Entropy of holomorphic and rational maps

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1 Topological entropy

Y compact metric $d : Y \times Y \to \mathbb{R}_+$

$X \subseteq Y$, $f : X \to X$ continuous

How complicated is the dynamics of f?

Topological Entropy $h(f)$ - AKM 65

Case I: X compact

$d_n(x, y) := \max_{0 \le i \le n-1} d(f^i(x), f^i(y))$

$N(f, \epsilon, n)$ maximal # of $\epsilon > 0$ separated points in metric $d_n(x, y)$

$h_X(f) := \lim_{\epsilon \searrow 0} \limsup_{n \to \infty} \frac{\log N(f,\epsilon,n)}{n}$

Maximum principle $h(f) = \max h_\mu(f)$

Example: $Y = \mathbb{CP}$

1. $X = Y$, $f : Y \to Y$ holomorphic Lyubich 83

   f rational $h(f) := h_Y(f) = \log \deg f$

2. $f = z^d$, $X = \{|z| \le R\}$

   $h_X(f) = \log d$ if $R = 1$

   $h_X(f) = 0$ if $R \in [0, 1)$

2 Entropy and Homology

X a compact smooth manifold of dimension m

$H_*(X) = H_*(X,\mathbb{Z}) = H_*(X,\mathbb{Q}) \oplus$ Torsion

$H^*(X) = H^*(X,\mathbb{R})$ - de Rham cohomology

$f : X \to X$ induces
$f_* : H_j(X,\mathbb{Z}) \to H_j(X,\mathbb{Z}), j = 1,\ldots,m$
$\rho_j(f_*)$ spectral radius of $f_*|H_j(X,\mathbb{Q})$,
$\rho(f_*) := \max_{1\le j\le m} \rho_j(f_*)$

f smooth $f^* : H^*(X) \to H^*(X)$

M. Shub conjecture 1974: $h(f) \ge \log \rho(f^*)$

Reason: if f is axiom A diffeo
$h(f) = \limsup_{n\to\infty} \frac{\log(\text{Fix } f^n)}{n}$

Lefschetz formula $|\text{Trace} f_*^n| \le \text{Fix } f^n$

3 Results for compact manifolds

X smooth compact of dimension m, $f : X \to X$

- Sinai 70's $f : T^m \to T^m, T^m := \mathbb{R}^m/\mathbb{Z}^m$
  $f(\mathrm{x}) = A\mathrm{x}, A \in \mathbb{Z}^{m\times m}$
  $h(f) = \sum_{i=1}^{m} \log^+ |\lambda_i(A)|$

  if A has k eigenvalues outside the unit disc
  $h(f) = \log \rho_k(f_*) = \log \rho(f_*)$

- F. Przytycki 1980 $f \in C^2 \Rightarrow h(f) \geq \log \deg f$

- Gromov ms 1977, pub. 2003 $f : \mathbb{CP}^m \to \mathbb{CP}^m$ holomorphic $h(f) = \log \deg f = \log \rho_m(f_*)$.

- Yomdin 1987: $h(f) \geq \log \rho(f_*)$ for smooth f

- Newhouse 1988 for smooth f $h(f)$ is the volume growth

- Friedland 1991 X projective variety f holomorphic
  $h(f) = \log \rho(f_*)$

- Friedland 1995 X Kähler manifold f holomorphic
  $h(f) = \log \rho(f_*)$

4 K3 surfaces

- S. Cantat 2001 Dynamique de automorphismes des surfaces K3, Acta Math.
- McMullen 2005 Dynamics on K3 surfaces: Salem numbers and Siegel disks

5 Polynomial automorphisms of $\mathbb{C}^2$

Friedland-Milnor: 1989 studied dynamics of polynomial automorphism Aut $\mathbb{C}^2$

continuous maps of $S^4 = \mathbb{C}^2 \cup \{\infty\}$

Generalized Hénon map $H(x, y) = (y, p(y) - bx)$

$p(y) = y^m + \sum_{i=2}^{m} a_i y^{m-i}, b \neq 0$

$\mathrm{Deg\ H = m > 1}$

(Hénon map-$\mathrm{Deg\ H} = 2$ Hénon - 69)

Dichotomy: for $f \in$ Aut $\mathbb{C}^2$ $\exists G \in$ Aut $\mathbb{C}^2$

- either $f = G \circ H_1 \circ \ldots \circ H_l \circ G^{-1}$
  then $h(f) = \log \prod_{i=1}^{l} \mathrm{Deg\ H_i}$
- or $f = G \circ E \circ G^{-1}$,
  $E = (ax + p(y), by + c), p \in \mathbb{C}[y], ab \neq 0$
  then $h(f) = 0$

Nonaffine automorphisms of $\mathbb{C}^2$ are rational maps of $\mathbb{CP}^2$

6 Rational maps

$Y \subseteq \mathbb{CP}^N$ smooth irreducible projective variety

$f : Y \to Y$ rational dominating nonholomorphic map

Problem: Define and compute entropy of f

- Z a countable union of strict irreducible subvarieties of Y, $X := Y \backslash Z$, $f : X \to X$ holomorphic:
  $Z_0 \subsetneqq Y$ is the variety where f is not defined
  $Z = \cup_{i=0}^{\infty} f^{-i}(Z_0)$

- For each compact $K \subset X$, $h(f, K)$ the exponential growth of ϵ-separated set in K
  $h_B(f) = \sup_{K \Subset X} h(f, K)$

- $h_0(f)$ defined as above for X compact

- $\mathcal{Y} := Y^{\mathbb{N}}$ is compact
  $d_{\mathcal{Y}}((u_i), (v_i)) = \sum_{i=1}^{\infty} 2^{-i} d(u_i, v_i)$
  $\sigma : \mathcal{Y} \to \mathcal{Y}, (y_i)_{i=1}^{\infty} \mapsto (y_{i+1})_{i=1}^{\infty}$
  $\mathcal{X}$-orbit space Closure $\{(f^{i-1}(x))_{i=1}^{\infty}, x \in X\}$
  $f : X \to X$ lifts to $\sigma : \mathcal{X} \to \mathcal{X}$ (complex space)
  $h_F(f) := h(\sigma | \mathcal{X})$

- $f^i : \mathcal{X} \to \mathcal{X}$ possible to define $(f^i)_* : \tilde{H}_*(Y) \to \tilde{H}_*(Y)$:
  $\tilde{H}_*(Y)$ spanned by all cycles not contained in Z
  $h_H(f) := \limsup_{i\to\infty} \frac{\log \rho((f^i)_*)}{i}$
  Example $f^n = G \circ (H_1 \circ \ldots \circ H_l)^n \circ G^{-1}$
  $\rho_1(f^n) = |\mathrm{Deg\ f^n}| =$
  $\mathrm{Deg\ (G)^2 (\prod_{i=1}^{l} Deg\ H_i)^n} \Rightarrow$
  $h_H(f) = \log \prod_{i=1}^{l} \mathrm{Deg\ H_i}$

- The graph construction: $\Gamma :=$ Closure $\{(x, f(x)) \in Y \times Y : x \in \mathcal{X}\}$
  $\Gamma^k := ((x_i)_{i=1}^k \in Y^k : (x_i, x_{i+1}) \in \Gamma$, $i = 1, \ldots, k-1$
  Γ^k has natural volume vol Γ^k induced by Y^k
  $h_G(f) := \limsup_{k\to\infty} \frac{\mathrm{vol}\ \Gamma^k}{k}$
  Volume Growth Gromov

7 Inequalities

A. $h_B(f) \leq h_0(f)$

Example $f(z) = z^2$, $X = \{z : |z| < 1\}$:
$h_B(f) = 0, h_0(f) = \log 2$

B. $h_0(f) \leq h_F(f)$

C. $h_F(f) \leq h_G(f)$ (Gromov)

D. $h_G(f) = h_H(f)$ (Dinh-Sibony)

Conjecture: For rational dominating maps all the above definitions of entropy are the same

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