Write down in the booklet your name, your ID # and sign it. Do all work in the exam booklet. SHOW ALL THE WORK AND JUSTIFY YOUR ANSWERS! No work - no credit. Calculators are allowed only as a help or a double check. For example finding the inverse of a matrix using a software does not count toward the credit. If you used a software write down by the answer "by calculator". Do not bother to recopy the questions. Put your copy of the test in exam booklet.

Receiving or giving aid on this examination will result in a failing grade in the examination and in the course, and is a cause for dismissal from the University.

1. Let
\[ A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 5 & 1 \\ -1 & -3 & -1 \end{pmatrix}. \]

a. (18 pts) Find the inverse of \( A \) using the Gauss-Jordan elimination.
b. (7 pts) Find \( 3 \times 2 \) matrix \( X \) such that \( AX = B \), \( B = \begin{pmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 2 \end{pmatrix} \).

2. (25 pts) Let \( u = (1, 2, 3)^T \), \( v = (-2, -5, 0)^T \), \( w = (0, -1, 6)^T \).

a. Find a basis in \( S = \text{span} \{u, v, w\} \).
b. What is the dimension of \( S \)?
c. Are \( u, v, w \) linearly independent? If not find a nontrivial linear combination of \( u, v, w \) which is equal to the zero vector.

3. Let \( L : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) be the following linear transformation:
\[ L((x_1, x_2)^T) = (-2x_1 + 6x_2, x_1 − 3x_2)^T. \]

a. (13 pts) Find bases for the kernel and the range of \( L \).
b. (12 pts) Find the representation matrix of \( L \) in the basis \( W = [(1, 1)^T, (2, 1)^T] \) of \( \mathbb{R}^2 \).

4. (25 pts) Find the line \( y = a + bx \) in \( \mathbb{R}^2 \) that best fits the points \((-2, 1), (-1, 0), (0, 1), (2, 2)\). (Use the least squares.)

5. (25 pts)
\[ A = \begin{pmatrix} 2 & 0 & 4 \\ 0 & 1 & 3 \\ 2 & -4 & 1 \\ -1 & 1 & 1 \end{pmatrix}. \]

Find the \( QR \) decomposition of \( A \). (Hint: \( Q \) and \( R \) have rational entries, i.e. no roots! If you do not remember exactly \( QR \) decomposition you can perform the Gram-Schmidt process on the columns of \( A \) to get a partial credit.)

6. (25 pts) Find the general solution of the following system of ODE:
\[
\begin{align*}
y'_1 &= y_1 + y_2 \\
y'_2 &= y_2 - y_3 \\
y'_3 &= -y_2 + y_3
\end{align*}
\]
7. (25pts) A town has 12,000 families and two grocery stores: Sam’s and Joe’s. Every family buys their produce exactly once a week. 20% who buy at any given week at Sam’s buy at Joe’s the next week. 40% who buy at any given week at Joe’s buy at Sam’s the next week. In the first week of 2002 half of the families of the town went shopping to each of the grocery store.

a. Find how many families shopped at Sam’s and in Joe’s in the second week.

b. How many families bought at Sam’s and Joe’s in the last week of 2002. (You may assume that this is the steady state solution, i.e. after many weeks.)

8. a. (5 pts) Let

\[ A = \begin{pmatrix} 5 - 4i & 3 + 2i \\ 3 - 2i & 5 + 4i \end{pmatrix}, \quad (i^2 = -1). \]

Find \( A^H \). Is \( A \) a Hermitian matrix?

b. (20 pts) Let

\[ B = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}. \]

Find an orthogonal matrix \( Q \) and a diagonal matrix \( D \) such that \( B = QDQ^T \).
1. (25 pts) Consider the following system of linear equations:

\[
\begin{align*}
    x - y + z - w &= a \\
    2x + y - z - 2w &= b \\
    -x - y - 5z + w &= c
\end{align*}
\]

(a) For which values of \(a, b, c\) the system is solvable?
(b) Assume that \(a, b, c\) are chosen so that the above system is solvable. Find a vector form of the solution.
(c) Find \(A_{\text{red}}\), where \(A\) is the coefficient matrix of the system.

2. (25 pts) Let \(A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 3 & 5 \\ 0 & 1 & 5 \end{pmatrix}\).

a. Find the inverse of \(A\) using Gauss elimination.
b. Find the adjoint of \(A\).
c. Solve the system \(A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}\).

3. (25 pts)

(a) Let \(U = \{v = (x, y, z)^T \in \mathbb{R}^3 : x + y + |z| = 0\}\).
Is \(U\) a subspace of \(\mathbb{R}^3\)? (Justify!)
(b) Let \(u = 1 - 3X + X^2, \ v = 1 + X - 2X^2, \ w = 6 - 2X - 6X^2\) three vectors in \(P_3\). Show that \(u, v, w\) are linearly dependent and find a dependancy relation among them.

4. (20 pts) Let \(A\) be the 3 \times 4 coefficient matrix of the linear system of the Problem 1.

a. Find a basis for \(\mathcal{N}(A)\) and compute \(n(A)\) (the nullity of \(A\)).
b. Find a basis for \(\mathcal{R}(A)\) and compute \(r(A)\) (the rank of \(A\)).
c. Find a basis for the row space of \(A\) and compute the dimension of the row space.

5. (25 pts) Let \(T : \mathbb{R}^3 \rightarrow \mathbb{R}^2\) be a linear transformation:

\[
T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3x - 2y + 7z \\ -x - 3y + 2z \end{pmatrix},
\]

and consider the bases
\[
A = \left( \begin{array}{c} 1 \\ 2 \\ \end{array} \right), \left( \begin{array}{c} 4 \\ 1 \\ 0 \end{array} \right), \left( \begin{array}{c} 3 \\ 0 \\ 0 \end{array} \right) \] and \(B = \left( \begin{array}{c} 2 \\ 1 \\ \end{array} \right), \left( \begin{array}{c} 5 \\ 3 \end{array} \right) \)

for \(\mathbb{R}^3\) and \(\mathbb{R}^2\) respectively.

a. Find the transition matrix from the standard basis \(\mathcal{E}_2\) in \(\mathbb{R}^2\) to the basis \(B\).
b. Find a matrix \(A\) such that \(T(x) = Ax\) for all \(x \in \mathbb{R}^3\).
c. Find the representation matrix of \(T\) in bases \(A, B\).
6. (25 pts) Let

$$u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}, \quad v_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 2 \\ 1 \\ 3 \\ 1 \end{pmatrix},$$

$$U = \text{span}(u_1, u_2), \quad V = \text{span}(v_1, v_2).$$

a. Show that $U + V = \text{span}(u_1, u_2, v_1, v_2)$.
b. Find a basis for $U + V$.
c. Is $U + V = U \oplus V$?

7. (30 pts) Let $A = \begin{pmatrix} 0 & -1 & 9 \\ 2 & 2 & 9 \\ 1 & 0 & 9 \\ 2 & -2 & 9 \end{pmatrix}$.

a. Find the $QR$ decomposition of $A$.
b. Find the least squares solution to $Ax = b = \begin{pmatrix} 45 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

c. Is $U + V = U \oplus V$?

8. (25 pts)
a. Consider the data points $(-2, 5), (-1, 0), (0, 0), (2, 15)$. Find the linear function $f(x) = a + bx$ which fits best the data.
b. Let $A$ and $B$ be two $10 \times 10$ matrices with $\text{Det} \ A = -2$ and $\text{Det} \ B = 3$. Find $\text{Det}(A^{-2}B^2A)$. (Justify your steps!)