MATH 425-Spring 2012
HOMEWORK ASSIGNMENTS

MATH 425 Linear Algebra II, Spring 2012
LCD-undergrad 24908; LCD-grad 24909,
MWF 10:00-10:50, Addams Hall 303

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1 HOMEWORK ASSIGNMENT 1
Assigned 1-9-12 – Due 1-18-12

Do the following problems from [5]: p'18: 1, 2, 3, 5, 6; p' 312: 1,2. Note about Petersen's notation: $\mathrm{Mat}_{m,n}$ is $\mathbb{C}^{m\times n}$; $|A|$ is $\det A$, the determinant of $A \in \mathbb{F}^{n\times n}$.

Additional problem. Let $z_1, z_2, \ldots, z_n \in \mathbb{C}$. The Vandermonde matrix is given as

$$V(z_1,\ldots,z_n) := \begin{bmatrix} 1 & z_1 & z_1^2 & \ldots & z_1^{n-1} \\ 1 & z_2 & z_2^2 & \ldots & z_2^{n-1} \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 1 & z_n & z_n^2 & \ldots & z_n^{n-1} \end{bmatrix} \in \mathbb{C}^{n\times n}.$$

Show that $\det V(z_1,\ldots,z_n)$, called the Vandermonde determinant is equal to $\prod_{1\le i<j\le n}(z_j - z_i)$.

2 HOMEWORK ASSIGNMENT 2
Assigned 1-23-12 – Due 2-3-12

a. 2 problems from §1.4.1, 4 problems from §1.4.2 from [1].
b. Let $\sigma \in \mathcal{S}_5$ be defined as $\sigma(1) = 3, \sigma(2) = 5, \sigma(3) = 1, \sigma(4) = 4, \sigma(5) = 2$. Find $\mathrm{sign}(\sigma)$.
c. $A, B \in \mathbb{F}^{m\times m}$ are called *congruent* if $A = TBT^\top$ for some $T \in \mathrm{GL}(n, \mathbb{F})$. Show

1. Congruence in $\mathbb{F}^{n\times n}$ is an equivalence relation.
2. Show that any two congruent matrices have the same rank

d. Assume that if $A \in \mathbb{F}^{n\times n}$ is a skew symmetric matrix.

1. Show that if n is odd and $\mathbb{F}$ has characteristic not equal to 2, i.e. $2 \neq 0$ in $\mathbb{F}$, then $\det A = 0$.

2. Show that if $\mathbb{F}$ has characteristic not equal to 2, then A is congruent to a block diagonal matrix $B = \mathrm{diag}(B_1, \ldots, B_k)$, where each block is either $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ or 1×1 0 matrix. **Hint**: Use a sequence of "elementary conjugation" given by $EAE^\top$ where E is an elementary matrix.
3. Show that if $\mathbb{F}$ has characteristics 2, then A is congruent to a block diagonal matrix $B = \mathrm{diag}(B_1, \ldots, B_k)$, where each block is either or 1×1 0 matrix, 1×1 identity or $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. (Note that $-1 = 1$ in $\mathbb{F}$.)
4. Given an example of $n \times n$ skew symmetric matrix, for a field with with charactersitic 2, whose determinant is nonzero for each $n \in \mathbb{N}$.
5. Assume that $\mathbb{F} = \mathbb{R}$. Then $\det A \geq 0$.

3 HOMEWORK ASSIGNMENT 3
Assigned 2-2-12 – Due 2-10-12

Problems 2-8, §1.6.1, (page 16) in [1]; Problems 1-8, (page 18) in [1].

4 HOMEWORK ASSIGNMENT 4
Assigned 2-6-12 – Due 2-15-12

Do the following problems

1. Let $\mathbf{u} = (1, -1, 1, -1)^\top, \mathbf{v} = (2, 0, -2, 1)^\top$. Find
    (a) The cosine of the angle between $\mathbf{u}$ and $\mathbf{v}$.
    (b) The scalar and the vector projection of $\mathbf{v}$ on $\mathbf{u}$.
    (c) A basis to the orthogonal complement of $\mathbf{U} := \mathrm{span}(\mathbf{u}, \mathbf{v})$.
    (d) The projection of the vector $(1, 1, 0, 0)^\top$ on $\mathbf{U}$ and $\mathbf{U}^\perp$.
2. Let $A \in \mathbb{R}^{4\times3}$. Assume that the vector $(1, -1, 1, -1)^\top$ is a vector in the column space of A. Is it possible that a vector $(2, 0, -2, 1)^\top$ is in the null space of $A^\top$? If yes give an example of such a matrix. If not, justify why.
3. Consider the overdetermined system

$$\begin{array}{rcccccl} & x_1 & + & x_2 & + & x_3 & = & 4 \\ - & x_1 & + & x_2 & + & x_3 & = & 0 \\ & & - & x_2 & + & x_3 & = & 1 \\ & x_1 & & & + & x_3 & = & 2 \end{array}$$

    (a) Is this system solvable?
    (b) Find the least squares solution of this system.
    (c) Find the projection of $(4, 0, 1, 2)^\top$ on the column space of the coefficient matrix $A \in \mathbb{R}^{4\times3}$ of this system.

4. Let $(-1,0),(0,1),(1,3),(2,9)$ be four points in the plane (x,y) Find
   (a) The best least squares fit by a linear function $y = ax + b$.
   (b) The best least squares by a quadratic polynomial $y = ax^2 + bx + c$.
   (c) Explain briefly why there exist a unique cubic polynomial $y = ax^3 + bx^2 + cx + d$ passing through these four points.
5. Let $a \le t_1 < t_2 < \ldots < t_n \le b$ be n points in the interval $[a,b]$. For any two continuous functions $f, g \in \mathrm{C}[a,b]$ define $\langle f,g\rangle := \sum_{i=1}^{n} f(t_i)g(t_i)$. Let P_m be the vector space of all polynomials of degree at most $m-1$.
   (a) Show that for $m \le n$ $\langle\ ,\ \rangle$ is an inner product on P_m.
   (b) Is $\langle\ ,\ \rangle$ an inner product on P_{n+1}? Justify!
6. For the inner product $\langle f,g\rangle := \int_{-1}^{1} f(x)g(x)dx$ on $\mathrm{C}[-1,1]$ Find the cosine of the angle between $f(x) = 1$ and $g(x) = e^x$.

Do the following problems from "Schaum's Outline of Linear ALgebra" by S. Lipschutz and M. Lipson, 4th edition, pages 258-261: 7.58, 7.60, 7.64, 7.71.

5 HOMEWORK ASSIGNMENT 5
Assigned 2-14-12 – Due 2-22-12

Do the following problems. The problems in Schaum are from pages 260–262.

1. Problem 7.75 from Schaum. In addition do the following
   (a) Find the QR decomposition of the matrix $A = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3]$.
   (b) Complete the orthonormal basis you found uisng the Gram-Schmidt problem to an orthonormal basis of $\mathbb{R}^4$.
2. Problem 7.76 part a in Schaum.
3. Problem 7.78 in Schaum.
4. Problem 7.91 in Schaum.
5. Problem 7.94 in Schaum.
6. Problem 1 page 23 in [1].
7. Problem 3 page 23 in [1]. **Hint**: try $a_{pq} = z^{pq}$, where $z = e^{\frac{2\pi\mathbf{i}}{n}}$.
8. Problem 5 page 23 in [1].

6 HOMEWORK ASSIGNMENT 6
Assigned 2-22-12 – Due 2-29-12

[1]: §2.3 page 28–29, Problems: 9(a,b,c), (special orthogonal means determinant one), 10a, 12.
[3]: §6.4 p'363-365, Problems: 4(a-f); 5(a,b,c,f),6,10,12,14.

7 HOMEWORK ASSIGNMENT 7
Assigned 2-26-12 – Due 3-7-12

I. Assume that A a real symmetric matrix. Denote by $\iota_+(A)$ be the number of positive eigenvalues, , $\iota_0(A)$ the number of zero eigenvalues, $\iota_-(A)$ be the number of negative eigenvalues. Denote $\iota(A) := (\iota_+(A), \iota_0(A), \iota_-(A))$. Show.

1. Show that $i_+(A)$ is the dimension of the unique subspace $\mathbf{U} \subset \mathbb{R}^n$ such that $\mathbf{x}^\top A\mathbf{x} > 0$ for each nonzero $\mathbf{x}$ in $\mathbf{U}$. (**Hint**: Use the convoy principle.)
2. Show that $i_-(A)$ is the dimension of the unique subspace $\mathbf{U} \subset \mathbb{R}^n$ such that $\mathbf{x}^\top A\mathbf{x} < 0$ for each nonzero $\mathbf{x}$ in $\mathbf{U}$.
3. rank $A = \iota_+(A) + \iota_-(A)$.
4. A symmetric $B \in \mathbb{R}^{n\times n}$ is called congruent to A if $B = QAQ^\top$ for some invertible matrix Q. Show that two symmetric matrices are congruent if and only if $\iota(A) = \iota(B)$. (This result is called the *Sylvester law of inertia*.)

II. State and prove the similar results in Problem I for a hermitian matrix. (This is the content of Problems 6 and 7 in [1, p'35-36].)

III. Let $A = [a_{pq}] \in \mathbb{C}^{n\times n}$ be a hermitian matrix. Rearrange the diagonal entries of A, $a_{11}, a_{22}, \ldots, a_{nn}$ in a nonincreasing way: $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n$. Show

1. $\lambda_1(A) \geq \alpha_1, \lambda_n(A) \leq \alpha_n$. **Hint**: Use the maximum and minimum characterization of $\lambda_1(A), \lambda_n(A)$.
2. Show that $\sum_{i=1}^k \alpha_i \leq \sum_{i=1}^n \lambda_i(A)$ for $k = 1, \ldots, n$. What happens for $k = n$? **Hint**: Use the convoy principle.
3. Show that $|\lambda_j(A)| \leq \sqrt{\sum_{p=q=1}^n |a_{pq}|^2}$ for each $j = 1, \ldots, n$. For which kind of matrices and for which j we have equality in this inequality?

[1]: §2.5 page 34–35, Problems: 1,3.

IV. Let $A = \begin{bmatrix} 1 & 1+i & 2-3i \\ 1-i & 3 & 3-2i \\ 2+3i & 3+2i & 2 \end{bmatrix}$.

1. Estimate from below and above $\lambda_1(A)$ using the results of Problem III.
2. Find the eigenvalues of 2×2 hermitian submatrix of A composed of the last two rows and columns of A.
3. Estimate from below $\lambda_1(A)$ using the Cauchy interlacing theorem, (Problem 3a on page 34 in [1]), and the results of part 2. Which estimate is better?
4. Use the Cauchy interlacing theorem and the results of part 2 to show that $\lambda_3(A)$ is negative. Use the inequalities in Problem III to estimate from below $\lambda_3(A)$.
5. Estimate $\lambda_2(A)$ from below and above using Cauchy interlacing theorem by considering the eigenvalues of 2×2 hermitian submatrix of A composed of the first two rows and columns of A. Compare this estimate with the estimate using the results of part 2.

V. For the following symmetric matrices find a diagonal matrix which is congruent to it. In each case determine how many positive negative and zero eigenvalues A has. Furthermore determine if there exist a lower triangular matrix L with one on the diagonal such that $A = LDL^\top$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 6 \\ 2 & 6 & 7 \end{bmatrix}, \begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & 3 \\ 1 & 3 & 2 \end{bmatrix}, \begin{bmatrix} 1 & -1 & 0 & 2 \\ -1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 2 \\ 2 & 0 & 2 & -1 \end{bmatrix}$$

8 HOMEWORK ASSIGNMENT 8 Assigned 3-09-12 – Due 3-16-12

1. Problem 1 [1, p' 41].
2. Problems 3,4,9 [1, p' 45], (In 9 you can assume that A is a normal matrix.)
3. Problems 1-6,8 [3, p' 380-382].

9 HOMEWORK ASSIGNMENT 9 Assigned 3-13-12 – Due 3-30-12

1. Problems 1, 3, 4, 5, 9 [1, p' 45–46].
2. Problems 1, 2,3 [1, p' 54],
3. Problems 1-6,8 [3, p' 380-382]. (If you did not do them yet!)

10 HOMEWORK ASSIGNMENT 10 Assigned 3-31-12 – Due 4-6-12

Problems 1, 5, page 58 in [1],
Problems 1c, 2, 4 page 64 in [1]. (A is called noderogatory if the minimal polynomial of A equal to the characteristic polynomial of A.)
Additional problems:

1. Let $A \in \mathbb{F}^{n\times n}, B \in \mathbb{F}^{m\times m}$. Assume that $f, g \in \mathbb{F}[t]$ are the minimal polynomials of A, B respectively. Form $C = \mathrm{diag}(A, B) := \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$. Let h be gcd, the greatest common divisor, of f and g, which is assumed to be monic. Show that $\frac{fg}{h}$ is the minimal polynomial of C.
2. Find the characteristic and the minimal polynomials of the following matrices

$$\begin{bmatrix} 2 & 2 & -5 \\ 3 & 7 & -15 \\ 1 & 2 & -4 \end{bmatrix}, \begin{bmatrix} 2 & 5 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 \\ 0 & 0 & 3 & 5 & 0 \\ 0 & 0 & 0 & 0 & 7 \end{bmatrix}.$$

3. Show that two similar matrices have the same minimal polynomial.

4. Let $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ have different characteristic polynomials, but the same minimal polynomial.

5. Show that the square matrices A and $A^\top$ have the same minimal polynomial.

6. Let $A \in \mathbb{F}^{n\times n}$ and assume that $f(t) \in \mathbb{F}[t]$ is an irreducible monic polynomial for which $f(A) = 0$. Show that f is the minimal polynomial of A.

11 HOMEWORK ASSIGNMENT 11 Assigned 4-8-12 – Due 4-16-12

Problems 1–3, 4b. (Weyr characteristic is defined Definition 3.28 on p'68 of [1]).

Problem 1. Suppose that the characteristic and the minimal polynomial of a linear operator T are as below. Find all possible Jordan canonical forms of T.

1. $f(t) = (t-2)^4(t-5)^3, g(t) = (t-2)^4(t-5)^3$,

2. $f(t) = (t-2)^4(t-5)^3, g(t) = (t-2)^2(t-5)^3$,

3. $f(t) = (t-2)^4(t-5)^3, g(t) = (t-2)(t-5)$.

Problem 2. Find all possible Jordan forms for all 8×8 matrices having $x^2(x-1)^3$ as a minimal polynomial.

Problem 3.

a. Show that if the charateristic polynomial of $A \in \mathbb{F}^{n\times n}$ splits to linear factors in $\mathbb{F}$, i.e. $\det(zI - A) = \prod_{j=1}^{n}(z - \lambda_j)$, then A is similar to $A^\top$.

b. Try to prove that for any $A \in \mathbb{F}^{n\times n}$, A is similar to $A^\top$. (**Hint**: Let $\mathbb{F}_1$ be a finite extension of $\mathbb{F}$, where $\det(zI - A)$ splits to linear factors. Then by part a, show that A and $A^\top$ are similar over $\mathbb{F}_1$. So there eixsts a matrix $X \in \mathbb{F}_1^{n\times n}$ such that $AX - XA^\top = 0$ and $\det X \neq 0$. Deduce now that one can choose X in $\mathbb{F}^{n\times n}$ such that $\det X \neq 0$.)

Problem 4. Recall that a matrix $A \in \mathbb{F}^{n\times n}$ is called diagonable if A is similar to a diagonal matrix over $\mathbb{F}$. A linear operator $T : \mathbf{V} \to \mathbf{V}$ is called diagonable if there is a basis in $\mathbf{V}$ such that T is represented by a diagonal matrix. Show

1. A is diagonable over $\mathbb{F}$ if and only if $\det(zI - A)$ splits to linear factors over $\mathbb{F}$, and the minimal characterstic polynomial of A has simple roots.

2. A is diagonable if the roots of $\det(zI - A)$ are in $\mathbb{F}$ whenever $(T - \lambda I)^m \mathbf{v} = \mathbf{0}$, for some positive integer m, then $(T - \lambda I)\mathbf{v} = \mathbf{0}$.

3. Suppose that the linear operator T is a projection, i.e. $T^2 = T$. Then T is diagonable.

4. Assume that $T, Q \in \mathbb{F}^{n\times n}$ are projections. Then T and Q are similar if and only if $\operatorname{rank} T = \operatorname{rank} Q$.

5. Let $n > 1$ be an integer, and consider the matrices $A = \mathbf{1}\mathbf{1}^\top \in \mathbb{F}^{n\times n}, \mathbf{1} = (1,\ldots,1)^\top \in \mathbb{F}^n$ and the diagonal matrix $\mathrm{diag}(n, 0, \ldots, 0) \in \mathbb{F}^{n\times n}$. Then A and B are similar if and only if the characteristics of $\mathbb{F}$ does not divide n.

12 HOMEWORK ASSIGNMENT 12
Assigned 4-16-12 – Due 4-25-12

A. Problem 1 on page 82 in [1]. (The system $\mathbf{x}_l = A_l\mathbf{x}_{l-1}$ is homogeneous.)
B. For the following matrices find the components of A as defined in Theorem 4.1 on page 75 in [1], find A^{100} and e^{At} using the components of A.

1. $\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$,

2. $\begin{bmatrix} 0 & 2 & -1 \\ 0 & -1 & 1 \\ 0 & -2 & 2 \end{bmatrix}$,

3. $\begin{bmatrix} 2 & 1 & -1 & 0 \\ 0 & 5 & -6 & -1 \\ 0 & 3 & -4 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$.

C. $A \in \mathbb{R}^{n\times n}$ is called a stochastic matrix if all entries of A are nonnegative and the sum of each row is 1. (I.e. each row of A is a probability vector.) Show

1. for each positive integer k A^k is a stochastic matrix.
2. A is power bounded. (See Definition 4.5 in [1].)
3. 1 is an eigenvalue of A.
4. Each Jordan block corresponding to eigenvalue of 1 is of order 1.
5. Each eigenvalue λ of A satisfies $|\lambda| \le 1$.
6. A is power convergent iff and only if each eigenvalue λ of A different from 1, $|\lambda| < 1$. (See Definition 4.5 in [1].)

References

[1] S. Friedland, Linear Algebra II, Lectures Notes, Spring 2012, http://www2.math.uic.edu/~friedlan/lectnotesM425S12.pdf

[2] G.H. Golub and C.F. Van Loan. Matrix Computation, *John Hopkins Univ. Press, 3rd Ed.*, Baltimore, 1996.

[3] S.J. Leon, *Linear Algebra with Applications*, Prentice Hall, 6th Edition, 2002.

[4] S. Lipschutz and M. Lipson, *Linear Algebra*, Fourth Edition, Schaum's Outlines, McGraw-Hill, 2009.

[5] P. Petersen, *Linear Algebra*, http://www.math.ucla.edu/∼petersen/linalg3.pdf