GRADING: Each problem 2pts. Each direction 1p. 2/7/10

Date 2/6/10

Solution to HW2 - MCS521 - SPRING 2010

A1 Show: (1) $\exists \underline{x} \geq 0 \quad A\underline{x} \leq \underline{b} \iff$ (2) for each $y \geq 0$ satisfying $y^T A \geq 0$ one has $\underline{y}^T \underline{b} \geq 0$

(1) $\Rightarrow$ (2) Suppose $y \geq 0$, $y^T A \geq 0$ & $\exists \underline{x} \geq 0, A\underline{x} \leq \underline{b}$.

Since $y^T \geq 0 \Rightarrow \underline{y}^T (A\underline{x}) \leq \underline{y}^T b$

$y^T(Ax) = (y^T \underline{A})\underline{x} \geq 0$ as $\underline{y}^T A \geq 0$ & $\underline{x} \geq 0$ QED.

$(1)^N \Rightarrow (2)^N$ Put (1) in Farkas's lemma form

$\begin{bmatrix} A \\ -I \end{bmatrix} \underline{x} \leq \begin{bmatrix} b \\ 0 \end{bmatrix}$ So this is not solvable hence dual problem solvable $\underline{z}^T = [\underline{y}^T, W^T] \geq \underline{0}$ $\quad \underline{z}^T \begin{bmatrix} A \\ -I \end{bmatrix} = \underline{0}$,

$0 > \underline{z}^T \begin{bmatrix} b \\ 0 \end{bmatrix} = \underline{y}^T \underline{b}$ $\quad y^T A - \underline{W}^T = 0 \iff \underline{y}^T A = W^T \geq 0$

I.e. If $\underline{W}^T = y^T A \geq 0$ we showed that $(2)^N$ holds

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A.2 (1) $\exists \ \underline{x} > 0$ s.t. (2) $A \ x = \underline{0} \iff$ (2) $\forall y$ s.t. $A^T \underline{y} \geq 0$ one has $\underline{y}^T A = 0$

(1) $\Rightarrow$ (2) Suppose that $A^T \underline{y} \geq 0$, i.e. $\underline{y}^T A \geq 0$

So

$0 = \underline{y}^T \underline{0} = (\underline{y}^T A) \underline{x}$ but $\underline{x} > 0 \Rightarrow \underline{y}^T A = 0$

$(1)^N \Rightarrow (2)^N$ $x \geq \epsilon \underline{1}$, $A \ \underline{x} = \underline{0}$ not solvable for any $\epsilon > 0$

$\underline{1} = (1, \ldots, 1)^T$ (Note that since $A\underline{x} = \underline{0}$ homogeneous we can assume $\epsilon = 1$!)

$\begin{bmatrix} A \\ -A \\ -I \end{bmatrix} \underline{x} \leq \begin{bmatrix} 0 \\ 0 \\ -\epsilon \underline{1} \end{bmatrix} = \epsilon \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} = \epsilon \underline{b}$ So not solvable for any $\epsilon > 0$.

By Farkas's lemma for inequalities means the dual solvable

$[\underline{y}_1^T, \underline{y}_2^T, \underline{w}^T] \begin{bmatrix} A \\ -A \\ -I \end{bmatrix} = 0$ and $\underline{z}^T \epsilon \underline{b} = -\epsilon \underline{y}_3^T \underline{1} < 0$

$\underline{z}^T \ldots \quad \underline{y} = \underline{y}_1 - \underline{y}_2 \quad \underline{y}^T A = \underline{y}_3^T \geq 0$ & $\underline{y}^T A \underline{1} > 0$

A.3 (1) $\underline{x} \neq \underline{0}$ satisfying $\underline{x} \geq 0$ $A\underline{x} = \underline{0}$ $\iff$ (2)

(2) $\nexists \underline{y}$ such that $\underline{y}^T A > 0$ Date

(1) Equivalent to solvability $\underline{x} \geq 0$, $A\underline{x} = \underline{0}$ $\underline{1}^T\underline{x} = 1$

Farkas's lemma for equalities (1)' $\begin{bmatrix} A \\ 1^T \end{bmatrix} \underline{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \iff$

(2') $\nexists$ $\underline{z}$ satisfying $\underline{z}^T \begin{bmatrix} A \\ 1^T \end{bmatrix} \geq \underline{0}$ and $\underline{z}^T \begin{bmatrix} 0 \\ 1 \end{bmatrix} < 0$.

$\underline{z}^T = [\underline{y}^T, t]$ $\quad \underline{y}^T A + t\underline{1} \geq \underline{0}$ $\quad t < 0 \Rightarrow \underline{y}^T A > \underline{0}$

$(\underline{y}^T A \geq -t\underline{1} > 0)$

$\boxtimes$

A4. $\exists \underline{x}$ s.t. (1) $A\underline{x} < \underline{b}$ $\iff$ (2) $\underline{y} = \underline{0}$ is the only solution to $\underline{y} \geq 0$, $\underline{y}^T A = \underline{0}$ $\underline{y}^T \underline{b} \leq 0$ $A \in \mathbb{R}^{m \times n}$

(1) $\Rightarrow$ (2) Assume $A\underline{x} < \underline{b}$ & $\underline{y} \geq 0$, $\underline{y}^T A = \underline{0}$, $\underline{y}^T \underline{b} \leq 0$

Suppose $\underline{y} \neq 0$ $\sum_{i=1}^m y_i((A\underline{x})_i - b_i) < 0$ as some $y_i > 0$

As $\underline{y}^T A = 0$ $\sum_{i=1}^m y_i (A\underline{x})_i = (\underline{y}^T A)\underline{x} = 0$. We obtained that $\underline{y}^T\underline{b} > 0$ $\Rightarrow$ SO No solution to $\underline{y} \gneq \underline{0}$, $\underline{y}^T A = \underline{0}$ and $\underline{y}^T \underline{b} \leq 0$,

(2)$^N \Rightarrow$ (1)^N Suppose first that there exists $\underline{y} \geq 0$, $\underline{y}^T A = \underline{0}$ and $\underline{y}^T \underline{b} < 0$. So $\underline{y} \neq 0$. By Farkas's lemma $A\underline{x} \leq \underline{b}$ not solvable. Hence $A\underline{x} < \underline{b}$ is not solvable.

It is left to discuss the case where $\exists \underline{y} \geq 0$ $\underline{y} \neq 0$ and $\underline{y}^T \underline{b} = 0$. Consider the system $A\underline{x} \leq \underline{b} - \varepsilon\underline{1}$ for some $\varepsilon > 0$. As $\underline{y}^T(\underline{b} - \varepsilon\underline{1}) = \underline{y}^T\underline{b} - \varepsilon \underline{y}^T\underline{1} = -\varepsilon \underline{y}^T \underline{1} < 0$ by Farkas's lemma $A\underline{x} \leq \underline{b} - \varepsilon\underline{1}$ is not solvable.

Since ε was an arbitrary positive $\Rightarrow$ $A\underline{x} < \underline{b}$ not solvable

$\boxtimes$

A5 Motzkin thm

Date

(1) $\exists\ \underline{x}$ satisfying $A\underline{x} < b$ and $A'\underline{x} \leq \underline{b}'$

iff (2) For all vectors $\underline{y}, \underline{y}' \geq 0$ one has

(i) if $\underbrace{\underline{y}^T A + (\underline{y}')^T A' = \underline{0}}_{(2a)}$ then $\underline{y}^T \underline{b} + (\underline{y}')^T \underline{b}' \geq 0$.

(ii) if $\underline{y}^T A + \underline{y}'^T A' = \underline{0}$ and $\underline{y} \neq 0$ then $\underline{y}^T b + \underline{y}'^T \underline{b}' > 0$

(1) => (2) easy. $\underline{y}^T A\,\underline{x} \leq \underline{y}^T \underline{b}$ (equality iff $\underline{y} = \underline{0}$)

$\underline{y}'^T A' x \leq \underline{y}'^T \underline{b}'$ as $\underline{y}, \underline{y}' \geq 0$.

Hence adding $(\underline{y}^T A + \underline{y}'^T A')\,x \leq \underline{y}^T \underline{b} + \underline{y}'^T \underline{b}'$

$\Rightarrow \quad 0 \leq \underline{y}^T \underline{b} + \underline{y}'^T \underline{b}'$

equality holds iff $\underline{y} = 0$.

$(1)^N \Rightarrow (2)^N$. Suppose first that (i) does not hold. Then Farkas's lemma yields that

$A\underline{x} \leq \underline{b}$ and $A'\underline{x} \leq \underline{b}'$ not solvable.

So assume that (i) holds, suppose first that

$\underline{y}^T A + (\underline{y}')^T A' = \underline{0}$, $\underline{y} \geq 0$, $\underline{y}' \geq 0 \Rightarrow \underline{y} = 0$.

Use Farkas's lemma to deduce that $A\underline{x} \leq \underline{b} - \varepsilon\underline{1}$, $A'\underline{x} \leq \underline{b}'$ solvable for any $\varepsilon \geq 0$. Hence (1) holds - contrary to the assumption that $(1)^N$ holds.

So now assume that $\exists\ \underline{y} \gneq 0, \underline{y}' \geq 0$

$\underline{y}^T A + \underline{y}'^T A' = 0$ and $\underline{y}^T \underline{b} + (\underline{y}')^T \underline{b}' \geq 0$

Consider the LP: $\underline{y} \geq 0$, $\underline{y}' \geq 0$, $\underline{y}^T A + (\underline{y}')^T A' = 0$, $\underline{y}^T \underline{1} = 1$,

$\underline{y}^T \underline{b} + (\underline{y}')^T \underline{b}' \geq 0$. Find $\inf \underline{y}^T \underline{b} + (\underline{y}')^T \underline{b}' = \beta$

Since the inequalities solvable and $\Rightarrow \beta \geq 0$, by duality theorem (Cor. A.3), minimum is achieved. If $\beta > 0$

Then use Farkas' lemma to see that $A\underline{x} \leq \underline{b} - \beta'\underline{1}$, $A'\underline{x} \leq \underline{b}'$ solvable. So this is not the case. Thus $\exists\ \underline{y}, \underline{y}' \geq 0$

$\underline{y}^T 1 = 1 \quad \underline{y}^T \underline{b} + (\underline{y}')^T \underline{b}' = 0$. QED.

Date

A.6 $\underline{x} \geq 0$, $A\underline{x} \leq b$ Equivalent to
$\begin{bmatrix} A \\ -I \end{bmatrix} \underline{x} \leq \begin{bmatrix} b \\ 0 \end{bmatrix} = \underline{b}'$

So $\max \underline{c}^T \underline{x}$ is dual to
$\underline{z}^T = [\underline{y}^T, \underline{u}^T] \geq 0$ $[y^T, u^T] \begin{bmatrix} A \\ -I \end{bmatrix} = \underline{c}^T \Leftrightarrow \underline{y}^T \underline{A} = \underline{c}^T + \underline{u}^T$

and $\min \underline{z}^T \underline{b}' = \min \underline{y}^T \underline{b}$.

Now note that $\underline{u}^T \geq 0 \Longleftrightarrow y^T A - \underline{c}^T \geq 0$

Hence the dual problem is $\underline{y}^T \geq \underline{0}$ $\underline{y}^T A \geq \underline{c}^T$
$\min \underline{y}^T \underline{b}$

Now apply the Duality Thm A.5. p'329

A9 (i) Suppose $A\underline{x} \leq b$ nonempty.
If (2) $\underline{y} \geq 0$ $\underline{y}^T A = \underline{c}^T$ solvable then by Duality Thm $\max\{\underline{c}^T\underline{x} : A\underline{x} \leq b\} = \min\{\underline{y}^T b : \underline{y} \geq 0, \underline{y}^T A = \underline{c}^T\}$

If (2) is not solvable then $\sup\{\underline{c}^T \underline{x} : A\underline{x} \leq b\} = \infty$
(I showed that in the class and it follows from Cor. A.3 p'327)

For empty set $y \geq 0$, $\underline{y}^T A = \underline{c}^T$ we define $\inf = \infty$

Similar discussion of $\underline{y}^T A = \underline{c}^T$, $\underline{y} \geq 0$ solvable.

(ii) $x_1 + x_2 \leq 1$ $A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$ $b = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$
$-x_1 - x_2 \leq -2$

$c^T = [1, 2]$ $[y_1, y_2] \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = [1, 2]$

both impossible to solve

A.12 $\overset{m \times n}{A} \underline{x} = \underline{b}$ Introduce additional variables $\underline{z} \geq 0$
$\underline{x} + \underline{z} = \underline{u}$ $\hat{A} = \begin{bmatrix} A & 0 \\ I & I \end{bmatrix} \begin{bmatrix} \underline{x} \\ \underline{z} \end{bmatrix} = \begin{bmatrix} \underline{b} \\ \underline{u} \end{bmatrix}$ $\underline{x} \geq 0$, $\underline{z} \geq 0$

$\max \underline{w}^T \underline{x} = \max [\underline{w}^T, 0] \begin{bmatrix} \underline{x} \\ \underline{z} \end{bmatrix}$. Dual system $[\underline{y}^T, \underline{v}^T] \begin{bmatrix} A & 0 \\ I & I \end{bmatrix} \geq [\underline{w}^T, 0]$

$\underline{y}^T A + \underline{v}^T \geq \underline{w}^T$ $\underline{v}^T \geq 0 \Rightarrow$

Note that rank $\hat{A}$ = rank $A + n$ $(A \in \mathbb{R}^{m \times n})$

So any fundamental $\hat{B}$ has rank $A + n$, so it is a solution of $A\underline{x} = \underline{b}$ $0 \leq \underline{x} \leq \underline{u}$. The dual solution determines $\underline{y}$, $\underline{v}$ and now ~~we~~ look which $(\underline{y}^T A + \underline{v})_i < w_i$

Explanation why in the simplex implementation. one only keeps $0 \le x \le u$

$$B_T = \begin{bmatrix} A_T & I \\ I & 0 \end{bmatrix}$$ where T is a ~~number~~ set of l.i. columns of A such that $A_T x_T = b_T$, $x_T \geq 0$.

$$B_T^T = \begin{bmatrix} A_T^T & I \\ 0 & I \end{bmatrix} \begin{bmatrix} y \\ v \end{bmatrix} = \begin{bmatrix} w_T \\ 0 \end{bmatrix}$$

so $v = 0$ and we get back to

$A_T^T y = w_T$ as without the constraint $x \le u$

Hence the condition $y^T A + v^T \geq w^T$

Reduces to $y^T A \geq w^T$ and we need to take i which violates the condition

$(y^T A)_i \geq w_i$ i.e. $(y^T A)_i < w_i$.