \mathbb{F}^{m \times n \times l}$: all tensors of rank $\leq r$

$\mathcal{R}_r(m, n, l)$ not closed variety for $r \geq 2$

Border rank of $\mathcal{T}$ the minimum $k$ s.t. $\mathcal{T}$ is a limit of $\mathcal{T}_j, j \in \mathbb{N}$, rank $\mathcal{T}_j = k$.

generic rank is the rank of a random tensor $\mathcal{T} \in \mathbb{C}^{m \times n \times l}$: $\text{grank}(m, n, l)$

typical rank is a rank of a random tensor $\mathcal{T} \in \mathbb{R}^{m \times n \times l}$.
Generic and typical ranks

\[ R_r(m, n, l) \subset \mathbb{F}^{m \times n \times l} : \text{all tensors of rank } \leq r \]

\[ R_r(m, n, l) \text{ not closed variety for } r \geq 2 \]

Border rank of \( T \) the minimum \( k \) s.t. \( T \) is a limit of \( T_j, j \in \mathbb{N}, \text{rank } T_j = k \).

generic rank is the rank of a random tensor \( T \in \mathbb{C}^{m \times n \times l} : \text{grank}(m, n, l) \)

typical rank is a rank of a random tensor \( T \in \mathbb{R}^{m \times n \times l} \).

typical rank takes all the values \( k = \text{grank}(m, n, l), \ldots, \text{mtrank}(m, n, l) \)
**Generic and typical ranks**

\( \mathcal{R}_r(m, n, l) \subset \mathbb{F}^{m \times n \times l} \): all tensors of rank \( \leq r \)

\( \mathcal{R}_r(m, n, l) \) not closed variety for \( r \geq 2 \)

Border rank of \( \mathcal{T} \) the minimum \( k \) s.t. \( \mathcal{T} \) is a limit of \( \mathcal{T}_j, j \in \mathbb{N}, \) rank \( T_j = k. \)

**generic rank** is the rank of a random tensor \( \mathcal{T} \in \mathbb{C}^{m \times n \times l} : \) grank \( (m, n, l) \)

**typical rank** is a rank of a random tensor \( \mathcal{T} \in \mathbb{R}^{m \times n \times l}. \)

**typical rank takes all the values** \( k = \text{grank}(m, n, l), \ldots, \text{mtrank}(m, n, l) \)

In all the examples we know \( \text{mtrank}(m, n, l) \leq \text{grank}(m, n, l) + 1 \)
THM: $\text{grank}_\mathbb{C}(m, n, l) = \min(l, mn)$ for $(m - 1)(n - 1) + 1 \leq l$. 
Generic rank of $\mathbb{C}^{m \times n \times l}$

**THM:** $\text{grank}_{\mathbb{C}}(m, n, l) = \min(l, mn)$ for $(m - 1)(n - 1) + 1 \leq l$.

**Reason:** For $l = (m - 1)(n - 1) + 1$ a generic subspace of matrices of dimension $l$ in $\mathbb{C}^{m \times n}$ intersect the variety of rank one matrices in $\mathbb{C}^{m \times n}$ at least at $l$ lines which contain $l$ linearly independent matrices.
Generic rank of $\mathbb{C}^{m \times n \times l}$

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**COR:** $\text{grank}_\mathbb{C}(2, n, l) = \min(l, 2n)$ for $2 \leq n \leq l$. 
THM: \( \text{grank}_C (m, n, l) = \min(l, mn) \) for \((m - 1)(n - 1) + 1 \leq l\).

Reason: For \( l = (m - 1)(n - 1) + 1 \) a generic subspace of matrices of dimension \( l \) in \( \mathbb{C}^{m \times n} \) intersect the variety of rank one matrices in \( \mathbb{C}^{m \times n} \) at least at \( l \) lines which contain \( l \) linearly independent matrices

COR: \( \text{grank}_C (2, n, l) = \min(l, 2n) \) for \( 2 \leq n \leq l \)

Dimension count for \( F = \mathbb{C} \) and \( 2 \leq m \leq n \leq l \leq (m - 1)(n - 1) + 1 \):
Generic rank of $\mathbb{C}^{m \times n \times l}$

**THM:** $\text{grank}_\mathbb{C}(m, n, l) = \min(l, mn)$ for $(m - 1)(n - 1) + 1 \leq l$.

**Reason:** For $l = (m - 1)(n - 1) + 1$ a generic subspace of matrices of dimension $l$ in $\mathbb{C}^{m \times n}$ intersect the variety of rank one matrices in $\mathbb{C}^{m \times n}$ at least at $l$ lines which contain $l$ linearly independent matrices.

**COR:** $\text{grank}_\mathbb{C}(2, n, l) = \min(l, 2n)$ for $2 \leq n \leq l$.

**Dimension count for $F = \mathbb{C}$ and $2 \leq m \leq n \leq l \leq (m - 1)(n - 1) + 1$:**

$f_r : (\mathbb{C}^m \times \mathbb{C}^n \times \mathbb{C}^l)^r \rightarrow \mathbb{C}^{m \times n \times l}, \ x \otimes y \otimes z = (ax) \otimes (by) \otimes ((ab)^{-1}z)$
Generic rank of $\mathbb{C}^{m \times n \times l}$

**THM:** $\text{grank}_{\mathbb{C}}(m, n, l) = \min(l, mn)$ for $(m - 1)(n - 1) + 1 \leq l$.

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**COR:** $\text{grank}_{\mathbb{C}}(2, n, l) = \min(l, 2n)$ for $2 \leq n \leq l$

**Dimension count for $F = \mathbb{C}$ and $2 \leq m \leq n \leq l \leq (m - 1)(n - 1) + 1$:**

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$$\text{grank}_{\mathbb{C}}(m, n, l)(m + n + l - 2) \geq mnl \Rightarrow \text{grank}_{\mathbb{C}}(m, n, l) \geq \left\lceil \frac{mnl}{(m+n+l-2)} \right\rceil$$
THM: \( \text{grank}_\mathbb{C}(m, n, l) = \min(l, mn) \) for \((m - 1)(n - 1) + 1 \leq l\).

Reason: For \( l = (m - 1)(n - 1) + 1 \) a generic subspace of matrices of dimension \( l \) in \( \mathbb{C}^{m \times n} \) intersect the variety of rank one matrices in \( \mathbb{C}^{m \times n} \) at least at \( l \) lines which contain \( l \) linearly independent matrices

COR: \( \text{grank}_\mathbb{C}(2, n, l) = \min(l, 2n) \) for \( 2 \leq n \leq l \)

Dimension count for \( F = \mathbb{C} \) and \( 2 \leq m \leq n \leq l \leq (m - 1)(n - 1) + 1 \):

\[
f_r : (\mathbb{C}^m \times \mathbb{C}^n \times \mathbb{C}^l)^r \to \mathbb{C}^{m \times n \times l}, \quad x \otimes y \otimes z = (ax) \otimes (by) \otimes ((ab)^{-1}z)
\]

\[
\text{grank}_\mathbb{C}(m, n, l)(m + n + l - 2) \geq mnl \Rightarrow \text{grank}_\mathbb{C}(m, n, l) \geq \left\lceil \frac{mnl}{m+n+l-2} \right\rceil
\]

Conjecture \( \text{grank}_\mathbb{C}(m, n, l) = \left\lceil \frac{mnl}{m+n+l-2} \right\rceil \)

for \( 2 \leq m \leq n \leq l < (m - 1)(n - 1) \) and \( (3, n, l) \neq (3, 2p + 1, 2p + 1) \)
Generic rank of $\mathbb{C}^{m \times n \times l}$

**THM:** $\text{grank}_{\mathbb{C}}(m, n, l) = \min(l, mn)$ for $(m - 1)(n - 1) + 1 \leq l$.

**Reason:** For $l = (m - 1)(n - 1) + 1$ a generic subspace of matrices of dimension $l$ in $\mathbb{C}^{m \times n}$ intersect the variety of rank one matrices in $\mathbb{C}^{m \times n}$ at least at $l$ lines which contain $l$ linearly independent matrices.

**COR:** $\text{grank}_{\mathbb{C}}(2, n, l) = \min(l, 2n)$ for $2 \leq n \leq l$.

**Dimension count for $\mathbb{F} = \mathbb{C}$ and $2 \leq m \leq n \leq l \leq (m - 1)(n - 1) + 1$:**

$$f_r : (\mathbb{C}^m \times \mathbb{C}^n \times \mathbb{C}^l)^r \to \mathbb{C}^{m \times n \times l}, \quad x \otimes y \otimes z = (ax) \otimes (by) \otimes ((ab)^{-1}z)$$

$$\text{grank}_{\mathbb{C}}(m, n, l)(m + n + l - 2) \geq mnl \Rightarrow \text{grank}_{\mathbb{C}}(m, n, l) \geq \left\lceil \frac{mnl}{m+n+l-2} \right\rceil$$

**Conjecture** $\text{grank}_{\mathbb{C}}(m, n, l) = \left\lceil \frac{mnl}{m+n+l-2} \right\rceil$ for $2 \leq m \leq n \leq l < (m - 1)(n - 1)$ and $(3, n, l) \neq (3, 2p + 1, 2p + 1)$.

**Fact:** $\text{grank}_{\mathbb{C}}(3, 2p + 1, 2p + 1) = \left\lceil \frac{3(2p+1)^2}{4p+3} \right\rceil + 1$.
Bilinear maps and product of matrices

bilinear map: \( \phi : U \times V \rightarrow W \)
Bilinear maps and product of matrices

**bilinear map:** \( \phi : U \times V \rightarrow W \)

\([u_1, \ldots, u_m], [v_1, \ldots, v_n], [w_1, \ldots, w_l] \) bases in \( U, V, W \)
Bilinear maps and product of matrices

bilinear map: \( \phi : U \times V \to W \)

\([u_1, \ldots, u_m], [v_1, \ldots, v_n], [w_1, \ldots, w_l]\) bases in U, V, W

\( \phi(u_i, v_j) = \sum_{k=1} t_{i,j,k} w_k, \mathcal{T} := [t_{i,j,k}] \in F^{m \times n \times l} \)
Bilinear maps and product of matrices

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\( T = \sum_{a=1}^{r} x_a \otimes y_a \otimes z_a, \quad r = \text{rank } T \)
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\[ T = \sum_{a=1}^{r} x_a \otimes y_a \otimes z_a, \quad r = \text{rank } T \]

\[ \phi(c, d) = \sum_{a=1}^{r} (c^\top x)(d^\top y) z_a, \quad c = \sum_{i=1}^{m} c_i u_i, \quad d = \sum_{j=1}^{n} d_j v_j \]
Bilinear maps and product of matrices

bilinear map: \( \phi : U \times V \rightarrow W \)

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\(\phi(u_i, v_j) = \sum_{k=1}^{l} t_{i,j,k} w_k\), \(T := [t_{i,j,k}] \in \mathbb{F}^{m \times n \times l}\)

\(T = \sum_{a=1}^{r} x_a \otimes y_a \otimes z_a\), \(r = \text{rank } T\)

\(\phi(c, d) = \sum_{a=1}^{r} (c^\top x)(d^\top y) z_a\), \(c = \sum_{i=1}^{m} c_i u_i\), \(d = \sum_{j=1}^{n} d_j v_j\)

Complexity: \(r\)-products
Bilinear maps and product of matrices

bilinear map: \( \phi : U \times V \rightarrow W \)

\[ [u_1, \ldots, u_m], [v_1, \ldots, v_n], [w_1, \ldots, w_l] \] bases in \( U, V, W \)

\[ \phi(u_i, v_j) = \sum_{k=1}^{l} t_{i,j,k} w_k, \quad \mathcal{T} := [t_{i,j,k}] \in \mathbb{F}^{m \times n \times l} \]

\[ \mathcal{T} = \sum_{a=1}^{r} x_a \otimes y_a \otimes z_a, \quad r = \text{rank } \mathcal{T} \]

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Complexity: \( r \)-products

Matrix product \( \tau : \mathbb{F}^{M \times N} \times \mathbb{F}^{N \times L} \rightarrow \mathbb{F}^{M \times L}, \ (A, B) \mapsto AB \)
Bilinear maps and product of matrices

**bilinear map:** $\phi : U \times V \rightarrow W$

$[u_1, \ldots, u_m], [v_1, \ldots, v_n], [w_1, \ldots, w_l]$ bases in $U, V, W$

$$\phi(u_i, v_j) = \sum_{k=1}^l t_{i,j,k} w_k, \quad T := [t_{i,j,k}] \in F^{m \times n \times l}$$

$$T = \sum_{a=1}^r x_a \otimes y_a \otimes z_a, \quad r = \text{rank } T$$

$$\phi(c, d) = \sum_{a=1}^r (c^\top x)(d^\top y) z_a, \quad c = \sum_{i=1}^m c_i u_i, \quad d = \sum_{j=1}^n d_j v_j$$

**Complexity:** $r$-products

**Matrix product** $\tau : F^{M \times N} \times F^{N \times L} \rightarrow F^{M \times L}$, $(A, B) \mapsto AB$

$M = N = L = 2$, $\text{grank}_C(4, 4, 4) = \lceil \frac{4 \times 4 \times 4}{4+4+4-2} \rceil = \lceil 6.4 \rceil = 7$
bilinear map: $\phi : U \times V \rightarrow W$

$[u_1, \ldots, u_m], [v_1, \ldots, v_n], [w_1, \ldots, w_l]$ bases in $U, V, W$

$\phi(u_i, v_j) = \sum_{k=1}^{l} t_{i,j,k} w_k$, $T := [t_{i,j,k}] \in F^{m \times n \times l}$

$T = \sum_{a=1}^{r} x_a \otimes y_a \otimes z_a$, $r = \text{rank } T$

$\phi(c, d) = \sum_{a=1}^{r} (c^\top x)(d^\top y)z_a$, $c = \sum_{i=1}^{m} c_i u_i$, $d = \sum_{j=1}^{n} d_j v_j$

Complexity: $r$-products

Matrix product $\tau : F^{M \times N} \times F^{N \times L} \rightarrow F^{M \times L}$, $(A, B) \mapsto AB$

$M = N = L = 2$, $\text{grank}_C(4, 4, 4) = \left\lceil \frac{4 \times 4 \times 4}{4 + 4 + 4 - 2} \right\rceil = \left\lceil 6.4 \right\rceil = 7$

Product of two $2 \times 2$ matrices is done by 7 multiplications
Known cases of rank conjecture

\[
grank(3, 2^p, 2^q) = \left\lceil \frac{12p^2}{4q} + 1 \right\rceil \quad \text{and} \quad grank(3, 2^p - 1, 2^q - 1) = \left\lceil 3 \left(\frac{2^p - 1}{2^q - 1}\right)^2 \right\rceil + 1
\]

\(\text{if } n \not\equiv 2 \pmod{3},\)

\(\text{if } n \equiv 0 \pmod{3},\)

\(\text{if } m \geq 4,\)

\(\text{if } n \geq 4,\)

\((l, 2^p, 2^q) \quad \text{if } l \leq 2^p \leq 2^q \text{ and } 2lp^2 + 2p + 2q - 2 \text{ is integer}\)

Easy to compute \(grank_C(m, n, l)\):

Pick at random \(w_r := (x_1, y_1, z_1, \ldots, x_r, y_r, z_r) \in (\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^l)^r\)

The minimal \(r \geq \left\lceil \frac{mnl}{m + n + l - 2} \right\rceil \)</n}

\(\text{s.t.} \quad \text{rank} J(f_r(w_r)) = mnl\)

is \(grank_C(m, n, l)\) (Terracini Lemma 1915)

Avoid round-off error:

\(w_r \in (\mathbb{Z}^m \times \mathbb{Z}^n \times \mathbb{Z}^l)^r\)

find \(\text{rank} J(f_r(w_r))\) exact arithmetic

I checked the conjecture up to \(m, n, l \leq 14\)
Known cases of rank conjecture

\[
\text{grank}(3, 2p, 2p) = \left\lceil \frac{12p^2}{4p+1} \right\rceil \quad \text{and} \quad \text{grank}(3, 2p - 1, 2p - 1) = \left\lceil \frac{3(2p-1)^2}{4p-1} \right\rceil + 1
\]
Known cases of rank conjecture

\[ \text{grank}(3, 2p, 2p) = \left\lfloor \frac{12p^2}{4p+1} \right\rfloor \text{ and } \text{grank}(3, 2p-1, 2p-1) = \left\lfloor \frac{3(2p-1)^2}{4p-1} \right\rfloor + 1 \]

\((n, n, n+2) \text{ if } n \neq 2 \pmod{3},\)
Known cases of rank conjecture

\[ \text{grank}(3, 2p, 2p) = \left\lceil \frac{12p^2}{4p+1} \right\rceil \quad \text{and} \quad \text{grank}(3, 2p - 1, 2p - 1) = \left\lceil \frac{3(2p-1)^2}{4p-1} \right\rceil + 1 \]

\[(n, n, n + 2) \text{ if } n \neq 2 \pmod{3}, \]
\[(n - 1, n, n) \text{ if } n = 0 \pmod{3}, \]
Known cases of rank conjecture

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\text{grank}(3, 2p, 2p) = \left\lceil \frac{12p^2}{4p+1} \right\rceil \quad \text{and} \quad \text{grank}(3, 2p - 1, 2p - 1) = \left\lceil \frac{3(2p-1)^2}{4p-1} \right\rceil + 1
\]

\( (n, n, n + 2) \) if \( n \neq 2 \) (mod 3),
\( (n - 1, n, n) \) if \( n = 0 \) (mod 3),
\( (4, m, m) \) if \( m \geq 4 \).
Known cases of rank conjecture

\( \text{grank}(3, 2p, 2p) = \left\lceil \frac{12p^2}{4p+1} \right\rceil \) and \( \text{grank}(3, 2p - 1, 2p - 1) = \left\lceil \frac{3(2p-1)^2}{4p-1} \right\rceil + 1 \)

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\((n, n, n)\) if \( n \geq 4 \)
Known cases of rank conjecture

grank(3, 2p, 2p) = \left\lceil \frac{12p^2}{4p+1} \right\rceil \text{ and } grank(3, 2p - 1, 2p - 1) = \left\lceil \frac{3(2p-1)^2}{4p-1} \right\rceil + 1

(n, n, n + 2) \text{ if } n \not\equiv 2 \, (\text{mod } 3),

(n - 1, n, n) \text{ if } n \equiv 0 \, (\text{mod } 3),

(4, m, m) \text{ if } m \geq 4,

(n, n, n) \text{ if } n \geq 4

(l, 2p, 2q) \text{ if } l \leq 2p \leq 2q \text{ and and } \frac{2lp}{l+2p+2q-2} \text{ is integer}
Known cases of rank conjecture

\[ \text{grank}(3, 2p, 2p) = \left\lceil \frac{12p^2}{4p+1} \right\rceil \quad \text{and} \quad \text{grank}(3, 2p - 1, 2p - 1) = \left\lceil \frac{3(2p-1)^2}{4p-1} \right\rceil + 1 \]

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\[(n, n, n) \text{ if } n \geq 4 \]
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Easy to compute \( \text{grank}_C(m, n, l) \):
Known cases of rank conjecture

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Easy to compute \(\text{grank}_{\mathbb{C}}(m, n, l)\):

Pick at random \(w_r := (x_1, y_1, z_1, \ldots, x_r, y_r, z_r) \in (\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^l)^r\)
Known cases of rank conjecture

\[ \text{grank}(3, 2p, 2p) = \lceil \frac{12p^2}{4p+1} \rceil \text{ and } \text{grank}(3, 2p - 1, 2p - 1) = \lceil \frac{3(2p-1)^2}{4p-1} \rceil + 1 \]

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\((n, n, n)\) if \(n \geq 4\)

\((l, 2p, 2q)\) if \(l \leq 2p \leq 2q\) and \(\frac{2lp}{l+2p+2q-2}\) is integer

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Pick at random \(w_r := (x_1, y_1, z_1, \ldots, x_r, y_r, z_r) \in (\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^l)^r\)

The minimal \(r \geq \lceil \frac{mnl}{(m+n+l-2)} \rceil\) s.t. \(\text{rank } J(f_r)(w_r) = mnl\)

is \(\text{grank}_C(m, n, l)\) (Terracini Lemma 1915)
Known cases of rank conjecture

\[ \text{grank}(3, 2p, 2p) = \lceil \frac{12p^2}{4p+1} \rceil \quad \text{and} \quad \text{grank}(3, 2p-1, 2p-1) = \lceil \frac{3(2p-1)^2}{4p-1} \rceil + 1 \]

\( (n, n, n+2) \) if \( n \neq 2 \) (mod 3),
\( (n-1, n, n) \) if \( n = 0 \) (mod 3),
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\( (n, n, n) \) if \( n \geq 4 \)
\( (l, 2p, 2q) \) if \( l \leq 2p \leq 2q \) and \( \frac{2lp}{l+2p+2q-2} \) is integer

Easy to compute \( \text{grank}_{\mathbb{C}}(m, n, l) \):

Pick at random \( \mathbf{w}_r := (\mathbf{x}_1, \mathbf{y}_1, \mathbf{z}_1, \ldots, \mathbf{x}_r, \mathbf{y}_r, \mathbf{z}_r) \in (\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^l)^r \)

The minimal \( r \geq \lceil \frac{mnl}{(m+n+l-2)} \rceil \) s.t. \( \text{rank} J(f_{r})(\mathbf{w}_r) = mnl \)

is \( \text{grank}_{\mathbb{C}}(m, n, l) \) (Terracini Lemma 1915)

Avoid round-off error:
\( \mathbf{w}_r \in (\mathbb{Z}^m \times \mathbb{Z}^n \times \mathbb{Z}^l)^r \) find \( \text{rank} J(f_{r})(\mathbf{w}_r) \) exact arithmetic
Known cases of rank conjecture

\[ \text{grank}(3, 2p, 2p) = \left\lceil \frac{12p^2}{4p+1} \right\rceil \quad \text{and} \quad \text{grank}(3, 2p - 1, 2p - 1) = \left\lceil \frac{3(2p-1)^2}{4p-1} \right\rceil + 1 \]

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Easy to compute \( \text{grank}_C(m, n, l) \):

Pick at random \( w_r := (x_1, y_1, z_1, \ldots, x_r, y_r, z_r) \in (\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^l)^r \)

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Avoid round-off error:
\( w_r \in (\mathbb{Z}^m \times \mathbb{Z}^n \times \mathbb{Z}^l)^r \) find \( \text{rank } J(f_r)(w_r) \) exact arithmetic

I checked the conjecture up to \( m, n, l \leq 14 \)
Generic rank III - the real case

For $\mathbf{m} \leq \mathbf{l}$

\[
\text{rank}(\mathbf{m}, \mathbf{n}, \mathbf{l}) = \text{grank}(\mathbf{m}, \mathbf{n}, \mathbf{l}) = \mathbf{m}\mathbf{n}.
\]

For $2 \leq \mathbf{m} \leq \mathbf{n} \leq \mathbf{l} < \mathbf{m}\mathbf{n} - 1$, there exist $V_1, \ldots, V_c(\mathbf{m}, \mathbf{n}, \mathbf{l}) \subset \mathbb{R}^{\mathbf{m} \times \mathbf{n} \times \mathbf{l}}$ pairwise distinct open connected semi-algebraic sets s.t.

\[
\text{Closure}\left(\bigcup_{i=1}^{c}(V_i(\mathbf{m}, \mathbf{n}, \mathbf{l}))\right) = \mathbb{R}^{\mathbf{m} \times \mathbf{n} \times \mathbf{l}} \quad \text{rank}T = \text{grank}(\mathbf{m}, \mathbf{n}, \mathbf{l}) \quad \text{for each } T \in V_1
\]

\[
\text{rank}T = \rho_i \quad \text{for each } T \in V_i
\]

\[
\{\rho_1, \ldots, \rho_c(\mathbf{m}, \mathbf{n}, \mathbf{l})\} = \{\text{grank}(\mathbf{m}, \mathbf{n}, \mathbf{l}), \ldots, \text{mtrank}(\mathbf{m}, \mathbf{n}, \mathbf{l})\}
\]

\[
\text{mtrank}(\mathbf{2}, \mathbf{n}, \mathbf{l}) = \text{grank}(\mathbf{2}, \mathbf{n}, \mathbf{l}) = \min(\mathbf{l}, \mathbf{2n})
\]

if $2 \leq \mathbf{n} < \mathbf{l}$ - one typical rank

\[
\text{mtrank}(\mathbf{2}, \mathbf{n}, \mathbf{n}) = \text{grank}(\mathbf{2}, \mathbf{n}, \mathbf{n}) + 1 = \mathbf{n} + 1 \quad \text{if } 2 \leq \mathbf{n}
\]

- two typical ranks

For $\mathbf{l} = (\mathbf{m} - 1)(\mathbf{n} - 1) + 1$ \exists $\mathbf{m}, \mathbf{n}$:

\[
c(\mathbf{m}, \mathbf{n}, \mathbf{l}) > 1, \text{ mtrank}(\mathbf{m}, \mathbf{n}, \mathbf{l}) \geq \text{grank}(\mathbf{m}, \mathbf{n}, \mathbf{l}) + 1
\]

Examples [3]

$\mathbf{m} = \mathbf{n} \geq 2, \mathbf{l} = (\mathbf{m} - 1)(\mathbf{n} - 1) + 1$.

$\mathbf{m} = \mathbf{n} = 4, \mathbf{l} = 11, 12$.
Generic rank III - the real case

For $mn \leq l$ $\text{mtrank}(m, n, l) = \text{grank}(m, n, l) = mn$. 

Examples [3]

$m = n \geq 2, l = (m - 1)(n - 1) + 1$.

$m = n = 4, l = 11, 12$.
Generic rank III - the real case

For \( mn \leq l \) \( \text{mrank}(m, n, l) = \text{grank}(m, n, l) = mn \).

For \( 2 \leq m \leq n \leq l < mn - 1 \), there exist \( V_1, \ldots, V_{c(m,n,l)} \subset \mathbb{R}^{m \times n \times l} \) pairwise distinct open connected semi-algebraic sets s.t.

\[ \text{Closure} \left( \bigcup_{i=1}^{c(m,n,l)} V_i \right) = \mathbb{R}^{m \times n \times l} \text{ for each } T \in V_1 \]

\[ \text{rank} T = \rho_i \text{ for each } T \in V_i \]

\[ \{ \rho_1, \ldots, \rho_{c(m,n,l)} \} = \{ \text{grank}(m, n, l), \ldots, \text{mrank}(m, n, l) \} \]
Generic rank III - the real case

For $mn \leq l$ \( \text{mtrank}(m, n, l) = \text{grank}(m, n, l) = mn \).

For $2 \leq m \leq n \leq l < mn - 1$, there exist $V_1, \ldots, V_{c(m,n,l)} \subset \mathbb{R}^{m \times n \times l}$ pairwise distinct open connected semi-algebraic sets s.t.

\[
\text{Closure}(\bigcup_{i=1}^{c(m,n,l)}) = \mathbb{R}^{m \times n \times l}
\]
Generic rank III - the real case

For $mn \leq l$ $m\text{trank}(m, n, l) = \text{grank}(m, n, l) = mn$.

For $2 \leq m \leq n \leq l < mn - 1$, there exist $V_1, \ldots, V_{c(m,n,l)} \subset \mathbb{R}^{m \times n \times l}$ pairwise distinct open connected semi-algebraic sets s.t.

$$\text{Closure}\left(\bigcup_{i=1}^{c(m,n,l)}\right) = \mathbb{R}^{m \times n \times l}$$

rank $\mathcal{T} = \text{grank}(m, n, l)$ for each $\mathcal{T} \in V_1$
Generic rank III - the real case

For $mn \leq l$ $\text{mtrank}(m, n, l) = \text{grank}(m, n, l) = mn$.

For $2 \leq m \leq n \leq l < mn - 1$, there exist $V_1, \ldots, V_{c(m,n,l)} \subset \mathbb{R}^{m \times n \times l}$ pairwise distinct open connected semi-algebraic sets s.t.

$$\text{Closure}(\bigcup_{i=1}^{c(m,n,l)}) = \mathbb{R}^{m \times n \times l}$$

$\text{rank } \mathcal{T} = \text{grank}(m, n, l)$ for each $\mathcal{T} \in V_1$

$\text{rank } \mathcal{T} = \rho_j$ for each $\mathcal{T} \in V_i$
Generic rank III - the real case

For \( mn \leq l \) \( \text{mtrank}(m, n, l) = \text{grank}(m, n, l) = mn \).

For \( 2 \leq m \leq n \leq l < mn - 1 \), there exist \( V_1, \ldots, V_{c(m,n,l)} \subset \mathbb{R}^{m \times n \times l} \) pairwise distinct open connected semi-algebraic sets s.t.

\[
\text{Closure}(\bigcup_{i=1}^{c(m,n,l)}) = \mathbb{R}^{m \times n \times l}
\]

\( \text{rank } \mathcal{T} = \text{grank}(m, n, l) \) for each \( \mathcal{T} \in V_1 \)

\( \text{rank } \mathcal{T} = \rho_j \) for each \( \mathcal{T} \in V_i \)

\( \{\rho_1, \ldots, \rho_{c(m,n,l)}\} = \{\text{grank}(m, n, l), \ldots, \text{mtrank}(m, n, l)\} \)
Generic rank III - the real case

For $mn \leq l$ $m\text{trank}(m, n, l) = \text{grank}(m, n, l) = mn$.

For $2 \leq m \leq n \leq l < mn - 1$, there exist $V_1, \ldots, V_{c(m,n,l)} \subset \mathbb{R}^{m \times n \times l}$ pairwise distinct open connected semi-algebraic sets s.t.

$\text{Closure}\left(\bigcup_{i=1}^{c(m,n,l)}\right) = \mathbb{R}^{m \times n \times l}$

$\text{rank } T = \text{grank}(m, n, l)$ for each $T \in V_1$

$\text{rank } T = \rho_i$ for each $T \in V_i$

$\{\rho_1, \ldots, \rho_{c(m,n,l)}\} = \{\text{grank}(m, n, l), \ldots, \text{mtrank}(m, n, l)\}$

$m\text{trank}(2, n, l) = \text{grank}(2, n, l) = \min(l, 2n)$ if $2 \leq n < l$ - one typical rank

$m\text{trank}(2, n, n) = \text{grank}(2, n, n) + 1 = n + 1$ if $2 \leq n$ - two typical ranks
Generic rank III - the real case

For \( mn \leq l \) \( \text{mrank}(m, n, l) = \text{grank}(m, n, l) = mn \).

For \( 2 \leq m \leq n \leq l < mn - 1 \), there exist \( V_1, \ldots, V_{c(m,n,l)} \subset \mathbb{R}^{m \times n \times l} \) pairwise distinct open connected semi-algebraic sets s.t.

\[
\text{Closure}\left( \bigcup_{i=1}^{c(m,n,l)} \right) = \mathbb{R}^{m \times n \times l},
\]

\( \text{rank } \mathcal{T} = \text{grank}(m, n, l) \) for each \( \mathcal{T} \in V_1 \)

\( \text{rank } \mathcal{T} = \rho_i \) for each \( \mathcal{T} \in V_i \)

\( \{\rho_1, \ldots, \rho_{c(m,n,l)}\} = \{\text{grank}(m, n, l), \ldots, \text{mrank}(m, n, l)\} \)

\( \text{mrank}(2, n, l) = \text{grank}(2, n, l) = \min(l, 2n) \) if \( 2 \leq n < l \) - one typical rank

\( \text{mrank}(2, n, n) = \text{grank}(2, n, n) + 1 = n + 1 \) if \( 2 \leq n \) - two typical ranks

For \( l = (m - 1)(n - 1) + 1 \) \( \exists m, n: \)

\( c(m, n, l) > 1, \text{mrank}(m, n, l) \geq \text{grank}(m, n, l) + 1 \)
Generic rank III - the real case

For \( mn \leq l \) \( \text{mtrank}(m, n, l) = \text{grank}(m, n, l) = mn \).

For \( 2 \leq m \leq n \leq l < mn - 1 \), there exist \( V_1, \ldots, V_{c(m,n,l)} \subset \mathbb{R}^{m \times n \times l} \) pairwise distinct open connected semi-algebraic sets s.t.

\[
\text{Closure}(\bigcup_{i=1}^{c(m,n,l)}) = \mathbb{R}^{m \times n \times l}
\]
rank \( T = \text{grank}(m, n, l) \) for each \( T \in V_1 \)
rank \( T = \rho_i \) for each \( T \in V_i \)
\( \{\rho_1, \ldots, \rho_{c(m,n,l)}\} = \{\text{grank}(m, n, l), \ldots, \text{mtrank}(m, n, l)\} \)

\( \text{mtrank}(2, n, l) = \text{grank}(2, n, l) = \min(l, 2n) \) if \( 2 \leq n < l \) - one typical rank
\( \text{mtrank}(2, n, n) = \text{grank}(2, n, n) + 1 = n + 1 \) if \( 2 \leq n \) - two typical ranks

For \( l = (m - 1)(n - 1) + 1 \) \( \exists m, n: \)
\( c(m, n, l) > 1, \text{mtrank}(m, n, l) \geq \text{grank}(m, n, l) + 1 \)

Examples [3]
Rank one approximations

\[
\langle A, B \rangle = \sum_{i,j,k} a_{i,j,k} b_{i,j,k}
\]

\[
\|T\| = \sqrt{\langle T, T \rangle}
\]

\[
\langle x \otimes y \otimes z, u \otimes v \otimes w \rangle = (u^\top x)(v^\top y)(w^\top z)
\]

X subspace of \( \mathbb{R}^{m \times n \times l} \), \( X_1, \ldots, X_d \) an orthonormal basis of \( X \)

\[
P_X(T) = \sum_{i=1}^d \langle T, X_i \rangle X_i
\]

\[
\|P_X(T)\|_2^2 = \sum_{i=1}^d \langle T, X_i \rangle_2^2
\]

\[
\|T\|_2^2 = \|P_X(T)\|_2^2 + \|T - P_X(T)\|_2^2
\]

Best rank one approximation of \( T \):

\[
\min_{x,y,z} \|T - x \otimes y \otimes z\| = \min_{\|x\| = \|y\| = \|z\| = 1} \|T - a x \otimes y \otimes z\|
\]

Equivalent: max \( \|x\| = \|y\| = \|z\| = 1 \sum_{i,j,k} t_{i,j,k} x_i y_j z_k = \lambda x \)

\( T \times y \otimes z := \sum_{j,k=1} t_{i,j,k} y_j z_k = \lambda x \)

\( T \times x \otimes z = \lambda y \), \( T \times x \otimes y = \lambda z \)

\( \lambda \) singular value, \( x, y, z \) singular vectors

How many distinct singular values are for a generic tensor?
Rank one approximations

\( \mathbb{R}^{m \times n \times l} \) IPS:

\[ \langle A, B \rangle = \sum_{i=j=k} a_{i,j,k} b_{i,j,k}, \quad \|T\| = \sqrt{\langle T, T \rangle} \]
\( \mathbb{R}^{m \times n \times l} \) \( \text{IPS} \):
\[
\langle A, B \rangle = \sum_{i=j=k}^{m,n,l} a_{i,j,k} b_{i,j,k}, \quad \|T\| = \sqrt{\langle T, T \rangle}
\]
\[
\langle x \otimes y \otimes z, u \otimes v \otimes w \rangle = (u^T x)(v^T y)(w^T z)
\]
Rank one approximations

\( \mathbb{R}^{m \times n \times l} \) IPS: 
\[
\langle A, B \rangle = \sum_{i=j=k}^{m,n,l} a_{i,j,k} b_{i,j,k}, \quad \| T \| = \sqrt{\langle T, T \rangle}
\]

\[
\langle x \otimes y \otimes z, u \otimes v \otimes w \rangle = (u^T x)(v^T y)(w^T z)
\]

**X** subspace of \( \mathbb{R}^{m \times n \times l} \), \( X_1, \ldots, X_d \) an orthonormal basis of **X**
Rank one approximations

\[ \mathbb{R}^{m \times n \times l} \text{ IPS: } \langle A,B \rangle = \sum_{i=j=k}^{m,n,l} a_{i,j,k} b_{i,j,k}, \quad \|T\| = \sqrt{\langle T, T \rangle} \]

\[ \langle x \otimes y \otimes z, u \otimes v \otimes w \rangle = (u^T x)(v^T y)(w^T z) \]

**X** subspace of \( \mathbb{R}^{m \times n \times l} \), \( \mathcal{X}_1, \ldots, \mathcal{X}_d \) an orthonormal basis of **X**

\[ P_X(T) = \sum_{i=1}^{d} \langle T, \mathcal{X}_i \rangle \mathcal{X}_i, \quad \|P_X(T)\|^2 = \sum_{i=1}^{d} \langle T, \mathcal{X}_i \rangle^2 \]
Rank one approximations

\[ \mathbb{R}^{m \times n \times l} \text{ IPS: } \langle A, B \rangle = \sum_{i=j=k}^{m,n,l} a_{i,j,k} b_{i,j,k}, \quad \|T\| = \sqrt{\langle T, T \rangle} \]

\[ \langle x \otimes y \otimes z, u \otimes v \otimes w \rangle = (u^T x)(v^T y)(w^T z) \]

**X** subspace of \( \mathbb{R}^{m \times n \times l} \), \( \mathcal{X}_1, \ldots, \mathcal{X}_d \) an orthonormal basis of **X**

\[ P_X(T) = \sum_{i=1}^{d} \langle T, \mathcal{X}_i \rangle \mathcal{X}_i, \quad \|P_X(T)\|^2 = \sum_{i=1}^{d} \langle T, \mathcal{X}_i \rangle^2 \]

\[ \|T\|^2 = \|P_X(T)\|^2 + \|T - P_X(T)\|^2 \]
Rank one approximations

\[ \mathbb{R}^{m \times n \times l} \text{ IPS: } \langle A, B \rangle = \sum_{i=j=k}^{m,n,l} a_{i,j,k} b_{i,j,k}, \quad \| T \| = \sqrt{\langle T, T \rangle} \]

\[ \langle x \otimes y \otimes z, u \otimes v \otimes w \rangle = (u^\top x)(v^\top y)(w^\top z) \]

**X** subspace of \( \mathbb{R}^{m \times n \times l} \), \( X_1, \ldots, X_d \) an orthonormal basis of **X**

\[ P_{X}(T) = \sum_{i=1}^{d} \langle T, X_i \rangle X_i, \quad \| P_{X}(T) \|^2 = \sum_{i=1}^{d} \langle T, X_i \rangle^2 \]

\[ \| T \|^2 = \| P_{X}(T) \|^2 + \| T - P_{X}(T) \|^2 \]

Best rank one approximation of \( T \):

\[ \min_{x,y,z} \| T - x \otimes y \otimes z \| \]

Equivalent:\n
\[ \max_{\| x \|=\| y \|=\| z \|=1} \| x \otimes y \otimes z \| \]
Rank one approximations

\[ \mathbb{R}^{m \times n \times l} \text{ IPS: } \langle A, B \rangle = \sum_{i=j=k}^{m,n,l} a_{i,j,k} b_{i,j,k}, \quad \| T \| = \sqrt{\langle T, T \rangle} \]

\[ \langle x \otimes y \otimes z, u \otimes v \otimes w \rangle = (u^T x)(v^T y)(w^T z) \]

\[ \text{X subspace of } \mathbb{R}^{m \times n \times l}, \quad \mathcal{X}_1, \ldots, \mathcal{X}_d \text{ an orthonormal basis of X} \]

\[ P_X(T) = \sum_{i=1}^{d} \langle T, \mathcal{X}_i \rangle \mathcal{X}_i, \quad \| P_X(T) \|^2 = \sum_{i=1}^{d} \langle T, \mathcal{X}_i \rangle^2 \]

\[ \| T \|^2 = \| P_X(T) \|^2 + \| T - P_X(T) \|^2 \]

Best rank one approximation of \( T \):

\[ \min_{x,y,z} \| T - x \otimes y \otimes z \| = \min_{\| x \|=\| y \|=\| z \|=1} \| T - a x \otimes y \otimes z \| \]
Rank one approximations

\[ \mathbb{R}^{m \times n \times l} \text{ IPS: } \langle A, B \rangle = \sum_{i=j=k}^{m,n,l} a_{i,j,k} b_{i,j,k}, \quad \| T \| = \sqrt{\langle T, T \rangle} \]

\[ \langle x \otimes y \otimes z, u \otimes v \otimes w \rangle = (u^T x)(v^T y)(w^T z) \]

**X** subspace of \( \mathbb{R}^{m \times n \times l} \), \( x_1, \ldots, x_d \) an orthonormal basis of **X**

\[ P_X(T) = \sum_{i=1}^{d} \langle T, x_i \rangle x_i, \quad \| P_X(T) \|^2 = \sum_{i=1}^{d} \langle T, x_i \rangle^2 \]

\[ \| T \|^2 = \| P_X(T) \|^2 + \| T - P_X(T) \|^2 \]

Best rank one approximation of \( T \):

\[ \min_{x,y,z} \| T - x \otimes y \otimes z \| = \min_{\| x \|=\| y \|=\| z \|=1} \| T - a \cdot x \otimes y \otimes z \| \]

Equivalent: \( \max_{\| x \|=\| y \|=\| z \|=1} \sum_{i=j=k}^{m,n,l} t_{i,j,k} x_i y_j z_k \)
Rank one approximations

\[ \mathbb{R}^{m \times n \times l} \text{ IPS: } \langle A, B \rangle = \sum_{i=j=k}^{m,n,l} a_{i,j,k} b_{i,j,k}, \quad \| T \| = \sqrt{\langle T, T \rangle} \]

\[ \langle x \otimes y \otimes z, u \otimes v \otimes w \rangle = (u^T x) (v^T y) (w^T z) \]

**X** subspace of \( \mathbb{R}^{m \times n \times l} \), \( \mathcal{X}_1, \ldots, \mathcal{X}_d \) an orthonormal basis of **X**

\[ P_X(T) = \sum_{i=1}^{d} \langle T, \mathcal{X}_i \rangle \mathcal{X}_i, \quad \| P_X(T) \|^2 = \sum_{i=1}^{d} \langle T, \mathcal{X}_i \rangle^2 \]

\[ \| T \|^2 = \| P_X(T) \|^2 + \| T - P_X(T) \|^2 \]

Best rank one approximation of \( T \):

\[ \min_{x,y,z} \| T - x \otimes y \otimes z \| = \min_{\| x \| = \| y \| = \| z \| = 1, a} \| T - a x \otimes y \otimes z \| \]

Equivalent: \[ \max_{\| x \| = \| y \| = \| z \| = 1} \sum_{i=j=k}^{m,n,l} t_{i,j,k} x_i y_j z_k \]

Lagrange multipliers: \[ T \times y \otimes z := \sum_{j=k=1} t_{i,j,k} y_j z_k = \lambda x \]

\[ T \times x \otimes z = \lambda y, \quad T \times x \otimes y = \lambda z \]
Rank one approximations

\[ \mathbb{R}^{m \times n \times l} \text{ IPS: } \langle A, B \rangle = \sum_{i=j=k}^{m,n,l} a_{i,j,k} b_{i,j,k}, \quad \|T\| = \sqrt{\langle T, T \rangle} \]

\[ \langle x \otimes y \otimes z, u \otimes v \otimes w \rangle = (u^\top x)(v^\top y)(w^\top z) \]

**X** subspace of \( \mathbb{R}^{m \times n \times l} \), \( X_1, \ldots, X_d \) an orthonormal basis of **X**

\[ P_X(T) = \sum_{i=1}^{d} \langle T, X_i \rangle X_i, \quad \|P_X(T)\|^2 = \sum_{i=1}^{d} \langle T, X_i \rangle^2 \]

\[ \|T\|^2 = \|P_X(T)\|^2 + \|T - P_X(T)\|^2 \]

Best rank one approximation of **T**:

\[ \min_{x,y,z} \|T - x \otimes y \otimes z\| = \min_{\|x\| = \|y\| = \|z\| = 1} \|T - a x \otimes y \otimes z\| \]

Equivalent: \( \max_{\|x\| = \|y\| = \|z\| = 1} \sum_{i=j=k}^{m,n,l} t_{i,j,k} x_i y_j z_j = \lambda x \)

Lagrange multipliers: \( T \times y \otimes z := \sum_{j=k=1} t_{i,j,k} y_j z_k = \lambda y \)
\( T \times x \otimes z = \lambda y, \quad T \times x \otimes y = \lambda z \)

\( \lambda \) singular value, **x, y, z** singular vectors
Rank one approximations

\[ \mathbb{R}^{m \times n \times l} \text{ IPS: } \langle A, B \rangle = \sum_{i=j=k}^{m,n,l} a_{i,j,k} b_{i,j,k}, \quad \|T\| = \sqrt{\langle T, T \rangle} \]

\[ \langle x \otimes y \otimes z, u \otimes v \otimes w \rangle = (u^\top x)(v^\top y)(w^\top z) \]

**X** subspace of \( \mathbb{R}^{m \times n \times l} \), \( \mathcal{X}_1, \ldots, \mathcal{X}_d \) an orthonormal basis of **X**

\[ P_X(T) = \sum_{i=1}^{d} \langle T, \mathcal{X}_i \rangle \mathcal{X}_i, \quad \|P_X(T)\|^2 = \sum_{i=1}^{d} \langle T, \mathcal{X}_i \rangle^2 \]

\[ \|T\|^2 = \|P_X(T)\|^2 + \|T - P_X(T)\|^2 \]

Best rank one approximation of \( T \):

\[ \min_{x,y,z} \|T - x \otimes y \otimes z\| = \min_{\|x\| = \|y\| = \|z\| = 1} a \|T - a x \otimes y \otimes z\| \]

Equivalent:

\[ \max_{\|x\| = \|y\| = \|z\| = 1} \sum_{i=j=k}^{m,n,l} t_{i,j,k} x_{i} y_{j} z_{k} \]

Lagrange multipliers:

\[ T \times y \otimes z := \sum_{j=k=1} t_{i,j,k} y_{j} z_{k} = \lambda x \]

\[ T \times x \otimes z = \lambda y, \quad T \times x \otimes y = \lambda z \]

\( \lambda \) singular value, \( x, y, z \) singular vectors

How many distinct singular values are for a generic tensor?
$\ell_p$ maximal problem and Perron-Frobenius
\( \sum_{i=1}^{n} |x_i|^p \) \right)^{\frac{1}{p}}
$\ell_p$ maximal problem and Perron-Frobenius

$\| (x_1, \ldots, x_n) \|^p := (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}$

Problem: $\max \|x\|_p = \|y\|_p = \|z\|_p = 1 \sum_{i=j=k}^{m,n,l} t_{i,j,k} x_i y_j z_k$
\( \ell_p \) maximal problem and Perron-Frobenius

\[
\| (x_1, \ldots, x_n) \|^p_p := (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}
\]

**Problem:** \( \max \| x \|=\| y \|=\| z \|=1 \sum_{i=j=k}^{m,n,l} t_{i,j,k} x_i y_j z_k \)

**Lagrange multipliers:** \( T \times y \otimes z := \sum_{j=k=1} t_{i,j,k} y_j z_k = \lambda x^{p-1} \)
\( \ell_p \) maximal problem and Perron-Frobenius

\[ \|(x_1, \ldots, x_n)\|_p := \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}} \]

Problem: \( \max \|x\|_p = \|y\|_p = \|z\|_p = 1 \sum_{i=j=k} t_{i,j,k} x_i y_j z_k \)

Lagrange multipliers: \( T \times y \otimes z := \sum_{j=k=1} t_{i,j,k} y_j z_k = \lambda x^{p-1} \)
\( T \times x \otimes z = \lambda y^{p-1}, \ T \times x \otimes y = \lambda z^{p-1} \) \( (p = \frac{2t}{2s-1}, \ t, s \in \mathbb{N}) \)
\( \ell_p \) maximal problem and Perron-Frobenius

\[ \|(x_1, \ldots, x_n)^\top\|_p := \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}} \]

**Problem:** \( \max \|x\|_p = \|y\|_p = \|z\|_p = 1 \sum_{i=j=k} t_{i,j,k} x_i y_j z_k \)

**Lagrange multipliers:** \( T \times y \otimes z := \sum_{j=k=1} t_{i,j,k} y_j z_k = \lambda x^{p-1} \)
\( T \times x \otimes z = \lambda y^{p-1}, \ T \times x \otimes y = \lambda z^{p-1} \) (\( p = \frac{2t}{2s-1}, t, s \in \mathbb{N} \))

\( p = 3 \) is most natural in view of homogeneity
\( \| (x_1, \ldots, x_n)^\top \|_p := (\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}} \)

**Problem:** \( \max_{\|x\|_p=\|y\|_p=\|z\|_p=1} \sum_{i=j=k}^{m,n,l} t_{i,j,k} x_i y_j z_k \)

**Lagrange multipliers:** \( \mathcal{T} \times y \otimes z := \sum_{j=k=1}^{t,i,j,k} t_{i,j,k} y_j z_k = \lambda x^{p-1} \)
\( \mathcal{T} \times x \otimes z = \lambda y^{p-1}, \mathcal{T} \times x \otimes y = \lambda z^{p-1} (p = \frac{2t}{2s-1}, t, s \in \mathbb{N}) \)

\( p = 3 \) is most natural in view of homogeneity

Assume that \( \mathcal{T} \geq 0 \). Then \( x, y, z \geq 0 \)

For which values of \( p \) we have an analog of Perron-Frobenius theorem?

Yes, for \( p \geq 3 \), No, for \( p < 3 \),

Friedland-Gauber-Han [1]
(\(R_1, R_2, R_3\))-rank approximation of 3-tensors

Fundamental problem in applications: Approximate well and fast

\(T \in \mathbb{R}^{m_1 \times m_2 \times m_3}\) by rank \((R_1, R_2, R_3)\)-3-tensor.

Best \((R_1, R_2, R_3)\)-approximation problem: Find \(U_i \subset \mathbb{F}^{m_i}\) of dimension \(R_i\) for \(i = 1, 2, 3\) with maximal:

\[\|P \otimes \otimes \otimes (T)\|\]

Relaxation method: Optimize on \(U_1, U_2, U_3\) by fixing all variables except one at a time.

This amounts to SVD (Singular Value Decomposition) of matrices:

Fix \(U_2, U_3\). Then \(V = U_1 \otimes (U_2 \otimes U_3) \subset \mathbb{R}^{m_1 \times (m_2 \cdot m_3)}\)

\[\max_{U_1} \|P V (T)\|\]

is an approximation in 2-tensors=matrices.

Use Newton method on Grassmannians - Eldén-Savas 2009 [1]

Shmuel Friedland Univ. Illinois at Chicago () Tensors
Fundamental problem in applications:
Approximate well and fast \( T \in \mathbb{R}^{m_1 \times m_2 \times m_3} \) by rank \((R_1, R_2, R_3)\)
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$(R_1, R_2, R_3)$-rank approximation of 3-tensors

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Find $U_i \subset \mathbb{F}^{m_i}$ of dimension $R_i$ for $i = 1, 2, 3$ with maximal $\|P_{U_1 \otimes U_2 \otimes U_3}(\mathcal{T})\|$. 
(\(R_1, R_2, R_3\))-rank approximation of 3-tensors

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Use Newton method on Grassmannians - Eldén-Savas 2009 [1]
Fast low rank approximation I:
Fast low rank approximations II:

Approximate $A \in \mathbb{R}^{m \times n}$ by $CUR$ where $C \in \mathbb{R}^{m \times p}$, $R \in \mathbb{R}^{q \times n}$ for some submatrices of $A$.

$$\min_{U \in \mathbb{C}^{p \times q}} \|A - CUR\|_F$$

Achieved for $U = C^\dagger AR^\dagger$ (corresponds to best $CUR$ approximation on the entries read).

Faster choice:

$$U = A[I, J]^\dagger$$

For given $A \in \mathbb{R}^{m \times n \times l}$, $F \in \mathbb{R}^{m \times p}$, $E \in \mathbb{R}^{n \times q}$, $G \in \mathbb{R}^{l \times r}$, where $\langle p \rangle \subset \langle n \rangle \times \langle l \rangle$, $\langle q \rangle \subset \langle m \rangle \times \langle l \rangle$, $\langle r \rangle \subset \langle m \rangle \times \langle l \rangle$.

$$\min_{U \in \mathbb{C}^{p \times q \times r}} \|A - U \times F \times E \times G\|_F$$

Achieved for $U = A \times E^\dagger \times F^\dagger \times G^\dagger$ (CUR approximation of $A$ obtained by choosing $E, F, G$ submatrices of unfolded $A$ in the mode 1, 2, 3).
Approximate $A \in \mathbb{R}^{m \times n}$ by CUR where $C \in \mathbb{R}^{m \times p}$, $R \in \mathbb{R}^{q \times n}$ for some submatrices of $A$. 
Approximate \( A \in \mathbb{R}^{m \times n} \) by \( CUR \) where \( C \in \mathbb{R}^{m \times p}, \ R \in \mathbb{R}^{q \times n} \) for some submatrices of \( A \).

\[
\min_{U \in \mathbb{C}^{p \times q}} \|A - CUR\|_F \text{ achieved for } U = C^\dagger AR^\dagger
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Approximate $A \in \mathbb{R}^{m \times n}$ by $CUR$ where $C \in \mathbb{R}^{m \times p}$, $R \in \mathbb{R}^{q \times n}$ for some submatrices of $A$.

$$\min_{U \in C_p \times q} \|A - CUR\|_F$$ achieved for $U = C^\dagger AR^\dagger$

Faster choice: $U = A[I, J]^\dagger$
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\min_{U \in \mathbb{C}^{p \times q}} \| A - CUR \|_F \quad \text{achieved for} \quad U = C^\dagger A R^\dagger
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\[
\min_{U \in \mathbb{C}^{p \times q \times r}} \| A - U \times F \times E \times G \|_F \text{ achieved for } U = A \times E^\dagger \times F^\dagger \times G^\dagger
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$CUR$ approximation of $A$ obtained by choosing $E, F, G$ submatrices of unfolded $A$ in the mode 1, 2, 3.
List of applications

- Face recognition
- Video tracking
- Factor analysis
List of applications

Face recognition
List of applications

Face recognition

Video tracking
List of applications

Face recognition

Video tracking

Factor analysis
Scaling of nonnegative tensors to tensors with given row, column and depth sums

\[ 0 \leq T = [t_{i,j,k}] \in \mathbb{R}^{m \times n \times l} \] has given row, column and depth sums:

\[ r = (r_1, \ldots, r_m)^\top, \quad c = (c_1, \ldots, c_n)^\top, \quad d = (d_1, \ldots, d_l)^\top > 0: \]
Scaling of nonnegative tensors to tensors with given row, column and depth sums

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  - \( \sum_{j,k} t_{i,j,k} = r_i > 0 \), \( \sum_{i,k} t_{i,j,k} = c_j > 0 \), \( \sum_{i,j} t_{i,j,k} = d_k > 0 \)
  - \( \sum_{i=1}^m r_i = \sum_{j=1}^n c_j = \sum_{k=1}^l d_k \)

Find necessary and sufficient conditions for scaling:

\[ \mathcal{T}' = \left[ t_{i,j,k} \right]_{x_i + y_j + z_k} \]

Solution: Convert to the minimal problem:

\[ \min \mathbf{r}^\top \mathbf{x} = \mathbf{c}^\top \mathbf{y} = \mathbf{d}^\top \mathbf{z} = 0 \]

Any critical point of \( f_{\mathcal{T}}(\mathbf{x}, \mathbf{y}, \mathbf{z}) \) gives rise to a solution of the scaling problem (Lagrange multipliers): \( f_{\mathcal{T}} \) is convex

\( f_{\mathcal{T}} \) is strictly convex implies \( \mathcal{T} \) is not decomposable: \( \mathcal{T} \neq \mathcal{T}_1 \oplus \mathcal{T}_2 \)
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\[
\begin{align*}
\mathbf{r} &= (r_1, \ldots, r_m)^\top, \quad \mathbf{c} = (c_1, \ldots, c_n)^\top, \quad \mathbf{d} = (d_1, \ldots, d_l)^\top > 0: \\
\sum_{j,k} t_{i,j,k} &= r_i > 0, \quad \sum_{i,k} t_{i,j,k} = c_j > 0, \quad \sum_{i,j} t_{i,j,k} = d_k > 0 \\
\sum_{i=1}^m r_i &= \sum_{j=1}^n c_j = \sum_{k=1}^l d_k
\end{align*}
\]

Find nec. and suf. conditions for scaling:

\( T' = [t_{i,j,k} e^{x_i+y_j+z_k}], \quad x, y, z \) such that \( T' \) has given row, column and depth sum
Scaling of nonnegative tensors to tensors with given row, column and depth sums

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Solution: Convert to the minimal problem:

\[
\min_{r^\top x = c^\top y = d^\top z = 0} f_T(x, y, z), \quad f_T(x, y, z) = \sum_{i,j,k} t_{i,j,k}e^{x_i+y_j+z_k}
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Scaling of nonnegative tensors to tensors with given row, column and depth sums

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Any critical point of \( f_T \) on \( S := \{r^\top x = c^\top y = d^\top z = 0\} \) gives rise to a solution of the scaling problem (Lagrange multipliers)
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\[ \mathcal{T}' = [t_{i,j,k} e^{x_i + y_j + z_k}], x, y, z \] such that \( \mathcal{T}' \) has given row, column and depth sum

Solution: Convert to the minimal problem:

\[ \min_{\mathbf{r}^\top \mathbf{x} = \mathbf{c}^\top \mathbf{y} = \mathbf{d}^\top \mathbf{z} = 0} f_{\mathcal{T}}(\mathbf{x}, \mathbf{y}, \mathbf{z}), \quad f_{\mathcal{T}}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \sum_{i,j,k} t_{i,j,k} e^{x_i + y_j + z_k} \]

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Solution: Convert to the minimal problem:

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Scaling of nonnegative tensors II

If $f_T$ is strictly convex and is $\infty$ on $\partial S$, $f_T$ achieves its unique minimum.

Equivalent to: the inequalities $x_i + y_j + z_k \leq 0$ if $t_i, j, k > 0$ and equalities $r^\top x = c^\top y = d^\top z = 0$ imply $x = 0$ $m$, $y = 0$ $n$, $z = 0$ $l$.

Fact: For $r = 1$ $m$, $c = 1$ $n$, $d = 1$ $l$, Sinkhorn scaling algorithm works. Newton method works, since the scaling problem is equivalent finding the unique minimum of strict convex function $H$. Hence, Newton method has a quadratic convergence versus linear convergence of Sinkhorn algorithm.

True for matrices too.

Are variants of Menon and Brualdi theorems hold in the tensor case? Yes for Menon, unknown for Brualdi.
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Shmuel Friedland Univ. Illinois at Chicago ()
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Fact: For \( r = 1_m, c = 1_n, d = 1_l \) Sinkhorn scaling algorithm works.

Newton method works, since the scaling problem is equivalent finding the unique minimum of strict convex function

Hence Newton method has a quadratic convergence versus linear convergence of Sinkhorn algorithm

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Are variants of Menon and Brualdi theorems hold in the tensor case?
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Are variants of Menon and Brualdi theorems hold in the tensor case?
Yes for Menon, unknown for Brualdi
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If $W$ contains identity matrix then $W$ space of commuting matrices

Strassen's condition hold for any $3 \times 3 \times 3$ subtensor of $T$: 

$$
\det(U(\text{adj}W)V - V(\text{adj}W)U) = 0,
$$

$U, V, W \in \mathbb{C}^{3 \times 3 \times 3}$

equations of degree 9

Friedland [5] one needs equations of degree 16
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S. Friedland, On tensors of border rank l in $\mathbb{C}^{m \times n \times l}$, arXiv:1003.1968v1.


L.H. Lim, Singular values and eigenvalues of tensors: a variational approach, CAMSAP 05, 1 (2005), 129-132.
