1 HOMEWORK ASSIGNMENT 1
Assigned 1-9-12 – Due 1-18-12

Do the following problems from [5]: p’18: 1, 2, 3, 5, 6; p’ 312: 1,2. Note about Petersen’s notation: Mat$_{m,n}$ is $\mathbb{C}^{m \times n}$; $|A|$ is det $A$, the determinant of $A \in \mathbb{F}^{n \times n}$.

Additional problem. Let $z_1, z_2, \ldots, z_n \in \mathbb{C}$. The Vandermonde matrix is given as

$V(z_1, \ldots, z_n) := \begin{bmatrix}
1 & z_1 & z_1^2 & \cdots & z_1^{n-1} \\
1 & z_2 & z_2^2 & \cdots & z_2^{n-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & z_n & z_n^2 & \cdots & z_n^{n-1}
\end{bmatrix} \in \mathbb{C}^{n \times n}.$

Show that det $V(z_1, \ldots, z_n)$, called the Vandermonde determinant is equal to $\prod_{1 \leq i < j \leq n}(z_j - z_i)$.

2 HOMEWORK ASSIGNMENT 2
Assigned 1-23-12 – Due 2-3-12

a. 2 problems from §1.4.1, 4 problems from §1.4.2 from [1].

b. Let $\sigma \in S_5$ be defined as $\sigma(1) = 3, \sigma(2) = 5, \sigma(3) = 1, \sigma(4) = 4, \sigma(5) = 2$. Find sign($\sigma$).

c. $A, B \in \mathbb{F}^{m \times m}$ are called congruent if $A = TBT^T$ for some $T \in \text{GL}(n, \mathbb{F})$. Show

1. Congruence in $\mathbb{F}^{n \times n}$ is an equivalence relation.

2. Show that any two congruent matrices have the same rank

d. Assume that if $A \in \mathbb{F}^{n \times n}$ is a skew symmetric matrix.

1. Show that if $n$ is odd and $\mathbb{F}$ has characteristic not equal to 2, i.e. $2 \neq 0$ in $\mathbb{F}$, then det $A = 0$. 

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2. Show that if $F$ has characteristic not equal to 2, then $A$ is congruent to a block diagonal matrix $B = \text{diag}(B_1, \ldots, B_k)$, where each block is either $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ or $1 \times 1$ identity matrix. **Hint:** Use a sequence of "elementary conjugation" given by $EAE^\top$ where $E$ is an elementary matrix.

3. Show that if $F$ has characteristics 2, then $A$ is congruent to a block diagonal matrix $B = \text{diag}(B_1, \ldots, B_k)$, where each block is either or $1 \times 1$ identity matrix or $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. (Note that $-1 = 1$ in $F$.)

4. Given an example of $n \times n$ skew symmetric matrix, for a field with characteristic 2, whose determinant is nonzero for each $n \in \mathbb{N}$.

5. Assume that $F = \mathbb{R}$. Then $\det A \geq 0$.

### 3 HOMEWORK ASSIGNMENT 3
**Assigned 2-2-12 – Due 2-10-12**

Problems 2-8, §1.6.1, (page 16) in [1]; Problems 1-8, (page 18) in [1].

### 4 HOMEWORK ASSIGNMENT 4
**Assigned 2-6-12 – Due 2-15-12**

Do the following problems

1. Let $u = (1, -1, 1, -1)^\top$, $v = (2, 0, -2, 1)^\top$. Find
   
   (a) The cosine of the angle between $u$ and $v$.
   (b) The scalar and the vector projection of $v$ on $u$.
   (c) A basis to the orthogonal complement of $U := \text{span}(u, v)$.
   (d) The projection of the vector $(1, 1, 0, 0)^\top$ on $U$ and $U^\perp$.

2. Let $A \in \mathbb{R}^{4 \times 3}$. Assume that the vector $(1, -1, 1, -1)^\top$ is a vector in the column space of $A$. Is it possible that a vector $(2, 0, -2, 1)^\top$ is in the null space of $A^\top$?
   If yes give an example of such a matrix. If not, justify why.

3. Consider the overdetermined system

   \[
   \begin{align*}
   x_1 + x_2 + x_3 &= 4 \\
   -x_1 + x_2 + x_3 &= 0 \\
   -x_2 + x_3 &= 1 \\
   x_1 + x_3 &= 2
   \end{align*}
   \]

   (a) Is this system solvable?
   (b) Find the least squares solution of this system.
   (c) Find the projection of $(4, 0, 1, 2)^\top$ on the column space of the coefficient matrix $A \in \mathbb{R}^{4 \times 3}$ of this system.
4. Let \((-1, 0), (0, 1), (1, 3), (2, 9)\) be four points in the plane \((x, y)\) Find

(a) The best least squares fit by a linear function \(y = ax + b\).
(b) The best least squares by a quadratic polynomial \(y = ax^2 + bx + c\).
(c) Explain briefly why there exist a unique cubic polynomial \(y = ax^3 + bx^2 + cx + d\) passing through these four points.

5. Let \(a \leq t_1 < t_2 < \ldots < t_n \leq b\) be \(n\) points in the interval \([a, b]\). For any two continuous functions \(f, g \in C[a, b]\) define \(\langle f, g \rangle := \sum_{i=1}^{n} f(t_i)g(t_i)\). Let \(P_m\) be the vector space of all polynomials of degree at most \(m - 1\).

(a) Show that for \(m \leq n\) \(\langle \cdot, \cdot \rangle\) is an inner product on \(P_m\).
(b) Is \(\langle \cdot, \cdot \rangle\) an inner product on \(P_{n+1}\)? Justify!

6. For the inner product \(\langle f, g \rangle := \int_{-1}^{1} f(x)g(x)dx\) on \(C[-1, 1]\) Find the cosine of the angle between \(f(x) = 1\) and \(g(x) = e^x\).

Do the following problems from “Schaum’s Outline of Linear Algebra” by S. Lipschutz and M. Lipson, 4th edition, pages 258-261: 7.58, 7.60, 7.64, 7.71.

5 \hspace{1cm} HOMEWORK ASSIGNMENT 5
Assigned 2-14-12 – Due 2-22-12

Do the following problems. The problems in Schaum are from pages 260–262.

1. Problem 7.75 from Schaum. In addition do the following
   (a) Find the QR decomposition of the matrix \(A = [v_1 \ v_2 \ v_3]\).
   (b) Complete the orthonormal basis you found using the Gram-Schmidt problem to an orthonormal basis of \(\mathbb{R}^4\).

2. Problem 7.76 part a in Schaum.
4. Problem 7.91 in Schaum.
5. Problem 7.94 in Schaum.
6. Problem 1 page 23 in [1].
7. Problem 3 page 23 in [1]. \textbf{Hint}: try \(a_{pq} = z^{pq}\), where \(z = e^{\frac{2\pi i}{n}}\).
8. Problem 5 page 23 in [1].

6 \hspace{1cm} HOMEWORK ASSIGNMENT 6
Assigned 2-22-12 – Due 2-29-12

[1]: §2.3 page 28–29, Problems: 9(a,b,c), (special orthogonal means determinant one), 10a, 12.
[3]: §6.4 p’363-365, Problems: 4(a-f); 5(a,b,c,f),6,10,12,14.
7 HOMEWORK ASSIGNMENT 7
Assigned 2-26-12 – Due 3-7-12

I. Assume that A a real symmetric matrix. Denote by \( i_+(A) \) be the number of positive eigenvalues, \( i_0(A) \) the number of zero eigenvalues, \( i_-(A) \) be the number of negative eigenvalues. Denote \( i(A) := (i_+(A), i_0(A), i_-(A)) \). Show.

1. Show that \( i_+(A) \) is the dimension of the unique subspace \( U \subset \mathbb{R}^n \) such that \( x^\top Ax > 0 \) for each nonzero \( x \) in \( U \). \textbf{(Hint: Use the convoy principle.)}

2. Show that \( i_-(A) \) is the dimension of the unique subspace \( U \subset \mathbb{R}^n \) such that \( x^\top Ax < 0 \) for each nonzero \( x \) in \( U \).

3. \( \text{rank } A = i_+(A) + i_0(A) + i_-(A) \).

4. A symmetric \( B \in \mathbb{R}^{n \times n} \) is called congruent to \( A \) if \( B = QAQ^\top \) for some invertible matrix \( Q \). Show that two symmetric matrices are congruent if and only if \( i(A) = i(B) \). (This result is called the Sylvester law of inertia.)

II. State and prove the similar results in Problem I for a hermitian matrix. (This is the content of Problems 6 and 7 in [1, p’35-36].)

III. Let \( A = [a_{pq}] \in \mathbb{C}^{n \times n} \) be a hermitian matrix. Rearrange the diagonal entries of \( A, a_{11}, a_{22}, \ldots, a_{nn} \) in a nonincreasing way: \( \alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n \). Show

1. \( \lambda_1(A) \geq \alpha_1, \lambda_n(A) \leq \alpha_n \). \textbf{Hint:} Use the maximum and minimum characterization of \( \lambda_1(A), \lambda_n(A) \).

2. Show that \( \sum_{i=1}^k \alpha_i \leq \sum_{i=1}^n \lambda_i(A) \) for \( k = 1, \ldots, n \). What happens for \( k = n \)? \textbf{Hint:} Use the convoy principle.

3. Show that \( |\lambda_j(A)| \leq \sqrt{\sum_{p=q=1}^n |a_{pq}|^2} \) for each \( j = 1, \ldots, n \). For which kind of matrices and for which \( j \) we have equality in this inequality?

[1]: §2.5 page 34–35, Problems: 1,3.

IV. Let \( A = \begin{bmatrix} 1 & 1 + i & 2 - 3i \\ 1 - i & 3 & 3 - 2i \\ 2 + 3i & 3 + 2i & 2 \end{bmatrix} \).

1. Estimate from below and above \( \lambda_1(A) \) using the results of Problem III.

2. Find the eigenvalues of \( 2 \times 2 \) hermitian submatrix of \( A \) composed of the last two rows and columns of \( A \).

3. Estimate from below \( \lambda_1(A) \) using the Cauchy interlacing theorem, (Problem 3a on page 34 in [1]), and the results of part 2. Which estimate is better?

4. Use the Cauchy interlacing theorem and the results of part 2 to show that \( \lambda_3(A) \) is negative. Use the inequalities in Problem III to estimate from below \( \lambda_3(A) \).

5. Estimate \( \lambda_2(A) \) from below and above using Cauchy interlacing theorem by considering the eigenvalues of \( 2 \times 2 \) hermitian submatrix of \( A \) composed of the first two rows and columns of \( A \). Compare this estimate with the estimate using the results of part 2.
V. For the following symmetric matrices find a diagonal matrix which is congruent to it. In each case determine how many positive negative and zero eigenvalues $A$ has. Furthermore determine if there exist a lower triangular matrix $L$ with one on the diagonal such that $A = LDL^\top$

\[
\begin{bmatrix}
1 & 0 & 2 \\
0 & 3 & 6 \\
2 & 6 & 7
\end{bmatrix}
, \quad \begin{bmatrix}
1 & -2 & 1 \\
-2 & 4 & 3 \\
1 & 3 & 2
\end{bmatrix}
, \quad \begin{bmatrix}
1 & -1 & 0 & 2 \\
-1 & 2 & 1 & 0 \\
0 & 1 & 1 & 2 \\
2 & 0 & 2 & -1
\end{bmatrix}
\]

8  HOMEWORK ASSIGNMENT 8  
Assigned 3-09-12 – Due 3-16-12

1. Problem 1 [1, p’ 41].

2. Problems 3,4,9 [1, p’ 45], (In 9 you can assume that $A$ is a normal matrix.)

3. Problems 1-6,8 [3, p’ 380-382].

9  HOMEWORK ASSIGNMENT 9  
Assigned 3-13-12 – Due 3-30-12

1. Problems 1, 3, 4, 5, 9 [1, p’ 45–46].

2. Problems 1, 2,3 [1, p’ 54],

3. Problems 1-6,8 [3, p’ 380-382]. (If you did not do them yet!)

10  HOMEWORK ASSIGNMENT 10  
Assigned 3-31-12 – Due 4-6-12

Problems 1, 5, page 58 in [1],
Problems 1c, 2, 4 page 64 in [1]. ($A$ is called noderogatory if the minimal polynomial of $A$ equal to the characteristic polynomial of $A$.)

Additional problems:

1. Let $A \in \mathbb{F}^{n \times n}, B \in \mathbb{F}^{m \times m}$. Assume that $f, g \in \mathbb{F}[t]$ are the minimal polynomials of $A, B$ respectively. Form $C = \text{diag}(A, B) := \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$. Let $h$ be gcd, the greatest common divisor, of $f$ and $g$, which is assumed to be monic. Show that $\frac{f}{h}$ is the minimal polynomial of $C$.

2. Find the characteristic and the minimal polynomials of the following matrices

\[
\begin{bmatrix}
2 & 2 & -5 \\
3 & 7 & -15 \\
1 & 2 & -4
\end{bmatrix}
, \quad \begin{bmatrix}
2 & 5 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 \\
0 & 0 & 4 & 2 & 0 \\
0 & 0 & 3 & 5 & 0 \\
0 & 0 & 0 & 0 & 7
\end{bmatrix}
\]
3. Show that two similar matrices have the same minimal polynomial.

4. Let \[
\begin{bmatrix}
1 & 1 & 0 \\
0 & 2 & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad \begin{bmatrix}
2 & 0 \\
0 & 1 & 0 \\
0 & 0 & 2
\end{bmatrix}
\]
be different characteristic polynomials, but the same minimal polynomial.

5. Show that the square matrices \(A\) and \(A^\top\) have the same minimal polynomial.

6. Let \(A \in \mathbb{F}^{n \times n}\) and assume that \(f(t) \in \mathbb{F}[t]\) is an irreducible monic polynomial for which \(f(A) = 0\). Show that \(f\) is the minimal polynomial of \(A\).

11 HOMEWORK ASSIGNMENT 11
Assigned 4-8-12 – Due 4-16-12

Problems 1–3, 4b. (Weyr characteristic is defined Definition 3.28 on p’68 of [1]).

Problem 1. Suppose that the characteristic and the minimal polynomial of a linear operator \(T\) are as below. Find all possible Jordan canonical forms of \(T\).

1. \(f(t) = (t - 2)^4(t - 5)^3, g(t) = (t - 2)^4(t - 5)^3\),
2. \(f(t) = (t - 2)^4(t - 5)^3, g(t) = (t - 2)^2(t - 5)^3\),
3. \(f(t) = (t - 2)^4(t - 5)^3, g(t) = (t - 2)(t - 5)\).

Problem 2. Find all possible Jordan forms for all 8 × 8 matrices having \(x^2(x - 1)^3\) as a minimal polynomial.

Problem 3.

a. Show that if the characteristic polynomial of \(A \in \mathbb{F}^{n \times n}\) splits to linear factors in \(\mathbb{F}\), i.e. \(\det(zI - A) = \prod_{j=1}^n(z - \lambda_j)\), then \(A\) is similar to \(A^\top\).

b. Try to prove that for any \(A \in \mathbb{F}^{n \times n}\), \(A\) is similar to \(A^\top\). (Hint: Let \(\mathbb{F}_1\) be a finite extension of \(\mathbb{F}\), where \(\det(zI - A)\) splits to linear factors. Then by part a, show that \(A\) and \(A^\top\) are similar over \(\mathbb{F}_1\). So there exists a matrix \(X \in \mathbb{F}_1^{n \times n}\) such that \(AX - XA^\top = 0\) and \(\det X \neq 0\). Deduce now that one can choose \(X\) in \(\mathbb{F}^{n \times n}\) such that \(\det X \neq 0\).)

Problem 4. Recall that a matrix \(A \in \mathbb{F}^{n \times n}\) is called diagonable if \(A\) is similar to a diagonal matrix over \(\mathbb{F}\). A linear operator \(T : V \to V\) is called diagonable if there is a basis in \(V\) such that \(T\) is represented by a diagonal matrix. Show

1. \(A\) is diagonable over \(\mathbb{F}\) if and only if \(\det(zI - A)\) splits to linear factors over \(\mathbb{F}\), and the minimal characteristic polynomial of \(A\) has simple roots.
2. \(A\) is diagonable if the roots of \(\det(zI - A)\) are in \(\mathbb{F}\) whenever \((T - \lambda I)^m v = 0\), for some positive integer \(m\), then \((T - \lambda I)v = 0\).
3. Suppose that the linear operator \(T\) is a projection, i.e. \(T^2 = T\). Then \(T\) is diagonable.
4. Assume that \(T, Q \in \mathbb{F}^{n \times n}\) are projections. Then \(T\) and \(Q\) are similar if and only if \(\text{rank } T = \text{rank } Q\).
5. Let \( n > 1 \) be an integer, and consider the matrices \( A = 11^\top \in \mathbb{F}^{n \times n} \), \( 1 = (1, \ldots, 1)^\top \in \mathbb{F}^n \) and the diagonal matrix \( \text{diag}(n, 0, \ldots, 0) \in \mathbb{F}^{n \times n} \). Then \( A \) and \( B \) are similar if and only if the characteristics of \( \mathbb{F} \) does not divide \( n \).

12 HOMEWORK ASSIGNMENT 12
Assigned 4-16-12 – Due 4-25-12

A. Problem 1 on page 82 in [1]. (The system \( x_l = A_l x_{l-1} \) is homogeneous.)

B. For the following matrices find the components of \( A \) as defined in Theorem 4.1 on page 75 in [1], find \( A^{100} \) and \( e^{At} \) using the components of \( A \).

1. \[
\begin{bmatrix}
1 & 1 \\
-1 & 3
\end{bmatrix},
\]

2. \[
\begin{bmatrix}
0 & 2 & -1 \\
0 & -1 & 1 \\
0 & -2 & 2
\end{bmatrix},
\]

3. \[
\begin{bmatrix}
2 & 1 & -1 & 0 \\
0 & 5 & -6 & -1 \\
0 & 3 & -4 & -1 \\
0 & 0 & 0 & -1
\end{bmatrix}.
\]

C. \( A \in \mathbb{R}^{n \times n} \) is called a stochastic matrix if all entries of \( A \) are nonnegative and the sum of each row is 1. (I.e. each row of \( A \) is a probability vector.) Show

1. for each positive integer \( k \) \( A^k \) is a stochastic matrix.

2. \( A \) is power bounded. (See Definition 4.5 in [1].)

3. 1 is an eigenvalue of \( A \).

4. Each Jordan block corresponding to eigenvalue of 1 is of order 1.

5. Each eigenvalue \( \lambda \) of \( A \) satisfies \( |\lambda| \leq 1 \).

6. \( A \) is power convergent iff and only if each eigenvalue \( \lambda \) of \( A \) different from 1, \( |\lambda| < 1 \). (See Definition 4.5 in [1].)

References


