Finite and infinite dimensional generalizations of Klyachko theorem

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Abstract. We describe the convex set of the eigenvalues of hermitian matrices which are majorized by sum of m hermitian matrices with prescribed eigenvalues. We extend our characterization to selfadjoint nonnegative (definite) compact operators on a separable Hilbert space. We give necessary and sufficient conditions on the eigenvalue sequence of a selfadjoint nonnegative compact operator of trace class to be a sum of m selfadjoint nonnegative compact operators of trace class with prescribed eigenvalue sequences.

§0. Introduction

The spectacular works of Klyachko [**Kly**] and Knutson-Tao [**K-T**] characterized completely the convex set of the eigenvalues of a sum of two hermitian matrices with prescribed eigenvalues. The work of Klyachko uses classical Schubert calculus, modern algebraic geometry: stable bundles over $\mathbf{P}^2$ and Donaldson theory, and representation theory. The work of Knutson and Tao combines combinatorial and geometrical tools using the honey comb model of Berenstein-Zelevensky [**B-Z**] to prove the saturation conjecture. The results of Klyachko and Knutson-Tao verified Horn conjecture from the sixties [**Hor**].

Our aim was to try to generalize the above results to selfadjoint compact operators in a separable Hilbert space. It turned out that to do that we needed to characterize the set of the eigenvalues of hermitian matrices, which are majorized by the sum of m hermitian matrices with prescribed eigenvalues. The above set is a polyhedron, which is characterized by the inequalities specified by Klyachko and an additional set of inequalities. This set of additional inequalites is induced by the exreme rays of a certain natural polyhedron associated with the original set of the eigenvalues of sum of two $n \times n$ hermitian matrices with prescribed eigenvalues. We do not know if this cone is related to the representation theory. For $n = 2$ one does not need this additional set of inequalities. We do believe that for $n \geq 3$ one needs additional inequalities and we give examples of such inequalities.

We then show that our results generalize naturally to selfadjoint nonnegative compact operators in a separable Hilbert space. These conditions is a set of countable inequalities which is the union of the inequalities for the $n \times n$ hermitian matrices for $n = 2, ...,$. It is an open question if in this setting we need additional set of inequalities as described above. Finally, we have a version of Klyachko theorem for selfadjoint nonnegative compact operators in the trace class. Our tools are basic results in linear programming and theory of selfadjoint operators.

§1. Statement of the results

Let

$$\mathbf{R}^n_{\geq} := \{x : \quad x = (x_1, ..., x_n) \in \mathbf{R}^n, \quad x_1 \geq x_2 \geq \cdots \geq x_n\}.$$

Set $< n >:= \{1, ..., n\}$ and denote by $|I|$ the cardinality of the set $I \subset < n >$. Let

$$x_I := \sum_{i \in I} x_i, \quad x \in \mathbf{R}^n, \quad I \subset < n >, \quad |I| \geq 1.$$

Let $\mathcal{H}_n, \mathcal{S}_n$ denote the set of $n \times n$ Hermitian and real symmetric matrices respectively. Assume that $A \in \mathcal{H}_n$. Denote by $\lambda(A) := (\lambda_1(A), ..., \lambda_n(A)) \in \mathbf{R}^n_{\geq}$ the eigenvalue vector corresponding to A. That is each $\lambda_i(A)$ is an eigenvalue of A, and the multiplicity of an eigenvalue t of A is equal to the number of coordinates of $\lambda(A)$ equal to t. We say that A is nonnegative definite and denote it by $A \geq 0$ iff $\lambda_n(A) \geq 0$. For $A, B \in \mathcal{H}_n$ we say that A is majorized by B and denote it by $A \leq B$ iff $B - A \geq 0$.

In [**Kly**] Klyachko stated the necessary and sufficient conditions on the $m+1$ sets of vectors

$$\lambda^j := (\lambda^j_1, ..., \lambda^j_n) \in \mathbf{R}^n_{\geq}, \quad j = 0, ..., m, \tag{1}$$

such that there exist $m+1$ Hermitian matrices $A_0, ..., A_m \in \mathcal{H}_n$ with the following properties: The vector λ^j is the eigenvalue vector of A_j for $j = 0, ..., m$ and

$$A_0 = \sum_{j=1}^m A_j. \tag{2}$$

Assume the nontrivial case $m, n > 1$. First, one has the trace condition

$$\lambda^0_{<n>} = \sum_{j=1}^m \lambda^j_{<n>}. \tag{3}$$

Second, there exists a finite collection of sets with the following properties:

$$\begin{aligned} &I_{0,k}, ..., I_{m,k} \subset < n >, \\ &|I_{0,k}| = |I_{1,k}| = \cdots |I_{m,k}| \leq n-1, \\ &k = 1, ..., N(n,m), \end{aligned} \tag{4}$$

such that

$$\lambda^0_{I_{0,k}} \leq \sum_{j=1}^m \lambda^j_{I_{j,k}}, \quad k = 1, ..., N(n,m). \tag{5}$$

The collections of sets (4) are characterized in terms of Schubert calculus. The special case $m = 2$ has a long history. Here the sets (4) were conjectured recursively by Horn [**Hor**]. This conjecture was recently proved by A. Knutson and T. Tao [**K-T**]. See Fulton [**Ful**] for the nice exposition of this subject. Fulton also notices that Klyachko theorem extends to the case where $A_0, ..., A_m \in \mathcal{S}_n$. See also J. Day, W. So and R. C. Thompson [**D-S-T**] for an earlier survey on the Horn conjecture.

To state our results it is convenient to introduce an equivalent statement of (5). First observe that the Wielandt inequalities [**Wie**] imply that for each $I \subset < n >, 0 < |I| < n$ there exists at least one $I_{0,k} = I$ for some $k \in < N(n,m) >$. (See §2.) Let

$$\begin{aligned} a_I &:= \min_{(I_{1,k}, ..., I_{m,k}), I_{0,k} = I} \sum_{j=1}^m \lambda^j_{I_{j,k}}, \quad I \subset < n >, \quad 0 < |I| < n, \\ a_{<n>} &:= \sum_{j=1}^m \lambda^j_{<n>}. \end{aligned} \tag{6}$$

Then (5) is equivalent to

$$\lambda^0_I \leq a_I, \quad I \subset < n >, \quad 0 < |I| < n. \tag{5'}$$

First we extend Klyachko theorem in finite dimension as follows:

Theorem 1. *For $m, n > 1$ the following conditions are equivalent:*
(a) The vectors (1) for $j = 0, ..., m$ satisfy the conditions (5′), the condition

$$\lambda^0_{<n>} \leq a_{<n>}, \tag{3'}$$

and the conditions

$$\sum_{i=1}^n w_i^l \lambda_i^0 \le -a_{<n>} + \sum_{I \subset <n>, 0<|I|<n} u_I^l a_I, \quad l = 1, ..., M(n), \tag{3''}$$

where w_i^l, u_I^l are given nonnegative numbers for $i = 1, ..., n, I \subset < n >, l = 1, ..., M(n)$.
(b) There exist a Hermitian matrix A_j with the eigenvalue vector $\lambda_j \in \mathbf{R}^n_{\ge}$ for $j = 0, ..., m$ such that

$$A_0 \le \sum_{j=1}^m A_j. \tag{2'}$$

Note that if $B \ge A$ and trace $B =$ trace A then $B = A$. Hence Klyachko theorem follows from Theorem 1. Furthermore, if λ^0 satisfies (3) and (5) then (3′′) hold. The inequalities (3′′) are determined by the extreme rays of a certain natural cone in $R^p, p = 2^n + 2n - 1$, which is described in the next section. For $n = 2$ the conditions (3′′) are not needed. We believe that (3′′) are needed for $n > 2$. Theorem 1 is closely related to the Completion Problem described in the beginning of §3.

The main purpose of this note to extend our Theorem 1 to the case where $A_0, ..., A_m$ are compact, selfadjoint nonnegative operators in a separable Hilbert space $\mathcal{H}$. Let $A, B : \mathcal{H} \to \mathcal{H}$ be linear, bounded, selfadjoint operators. We let $A \le B$ ($A < B$) iff $0 \le B - A$ ($0 < B - A$), i.e. $B - A$ is nonnegative (respectively positive). Let $A : \mathcal{H} \to \mathcal{H}$ be a bounded, selfadjoint, nonnegative, compact operator on a separable Hilbert space. Then $\mathcal{H}$ has an orthonormal basis $\{e_i\}_1^\infty$ such that

$$\begin{aligned} &Ae_i = \lambda_i e_i, \quad \lambda_i \ge 0, \quad i = 1, ..., \\ &\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n \cdots, \\ &\lim_{n\to\infty} \lambda_i = 0. \end{aligned} \tag{7}$$

We say that $\{\lambda_i\}_{i=1}^\infty$ is the eigenvalue sequence of A. (Note that a nonnegative A has a finite range iff $\lambda_i = 0$ for some $i \ge 1$.) A (as above) is said to be in the trace class [**Kat**, X.1.3] if $\sum_{i=1}^\infty \lambda_i < \infty$. Then trace $A := \sum_{i=1}^\infty \lambda_i$.

Theorem 2. *For $m > 1$ the following conditions are equivalent:*
(a) For $j = 0, ..., m$, let $\{\lambda_i^j\}_{i=1}^\infty$ be decreasing sequences of nonnegative numbers converging to zero, such that (5′), (3′) and (3′′) holds for $m + 1$ vectors $\lambda^j := (\lambda_1^j, ..., \lambda_n^j), j = 0, ..., m$ for any $n > 1$.
(b) There exist a linear, bounded, selfadjoint, nonnegative, compact operators $A_j : \mathcal{H} \to \mathcal{H}$ with the eigenvalue sequence $\{\lambda_i^j\}_{i=1}^\infty$ for $j = 0, ...m$, such that (2′) holds.

Klyachko theorem can be extended to the infinite dimensional case as follows:

Theorem 3. *For $m > 1$ the following conditions are equivalent:*
(a) For $j = 0, ..., m$, let $\{\lambda_i^j\}_{i=1}^\infty$ be decreasing sequences of nonnegative numbers converging to zero, such that (5′), (3′) and (3′′) holds for $m + 1$ vectors $\lambda^j := (\lambda_1^j, ..., \lambda_n^j), j = 0, ..., m$ for any $n > 1$. Furthermore

$$\sum_{i=1}^\infty \lambda_i^0 = \sum_{j=1}^m \sum_{i=1}^\infty \lambda_i^j < \infty \tag{8}$$

(b) There exist a linear, bounded, selfadjoint, nonnegative, compact operators in the trace class $A_j : \mathcal{H} \to \mathcal{H}$ with the eigenvalue sequence $\{\lambda_i^j\}_{i=1}^\infty$ for $j = 0, ...m$, such that (2) holds.

Remark 1. *All the matrices and operators in Theorems* 1, 2 *and* 3 *can be presented as finite or inifinite real symmetric matrices in corresponding orthornomal bases.*

It is an open problem whether the conditions $(3'')$ in part (a) of Theorems 2 and 3 can be omitted. It is easy to show that in part (a) of Theorems 2 and 3 it is enough to assume the validity of $(5'), (3')$ and $(3'')$ for any infinite increasing sequence $1 < n_1 < n_2 < \cdots$ instead for any $n > 1$.

We describe briefly the organization of the paper. In §2 we prove Theorem 1. In §3 we prove the implication $(b) \Rightarrow (a)$ in Theorem 2, and we deduce Theorem 3 from Theorem 2. The use of functional analysis in §3 was kept to the minimum. Our main tool is the convoy principle, e.g. [**Fri**] and the references therein. In §4 we prove the implication $(a) \Rightarrow (b)$ of Theorem 2. This needs the spectral decomposition theorem for a linear, bounded, selfadjoint operator and the notion of the weak convergence.

We remark that it is straightforward to rephrase our results to the nonpositive analogs of Theorems 2 and 3. However, difficulties arise when one tries to generalize Theorems 2 and 3 to indefinite, selfadjoint, compact operators. Similar problems arise in the joint paper of G. Porta and the author in [**F-P**].

§2. Finite dimensional case

The following result is well known, e.g. [**Gan**, §X.7], and it follows from the Weyl's inequalities

$$\lambda^0_{i+j-1} \le \lambda^1_i + \lambda^2_j \quad \text{for} \quad i+j-1 \le n,$$

which are a special case of (5):

Lemma 1. *The following are equivalent:*
(a) The vectors $\lambda^0, \lambda^1 \in \mathbf{R}^n_{\ge}$ satisfy $\lambda^0 \le \lambda^1$.
(b) There exist a Hermitian matrix $A_j \in \mathcal{H}_n$ with the eigenvalue vector λ^j for $j = 0, 1$ such that $A_0 \le A_1$.

The following remark will be useful in the sequel:

Remark 2. Let $\lambda^0 = (\lambda^0_1, ..., \lambda^0_n) \in \mathbf{R}^n_{\ge}$ be the eigenvalue vector of A_0. Then $A_0 = U\ diag(\lambda^0_1, ..., \lambda^0_n)\ U^*$ for some unitary matrix U. Assume that λ^1 satisfy the condition (a) of Lemma 1. Then λ^1 is the eigenvalue vector of $A_1 := U\ diag(\lambda^1_1, ..., \lambda^1_n)\ U^*$ and $A_0 \le A_1$.

Let $A, B \in \mathcal{H}_n$. Then Wielandt inequalities state [**Wie**]:

$$\lambda_I(A+B) \le \lambda_I(A) + \lambda_{<|I|>}(B)$$

for any nonempty subset I of $< n >$. In particular

$$\lambda_I(\sum_{j=1}^m A_j) \le \lambda_I(A_1) + \sum_{j=2}^m \lambda_{<|I|>}(A_j).$$

Hence, each strict subset I of $< n >$ is equal to some $I_{0,k}$ given in (4) as we claimed.

Let $y = (y_1, ..., y_n) \in \mathbf{R}^n$ satisfy the inequalities

$$\begin{aligned} -y_i + y_{i+1} \le 0, \quad i = 1, ..., n-1, \\ y_I \le a_I, \quad \emptyset \ne I \subset < n > . \end{aligned} \tag{9}$$

The above system of inequalities can be written as

$$Uy^T \le b^T,$$

where U is an $(2^n + n - 2) \times n$ matrix with entries in the set $\{0, 1, -1\}$ induced by the above inequalities in the above order. More precisely, $U^T = (U_1^T, U_2^T), b = (0, a)$, where U_1, U_2, b are $(n-1) \times n$, $(2^n - 1) \times n$ and $1 \times (2^n + n - 2)$ matrices respectively. Moreover,

$$a := (a_I)_{I \subset <n>, 0<|I|} \in \mathbf{R}^{2^n - 1}$$

is a row vector. Note that the first $n-1$ inequalities (9) are equivalent to the condition that $y \in \mathbf{R}^n_{\geq}$. Then next $2^n - 2$ conditions are the inequalities of the type (5′). The last inequality of (9) is of the type (3′). Klyachko's conditions for the vector $x = (x_1, ..., x_n) = \lambda^0$ are equivalent to:

$$\begin{aligned} Ux^T &\leq b^T, \\ -x_{<n>} &\leq -a_{<n>}. \end{aligned} \tag{9'}$$

Lemma 2. *Let $\lambda^1, ..., \lambda^m \in \mathbf{R}^n_{\geq}$ be given. Assume that $b = (0, a)$, where the coordinates of the vector $a \in \mathbf{R}^{2^n-1}$ are given by (6). Let $y \in \mathbf{R}^n_{\geq}$. Then there exist $A_0, A_1, ..., A_m \in \mathcal{H}_n$ with the corresponding eigenvalue vectors $y, \lambda^0, ..., \lambda^m$ satisfying (2′) iff the following system in x is solvable:*

$$\begin{aligned} Ux^T &\leq b^T, \\ -x_{<n>} &\leq -a_{<n>}, \\ -x_i &\leq -y_i, \quad i = 1, ..., n. \end{aligned} \tag{10}$$

Proof. Suppose first that (2′) holds. Let $\tilde{A}_0 := \sum_{j=1}^m A_j$ and assume that x is the eigenvalue vector of $\tilde{A}_0$. Then Klyachko's theorem yields the inequalities (9′). As $A_0 \leq \tilde{A}_0$ Lemma 1 yields the inequalities $x \geq y$. Hence (10) holds.

Assume now that x satisfies (10). Then x satisfies (9′). Klyachko's theorem yields that there exists $\tilde{A}_0, A_1, ..., A_m \in \mathcal{H}_n$ with the respective eigenvalue vectors $x, \lambda^1, ..., \lambda^m$ such that $\tilde{A}_0 = \sum_{j=1}^m A_j$. Note that the last n conditions of (10) state that $x \geq y$. A trivial variation of Remark 2 yields the existence of $A_0 \leq \tilde{A}_0$ so that y is the eigenvalue vector of A_0. $\diamond$.

System (10) of can be written in a matrix form as

$$\begin{aligned} &Vx^T \leq c^T, \\ &V^T = (U^T, -e^T, -I), \quad c = (b, -a_{<n>}, -y), \quad e = (1, ..., 1) \in \mathbf{R}^n. \end{aligned} \tag{10'}$$

A variant of Farkas lemma, [**Sch**, 7.3] yields that the solvability of the above system is equivalent to the implication

$$z \geq 0, \quad zV = 0 \quad \Rightarrow \quad zc^T \geq 0. \tag{11}$$

Here $z = (t, u, v, w)$ is a row vector which partitioned as V. Hence

$$t = (t_1, ..., t_{n-1}), \quad u = (u_I)_{I \subset <n>, 0 < |I|}, \quad v \in \mathbf{R}, \quad w = (w_1, ..., w_n).$$

Lemma 3. *Any solution $z = (t, u, v, w)$ to the system $zV = 0$ is equivalent to the identity in n variables $x = (x_1, ..., x_n)$:*

$$\sum_{I \subset <n>, 0 < |I|} u_I x_I = \sum_{i=1}^{n-1} t_i (x_i - x_{i+1}) + \sum_{i=1}^{n} (w_i + v) x_i. \tag{12}$$

The cone of nonnegative solutions $zV = 0, z \geq 0$ is finitely generated by the extremal vectors of the following three types

$$\begin{aligned} &z^{l,1} := (t^{l,1}, u^{l,1}, 0, w^{l,1}), \quad l = 1, ..., M_1(n), \\ &z^{l,2} := (t^{l,2}, u^{l,2}, 1, 0), \quad l = 1, ..., M_2(n), \\ &z^{l,3} := (t^l, u^l, 1, w^l), \quad u^l_{<n>} = 0, \quad w^l \neq 0, \quad l = 1, ..., M(n). \end{aligned} \tag{13}$$

The number of nonzero coordinates in any extremal vector of the form (13) is at most $n+1$. Furthemore, the set of extremal vectors of the form $z^{l,3}$ is empty for $n = 2$.

Proof. Let $z = (t, u, v, w)$ satisfy (12). Equate the coefficient of x_i in (12) to deduce that the $i - th$ coordinate of the vector zV is equal to zero. Hence $zV = 0$. Similarly $zV = 0$ implies the identity (12). The Farkas-Minkowski-Weyl theorem [**Sch.**, 7.2] yields that the cone $zV = 0, z \geq 0$ is finitely generated. First we divide the extreme vectors $z = (t, u, v, w)$ to two sets $v = 0$ and $v \neq 0$. We normalize the second set by letting $v = 1$. We divide the second set to the subsets $w = 0$ and $w \neq 0$. Note that the subset $w = 0$ contains an extremal vector $\zeta = (0, v, 1, 0)$ where $v_{<n>} = 1$ and all other coordinates are equal to zero. Hence the extremal vector in the second subset $(t^l, u^l, 1, w^l), w^l \neq 0$ satisfies $u^l_{<n>} = 0$.

Let z be an extremal ray of the cone $zV = 0, z \geq 0$. Assume that z has exactly p nonvanishing coordinates. Let $\hat{U}$ be $p \times n$ submatrix of U corresponding to the nonzero elements of z. Let $wV = 0$ and assume that $w_i = 0$ if $z_i = 0$. Then the nonzero coordinates of w satisfy n equations. As z is an extremal ray it follows that $w = \alpha z$ for some $\alpha \in \mathbf{R}$. Hence the p columns of $\hat{U}$ span $p - 1$ dimensional subspace, i.e. rank $\hat{U} = p - 1 \leq n$.

Consider an extremal vector $z^{l,3}$. By the definition $v = 1$, $w^l \neq 0$ and $u^l_{<n>} = 0$. Use (12) to deduce that $u^l \neq 0$. Assume first that $n = 2$. Since $z^{l,3}$ has at most 3 nonzero coordinates, we deduce that each vector $u^l, v^l = 1, w^l$ has exactly one nonzero coordinate. As $u^l_{<2>} = 0$ (12) can not hold. $\diamond$

Lemma 4. *Let $n > 1$ and $a := (a_I)_{I \subset <n>, 1 \leq |I|}$ be a given vector. Define*

$$K(a) := \{x \in \mathbf{R}^n_{\geq} : \quad x_I \leq a_I, \quad I \subset <n>, \quad 0 < |I| < n, \quad x_{<n>} = a_{<n>}\}.$$

Assume that $K(a)$ is nonempty. Define

$$K'(a) := \{y \in \mathbf{R}^n_{\geq} : \quad \exists x \in K(a), \quad y \leq x\}.$$

Then $K'(a)$ is polyhedral set given by the $(5'), (3')$ and $(3'')$ with $y = \lambda^0$.

Proof. Farkas lemma yields that $y \in K'(a)$ iff (11) holds, where c is defined in $(10')$. Assume first that $y \in K'(a)$. Then $(5'), (3')$ hold. The system $zV = 0$ is equivalent to $(t, u)U = ve + w$, where U is the matrix representing the system (9). Hence

$$\begin{aligned} zc^T =& (t, u)b^T - va_{<n>} - wy^T = \\ & (t, u)b^T - va_{<n>} - ((t, u)U - ve)y^T = (t, u)(b^T - Uy^T) + v(ey^T - a_{<n>}), \\ zc^T =& ua^T - va_{<n>} - wy^T. \end{aligned} \tag{14}$$

Use (11) and the second equality of (14) for the vectors $z^{l,3}, l = 1, ..., M(n)$ to deduce $(3'')$.

Assume now that y satisfies $(5'), (3')$ and $(3'')$. Observe that $(5'), (3')$ are equivalent to $b^T \geq Uy^T$. We claim that (11) holds. Suppose first that $z = (t, v, 0, w) \geq 0$. From the last part of the first equality (14) we deduce that $zc^T \geq 0$. Assume next that $z = (t, v, 1, 0)$. Choose $y \in K(a)$. Clearly, $y \in K'(a)$. Then (11) holds for this particular choice of y. Use the second equality of (14) to get $ua^T - va_{<n>} \geq 0$. Thus, it is enough to prove (11) for the extreme points of the cone $zV = 0, z \geq 0$ of the third type $z^{l,3}, l = 1, ..., M(n)$. These are exactly the conditions $(3'')$. $\diamond$

Proof of Theorem 1. Assume the condition (b) of Theorem 1. Let $\tilde{A}_0 := \sum_{j=1}^m A_j$ and $\tilde{\lambda}^0$ be the eigenvalue vector of $\tilde{A}_0$. Then Lemma 1 yields $\tilde{\lambda}^0 \geq \lambda^0$. Use Klyacko theorem and Lemma 4 to deduce the conditions (a) of Theorem 1.

Assume the conditions (a) of Theorem 1. Lemma 4 implies the existence of $\tilde{\lambda}^0 \in K(a)$ such that $\lambda^0 \leq \tilde{\lambda}^0$. Lemma 2 yields the condition (b) of Theorem 1. $\diamond$

Proposition 1. *Let $n \geq 3$ and assume that I, J be two proper subsets of $< n >$ such that*

$$I \cup J = < n >, \quad I \cap J = \{i\}, \quad 1 \leq i \leq n.$$

Then the equality

$$x_I + x_J = x_{<n>} + x_i$$

corresponds to an extremal ray $z^{l,3} = (0, u, 1, w^l)$, *where* u *has two nonzero coordinates (equal to* 1 *and corresponding to sets* I, J*) and* w *has one nonzero coordinate (*$w_i = 1$*). Hence the corresponding inequality* $(3'')$ *is given by:*

$$\lambda_i^0 \le -a_{<n>} + a_I + a_J. \qquad (3_{IJ})$$

Proof. It is ehough to show that that $4 \times n$ matrix $\hat{U}$ appearing in the proof of Lemma 3 has rank 3. Without the loss of generality we may assume that $\{1,2\} \subset I_1, \{2,3\} \subset I_2$, i.e. $i = 2$. It is enough to show that the 4×3 submatrix $\tilde{U}$, composed of the first three columns of $\hat{U}$ has rank 3. That is, one may assume that $n = 3$ and a straightforward calculation shows that rank $\hat{U} = 3$. Use the above extremal ray $z^{l,3}$ in $(3'')$ to obtain (3_{IJ}). $\diamond$

We believe that for any $n \ge 3$ at least one of the inequalities of the form (3_{IJ}) does not follow from $(5')$ and $(3')$.

§3. Convoy principle

Let $\mathcal{H}$ be a separable Hilbert space with an inner product $< u, v > \in \mathbf{C}$ for $u, v \in \mathcal{H}$. Let $A : \mathcal{H} \to \mathcal{H}$ be a linear, bounded, selfadjoint operator. Let $V \subset \mathcal{H}$ be an n-dimensional subspace. Pick an orthonormal basis $f_1, ..., f_n \in V$. Denote by $A(f_1, ..., f_n) \in \mathcal{H}_n$ the $n \times n$ matrix whose (i, j) entry is $< Af_i, f_j >$. Let

$$\lambda_1(A, V) \ge \lambda_2(A, V) \ge \cdots \ge \lambda_n(A, V)$$

be the n eigenvalues of the Hermitian matrix $(< Af_i, f_j >)_1^n$. Clearly, the above eigenvalues do not depend on a particular choice of an orthonormal basis $f_1, ..., f_n$ of V. We now recall the convoy principle, e.g. [**Fri**].

Lemma 5. *Let* $A : \mathcal{H} \to \mathcal{H}$ *be a bounded, linear, selfadjoint, nonnegative, compact operator with the eigenvalue sequence* $\{\lambda_i\}_{i=1}^\infty$. *Let* $n \ge k \ge 1$ *be any integers. Assume that* $V \subset \mathcal{H}$ *is any* n*-dimensional subspace. Then* $\lambda_k(A, V) \le \lambda_k$ *and this inequality is sharp.*

Proof. Choose an orthonormal basis $f_1, ..., f_n$ of V so that $(< Af_i, f_j >)_1^n$ is the diagonal matrix diag $(\lambda_1(A, V), ..., \lambda_n(A, V))$. Let $f = \sum_{i=1}^k \alpha_i f_i \ne 0$ such $< f, e_i >= 0, i = 1, ..., k-1$, where $\{e_i\}_1^\infty$ is an orthonormal basis of $\mathcal{H}$ given in (7). Deduce from (7) and from the choice of $f_1, ..., f_n$ that

$$\lambda_k(A, V) \le \frac{< Af, f >}{< f, f >} \le \lambda_k.$$

For $V =$span$(e_1, ..., e_n)$ we obtain that $\lambda_k(A, V) = \lambda_k$. $\diamond$

The Completion Problem. Let $\lambda^0, ..., \lambda^m \in \mathbf{R}^n_\ge$ be given. Find $\theta^0, ..., \theta^m \in \mathbf{R}^l_\ge$ for some $l \ge 1$ with the following properties:
(a) Each row vector (λ^j, θ^j) belongs to $\mathbf{R}^{n+l}_\ge$ for $j = 0, ..., m$.
(b) There exists $A_j \in \mathcal{H}_{n+l}$ such that (λ^j, θ^j) is its eigenvalue vector for $j = 0, ..., m$ and $A_0 = \sum_{j=1}^m A_j$.

Proposition 2. *Assume that the Completion Problem is solvable. Then* $(5'), (3'), (3'')$ *hold.*

Proof. Without loss of generality assume that $A_0 =$ diag $(\lambda_1^0, ..., \lambda_n^0, \theta_1^0, ..., \theta_l^0)$. Let B_j be the principal submatrix of A_j corresponding to the first n row and columns for $j = 0, ..., m$. Clearly, $B_0 = \sum_{j=1}^m B_j$ and $\lambda^0 = \lambda(B_0)$. The convoy principle yields $\lambda(B_j) \le \lambda^j, j = 1, ..., m$. Remark 2 implies the existence of $C_j \in \mathcal{H}_n$ such that $\lambda(C_j) = \lambda^j$ and $C_j \ge B_j$ for $j = 1, ..., m$. Hence $B_0 \le \sum_{j=1}^m C_j$ and $(5'), (3'), (3'')$ follow from Theorem 1. $\diamond$

It is an open problem if the conditions $(5'), (3'), (3'')$ imply always the solvability of the Completion Problem. We refer to [**D-S-T**, §3.7] for a related completion problem.

Proof $(b) \Rightarrow (a)$ **in Theorem 2.** Denote by $(5'_n), (3'_n), (3''_n)$ the inequalities $(5'), (3'), (3'')$ respectively. Assume that $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis of $\mathcal{H}$ so that

$$A_0 e_i = \lambda_i^0 e_i, \quad \lambda_i \geq 0, \quad i = 1, ...,.$$

Let V_n :=span$(e_1, ..., e_n)$ and set

$$B_k := (< A_k e_i, e_j >)_1^n \in \mathcal{H}_n, \quad k = 0, ..., m.$$

The assumption $A_0 \leq \sum_{k=1}^m A_j$ yields $B_0 \leq \sum_{k=1}^m B_j$. Recall that the eigenvalue vector of B_j is $\lambda(B_j) := (\lambda_1(A_j, V_n), ..., \lambda_n(A_j, V_n))$ for $j = 0, ..., m$. Note that the choice of V_n yield that $\lambda_i(A_0, V_n) = \lambda_i^0, i = 1, ..., n$. Use Lemma 5 for upper estimates of $\{\lambda_i(A_j, V_n)\}_1^n$ to deduce that $\lambda^j := (\lambda_1^j, ..., \lambda_n^j) \geq \lambda(B_j)$ for $j = 1, ..., m$. Remark 2 yields the existence of $C_j \in \mathcal{H}_n$, with the eigenvalue vector λ^j, such that $C_j \geq B_j$ for $j = 1, ..., m$. Hence $B_0 \leq \sum_{j=1}^m C_j$. Theorem 1 yields part (a) of Theorem 2. ⋄

We defer the proof of the implication $(a) \Rightarrow (b)$ to the next section. To show that Theorem 3 is a simple corollary of Theorem 2 we bring the proof of the following Lemma, which is well known to the experts:

Lemma 6. *Let $A : \mathcal{H} \to \mathcal{H}$ be a linear, bounded, selfadjoint, nonnegative, compact operator given by* (7). *Then A is in trace class iff for some orthonormal basis $\{f_i\}_1^{\infty}$ the nonnegative series $\sum_{i=1}^{\infty} < Af_i, f_i >$ converges. Furthermore, if A is in the trace class then*

$$\sum_{i=1}^{\infty} < Af_i, f_i >= \sum_{i=1}^{\infty} \lambda_i.$$

Proof. Let V_n =span$(f_1, ..., f_n), n = 1, ...,$. Assume first that A is in the trace class. Lemma 5 yields

$$\sum_{i=1}^{n} < Af_i, f_i >= \sum_{i=1}^{n} \lambda_i(A, V_n) \leq \sum_{i=1}^{n} \lambda_i \leq \text{trace } A.$$

Hence the nonnegative series $\sum_{i=1}^{\infty} < Af_i, f_i >$ converges. Assume now that the nonnegative series $\sum_{i=1}^{\infty} < Af_i, f_i >$ converges. Since $\lim_{n\to\infty}$ dist $(V_n, e_k) = 0$ a straightforward argument yields (or see Lemma 7)

$$\lim_{n\to\infty} \lambda_k(A, V_n) = \lambda_k, \quad k = 1, ...,.$$

Lemma 5 yields that the sequence $\{\lambda_k(A, V_n)\}_{n=k}^{\infty}$ is a nondecreasing sequence that converges to λ_k for any $k \geq 1$. Hence, for a given positive integer k and $\epsilon > 0$ there exists $N(k, \epsilon)$ so that

$$\sum_{i=1}^{k} \lambda_i \leq \epsilon + \sum_{i=1}^{k} \lambda_i(A, V_n) \leq \epsilon + \sum_{i=1}^{n} \lambda_i(A, V_n) = \epsilon + \sum_{i=1}^{n} < Af_i, f_i >\leq \epsilon + \sum_{i=1}^{\infty} < Af_i, f_i >$$

for any $n > N(k, \epsilon)$. Fix k. We then deduce

$$\sum_{i=1}^{k} \lambda_i \leq \sum_{i=1}^{\infty} < Af_i, f_i > .$$

As k is arbitrary we obtain

$$\text{trace } A \leq \sum_{i=1}^{\infty} < Af_i, f_i > .$$

Hence A is in the trace class. Assume that A is in the trace class. The above arguments yield the equality in the above inequality. $\diamond$

Proof of Theorem 3. We assume the validity of Theorem 2.
$(a) \Rightarrow (b)$. Theorem 2 yields the existence of $A_0, ..., A_m$ selfadjoint, nonnegative, compact operators with the prescribed eigenvalue sequences so that $A_0 \le \sum_{j=1}^m A_j$. Assumption (8) yields that $A_0, ..., A_m$ are in the trace class. Let $A := \sum_{j=1}^m A_j - A_0$. Then A is a selfadjoint, nonnegative, compact operator. Lemma 6 yields that A is in the trace class. Then (8) implies that trace $A = 0$, hence $A = 0$.
$(b) \Rightarrow (a)$. Theorem 2 yields $(5'_n), (3'_n), (3''_n), n = 1, ...,$. As $A_0, ..., A_m$ are in the trace class and $A_0 = \sum_{j=1}^m A_j$ Lemma 6 yields (8). $\diamond$

§4. Completion of the Proof of Theorem 2.

Recall that $||x|| := \sqrt{< x, x >}$ is the standard norm on $\mathcal{H}$ induced by its inner product. Let $A : \mathcal{H} \to \mathcal{H}$ be a linear, bounded, selfadjoint operator. Let $||A|| := \sup_{||x|| \le 1} ||Ax||$ be the norm of A. Then for any n-dimensional subspace $V \subset \mathcal{H}$ we have that

$$|\lambda_i(A, V)| \le ||A||, \quad i = 1, ..., n.$$

Let

$$\lambda_k(A, \mathcal{H}) := \sup_{V \subset \mathcal{H}, \dim V = k} \lambda_k(A, V), \quad k = 1, ..., .$$

For a nonnegative compact A of the form (7) Lemma 5 yields that

$$\lambda_k(A, \mathcal{H}) = \lambda_k, \quad k = 1, ..., .$$

Lemma 7. *Let $A : \mathcal{H} \to \mathcal{H}$ be a linear, bounded, selfadjoint operator. Then the sequence $\{\lambda_i(A, \mathcal{H})\}_1^\infty$ is a nonincreasing sequence which lies in $[-||A||, ||A||]$. Let $\{f_i\}_1^\infty$ be any orthonormal basis in $\mathcal{H}$. Set V_n=span$(f_1, ..., f_n)$ for $n = 1, ...,$. Then the sequence $\{\lambda_k(A, V_n)\}_{n=k}^\infty$ is an incresing sequence which converges to $\lambda_k(A, \mathcal{H})$ for each $k = 1, 2, ...,$.*

Proof. Fix a complete flag

$$W_1 \subset W_2 \subset \cdots \subset W_i \cdots, \quad \dim W_i = i, \quad i = 1, ...,$$

of subspaces in $\mathcal{H}$. Then the convoy principle for matrices yields that

$$\lambda_i(A, W_{i+1}) \ge \lambda_i(A, W_i) \ge \lambda_{i+1}(A, W_{i+1}), \quad i = 1, 2, ..., .$$

(These inequalities are natural extensions of the Cauchy interlacing inequalites for matrices.) Hence the sequence $\{\lambda_i(A, \mathcal{H})\}_1^\infty$ is a nonincreasing sequence which lies in $[-||A||, ||A||]$. Furthermore we obtain that $\{\lambda_k(A, V_n)\}_{n=k}^\infty$ is a nondecreasing sequence. From the definition of $\lambda_k(A, \mathcal{H})$ we immediately deduce that

$$\lambda_k(A, V_n) \le \lambda_k(A, \mathcal{H}), \quad n = k, k + 1, ..., .$$

Let

$$\tilde{\lambda}_k := \lim_{n \to \infty} \lambda_k(A, V_n), \quad k = 1, ..., .$$

Hence $\tilde{\lambda}_k \le \lambda_k(A, \mathcal{H}), k = 1, ...,$. We claim that for any k-dimensional subspace $W \subset \mathcal{H}$

$$\tilde{\lambda}_k \ge \lambda_k(A, W).$$

Assume that $g_1, ..., g_k$ is an orthonormal basis in W so that the matrix $(< Ag_i, g_j >)_1^k$ is the diagonal matrix diag$(\lambda_1(A, W), ..., \lambda_k(A, W))$. Let $P_n : \mathcal{H} \to V_n$ be the orthogonal projection on V_n. That is

$$P_n x = \sum_{i=1}^{n} < x, f_i > f_i.$$

Then $\lim_{n\to\infty} P_n x = x$ for every $x \in \mathcal{H}$, i.e. P_n converges in the strong topology. Hence, for $n > N$, $P_n g_1, ..., P_n g_k$ are linearly idependent. Let $g_{1,n}, ...g_{k,n} \in V_n$ be the k orthonormal vectors obtained from $P_n g_1, ..., P_n g_n$ using the Gram-Schmidt process. We can renormalize $g_{1,n}, ..., g_{k,n}$ (by multiplying them by suitable complex numbers of length 1) so that

$$\lim_{n\to\infty} g_{i,n} = g_i, \quad i = 1, ..., k.$$

Hence the matrix $(< Ag_{i,n}, g_{j,n} >)_{i,j=1}^k$ converges the diag $(\lambda_1(A, W), ..., \lambda_k(A, W))$. Let W_n = span $(g_{1,n}, ..., g_{k,n})$. Then

$$\lim_{n\to\infty} \lambda_k(A, W_n) = \lambda_k(A, W).$$

As $W_n \subset V_n$ the convoy principle implies

$$\lambda_k(A, W_n) \le \lambda_k(A, V_n) \le \tilde{\lambda}_k.$$

Hence

$$\lambda_k(A, W) \le \tilde{\lambda}_k \quad \Rightarrow \quad \lambda_k(A, \mathcal{H}) \le \tilde{\lambda}_k \Rightarrow \quad \lambda_k(A, \mathcal{H}) = \tilde{\lambda}_k.$$

$\diamond$

Let A satisfy the assumption of Lemma 7. Denote by $\sigma(A)$ the spectrum of A. Then $\sigma(A)$ is a compact set located in the closed interval $[-||A||, ||A||]$. Recall the spectral decomposition of A:

$$A = \int_{[-||A||,||A||]} x dE(x).$$

Here $E(x), x \in \mathbf{R}, 0 \le E(x) \le I$ is the resolution of the identity of commuting increasing family of orthogonal projections induced by A, which is continuous from the right. Hence $E(-||A|| - 0) = 0$ and $E(||A|| + 0) = I$. Consult for example with the classical book [**A-N**] or a modern book [**R-S**]. Note that

$$I = \int_{[-||A||,||A||]} dE(x)$$

For a measurable set $T \subset \mathbf{R}$ denote by $P(A, T)$ the spectral projection of A on T:

$$P(A, T) := \int_T dE(x).$$

We let dim $P(A, T)$ be the dimension of the closed subspace $P(A, T)\mathcal{H}$. Note that $0 \le$ dim $P(A, T) \le \infty$. Observe that dim $P(A, (a, b))$ is finite and positive iff $\sigma(A) \cap (a, b)$ consists of a finite number of eigenvalues of A, each one with a finite dimensional eigenspaces. We say that $\mu(A)$ is the first accumulation point of the spectrum of A if

$$\dim P(A, (\mu(A) + \epsilon, \infty)) < \infty, \quad \dim P((\mu(A) - \epsilon, \infty)) = \infty$$

for every positive ϵ. $\mu(A)$ must be either a point of the continuous spectrum or a point spectrum with an infinite corresponding eigenspace. (It is a maximal point in $\sigma(A)$ with this property.)

Lemma 8. *Let $A : \mathcal{H} \to \mathcal{H}$ be a linear, bounded, selfadjoint operator. Then the nonincreasing sequence $\{\lambda_i(A, \mathcal{H})\}_1^\infty$ converges to $\mu(A)$.*

Proof. Suppose first that dim $P(A,(a,b)) > 0$. Let

$$A(a,b) := \int_{(a,b)} x dE(x).$$

Then

$$a \le \frac{< Ax, x >}{< x, x >} \le b, \quad 0 \ne x \in P(A,(a,b))\mathcal{H}.$$

Let $\epsilon > 0$. Let $f_1, ..., f_{k-1}$ be an orthonormal basis $V = P(A, (\mu(A) + \epsilon, \infty))\mathcal{H}$. (If $k = 1$ then $V = 0$.) Hence $V^\perp = P(A, (-\infty, \mu(A) + \epsilon])\mathcal{H}$. Let $W \subset \mathcal{H}$ be any subspace of dimension k. Then $V^\perp \cap W$ contains a nonzero vector $x \in P(A, (-\infty, \mu(A) + \epsilon])\mathcal{H}$. Then convoy principle and the above observation yield that

$$\lambda_k(A, W) \le \mu(A) + \epsilon.$$

Hence

$$\lambda_k(A, \mathcal{H}) \le \mu(A) + \epsilon.$$

Recall that $U := P(A, (\mu(A) - \epsilon, \infty))\mathcal{H}$ is infinite dimenisonal. Let $W \subset U$, dim $W = l$. Then the convoy principle and the above observation yield that

$$\lambda_l(A, W) \ge \mu(A) - \epsilon.$$

Hence $\lambda_l(A, \mathcal{H}) \ge \mu(A) - \epsilon$. This inequality true for any l. Hence

$$\lim_{l\to\infty} \lambda_l(A, \mathcal{H}) \ge \mu(A) - \epsilon.$$

Since ϵ was an arbitrary positive number we deduce the lemma. $\diamond$

Corollary. *Let $A : \mathcal{H} \to \mathcal{H}$ be a linear, bounded, selfadjoint, nonnegative operator. Then A is compact iff the nonincreasing sequence $\{\lambda_i(A, \mathcal{H})\}_1^\infty$ converges to* 0.

Proof of $(a) \Rightarrow (b)$ in Theorem 2. Fix an orthonormal basis $\{e_i\}_{i=1}^\infty$ of $\mathcal{H}$. Assume the condition (a) of Theorem 2. Theorem 1 yields the existence of $m+1$ $n \times n$ Hermitian (real symmetric) matrices $B_{j,n}, j = 0, ..., m$ with the eigenvalues $\{\lambda_i^j\}_{i=1}^n, j = 0, ..., m$ respectively, such that $B_{0,n} \le \sum_{j=1}^m B_{j,n}$. Note that any entry of the matrix $B_{j,n}$ is bounded in absolute value by λ_1^j. Moreover we assume that $B_{0,n}$ =diag$(\lambda_1^0, ..., \lambda_n^0)$. Define a nonnegative, compact operators with a finite range by infinite block diagonal matrix $A_{j,n}$:=diag$(B_{j,n}, 0)$ in the orhonormal basis $\{e_i\}_1^\infty$ for $j = 0, ..., m$. We still have the inequality $A_{0,n} \le \sum_{j=1}^m A_{j,n}$. The first n eigenvalues of $A_{j,n}$ are $\{\lambda_i^j\}_{i=1}^n$ while others eigenvalues are 0. As the entry of each matrix $B_{j,n}$ is uniformly bounded, there exists a subsequence $n_l \to \infty$ so that each (p, q) entry of A_{j,n_l} converges a complex number $a_{pq,j}$ as $l \to \infty$. We claim that each infinite Hermitian matrix $C_j := (a_{pq,j})_{p,q=1}^\infty$ represents a linear, bounded, selfadjoint, and nonnegative operator. This trivially holds for C_0, which is an infinite diagonal matrix diag$(\lambda_1^0, ...)$. Since any principal submatrix of $A_{j,n}$ is nonnegative and its norm is bounded above by λ_1^j, each principal submatrix of C_j is nonnegative and its norm is bounded above by λ_1^j. [**A-N**, §26] yields that $C_1, ..., C_m$ represent linear, bounded, selfadjoint and nonnegative operators. Let V_k =span$(e_1, ..., e_k)$. From the definition of $C_1, ..., C_k$ we deduce

$$\lambda_i(C_j, V_k) \le \lambda_i^j, \quad i = 1, ..., k.$$

Lemma 7 yields that

$$\lambda_i(C_j, \mathcal{H}) \le \lambda_i^j, \quad i = 1, ..., \quad j = 1, ..., m.$$

Corollary to Lemma 8 yields that $C_1, ..., C_m$ are compact. As any principal submatrix of $A_{0,n}$ is majorized by the corresponding sum of submatrices $A_{j,n}, j = 1, ..., m$ we deduce that any principal submatrix of C_0 is majorized by the corresponding sum of submatrices $C_j, j = 1, ..., m$. Hence $C_0 \le \sum_{j=1}^m C_j$. Let

$$C_j e_{i,j} = \lambda_i(C_j, \mathcal{H}) e_{i,j}, \quad i = 1, ...,$$

be eigenvectors and the eigenvalues of C_j such that $\{e_{i,j}\}_{i=1}^{\infty}$ is an orthonormal basis of $\mathcal{H}$. Let $A_0 = C_0$ and define A_j by the equality

$$A_j e_{i,j} = \lambda_i^j e_{i,j}, \quad i = 1, ..., \quad j = 1, ..., m.$$

Then each $A_j \geq C_j, j = 1, ..., m$ and part (b) of Theorem 2 follows. $\diamond$

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