Invariant measures of groups of homeomorphisms and Auslander's conjecture

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September 26, 1994

Abstract. We give sufficient conditions for a group of homeomorphisms of a compact Hausdorff space to have an invariant probability measure. For a complex projective space $\mathbf{CP}^n$ we give a necessary condition for a subgroup of $Aut(\mathbf{CP}^n)$ to have an invariant probability measure. We discuss two approaches to Auslander's conjecture.

§0. Introduction

Let X be a compact Hausdorff space, $\mathcal{B}$ its Borel σ algebra generated by open sets and $\Pi(X)$ the set of all σ-additive probability measures on X. Assume that $T : X \to X$ is a continuous transformation. Then Krylov-Bogolyubov's theorem claims that T has an invariant probability measure $\mu \in \Pi(X)$. Assume that $\mathcal{G}$ is a group of homeomorphisms of X. The main problem discussed here is when $\mathcal{G}$ has an invariant probability measure. In Section 1 we recall the known result that $\mathcal{G}$ has an invariant measure if $\mathcal{G}$ is amenable. See [**Gre**] or [**Zim**]. We show that for our particular problem one can relax slightly the amenability conditions on $\mathcal{G}$. That is, it is enough assume that $\mathcal{G}$ is an amenable extension of $\mathcal{G}_0$ which has an invariant measure.

We then study necessary conditions for $\mathcal{G} \subset Aut(\mathbf{CP}^n)$ to have an invariant measure. These conditions give rise to sufficient conditions on finitely generated group $\Gamma \subset GL_{n+1}(\mathbf{C})$ to have a free subgroup on two generators. In the last section we discuss Auslander's conjecture.

§1. Invariant measures

Let $\mathcal{G}$ be a group. Denote by $B(\mathcal{G})$ the space of all bounded complex-valued function on G equipped with the sup norm $\|f\|_\infty$. Then $\mathcal{G}$ acts from the left on $B(\mathcal{G})$. $\mathcal{G}$ is called amenable if it has a nonnegative bounded linear functional $\mu : B(\mathcal{G}) :\to \mathbf{R}, \mu(1_\mathcal{G}) = 1$ which is invariant under the (left) action of $\mathcal{G}$. (This functional is called a left invariant mean LIM.) This notion was introduced by von Neumann [**Neu**] to overcome the Hausdorff-Banach-Tarski paradox [**B-T**]. Von Neumann showed that:
(I) any abelian group is amenable;
(II) subgroups, factor groups, groups extensions and direct union of amenable groups are amenable.

In particular any solvable group is amenable. It is known that an amenable group can not have a free subgroup on two generators.

Let $\mathcal{G}$ be a topological group. A continuous function $f : \mathcal{G} :\rightarrow \mathbf{C}$ is called *right uniformly continuous* if, given $\epsilon > 0$, there exists a neighborhood $U(\epsilon)$ of identity such that:

$$|f(x) - f(yx)| < \epsilon \; \forall x \in \mathcal{G}, y \in U(\epsilon).$$

Denote by $UCB_r(\mathcal{G})$ the Banach space of right uniformly continuous bounded functions on $\mathcal{G}$. We say that $\mathcal{G}$ has the *fixed point property* if, whenever $\mathcal{G}$ acts affinely on a compact convex set S in a locally convex convex space E, with the map $\mathcal{G} \times S \rightarrow S$ continuous, there is a point s_0 fixed under the action of G. Rickert [**Ric**] showed that a locally compact group $\mathcal{G}$ has a fixed point property iff $\mathcal{G}$ has LIM on $UCB_r(\mathcal{G})$. Thus, locally compact group $\mathcal{G}$ is called amenable iff it has a fixed point property. (Compare with [**Zim**].) It then follows that the properties (I-II) hold.

Definition 1.1. *Let $\mathcal{G}$ be a locally compact group and $\mathcal{G}_0$ its subgroup. $\mathcal{G}$ is called an amenable extension of $\mathcal{G}_0$ if there exists a normal subgroup $\mathcal{H} \subset \mathcal{G}_0$ which is normal in $\mathcal{G}$ so that $\mathcal{G}/\mathcal{H}$ is amenable.*

Theorem 1.2. *Let $\mathcal{G}$ be a locally compact group which acts affinely on a compact convex set S in a locally convex space E. Assume that $s_0 \in S$ is a fixed point of $T \subset \mathcal{G}$. Let $\mathcal{G}_0$ be a subgroup generated by T. If $\mathcal{G}$ is an amenable extension of $\mathcal{G}_0$ then $\mathcal{G}$ has a fixed point in S.*

Proof. Let $\mathcal{H} \subset \mathcal{G}_0$ be a normal subgroup of $\mathcal{G}$ such that $\mathcal{G}/\mathcal{H}$ is amenable. Set $S_0 = \{s : s = gs_0, g \in \mathcal{G}\}$. As $\mathcal{H}$ fixes s_0 and is normal in $\mathcal{G}$ it follows that $\mathcal{H}$ fixes every element in S_0. Let $S' \subset S$ be the closure of the convex hull of S_0. Thus $\mathcal{H}$ fixes every element in S'. Hence, $\mathcal{G}_1 = \mathcal{G}/\mathcal{H}$ is a locally compact group which acts affinely on a compact convex set S'. As $\mathcal{G}_1$ can be assumed to be amenable we deduce that $\mathcal{G}_1$ has a fixed point $s_1 \in S'$. Clearly, s_1 is a fixed point of $\mathcal{G}$ in S. $\diamond$

As any finite group is amenable we deduce.

Corollary 1.3. *Let $\mathcal{G}$ be a locally compact group which acts affinely on a compact convex set S in a locally convex space E. Assume that $\mathcal{G}_0 \subset \mathcal{G}$ is a subgroup of a finite index. Then $\mathcal{G}$ has a fixed point in S iff $\mathcal{G}_0$ has a fixed point in S.*

In what follows we consider the following situation. (Compare with [**Zim**].) Let X be a compact Hausdorff space. Let $C(X)$ be the Banach space of complex valued continuous functions on X with the sup norm and let $C^*(X)$ be its conjugate space. Riesz representation theorem claims that any $\phi \in C^*(X)$ is represented by the unique finite σ-additive measure μ on $(X, \mathcal{B})$:

$$\phi(f) = \int_X f(x) d\mu(x) = \mu(f), f \in C(X).$$

Furthermore, $|\phi|$ - the norm of ϕ is equal to $|\mu|$ - the total variation of μ. Let $\Pi(X)$ be the set of all probability measures on X. Then $\Pi(X)$ is a convex compact set in w^* topology (the *weak** topology). Consult for example with [**D-S**, Ch. 5]. If X is a compact metric space then $C(X)$ is separable and $\Pi(X)$ is a compact convex metrisable space, e.g. [**Wal**, Ch. 6]. Thus, we are studying the case where $S = \Pi(X)$ and $E = C(X)^*$ with w^* topology.

Assume that $T : X \to X$ is a continuous transformation. Then T induces the linear transformation $\hat{T} : C(X) \to C(X)$ by the natural action $(\hat{T}f)(x) = f(Tx)$. Note that $|\hat{T}| = 1$. Let $\hat{T}^* : C(X)^* \to C(X)^*$ be the induced conjugate operator. Thus, for any finite σ-additive measure μ, $\hat{T}^*\mu$ is a finite σ-additive measure. Furthermore,

$$\hat{T}^*\mu(f) = \mu(\hat{T}f) = \int_X f(Tx)d\mu(x), f \in C(X).$$

Clearly, $|\hat{T}^*| = 1$. Set

$$P_n(T) = \frac{\sum_1^n (\hat{T}^*)^i}{n} : C^*(X) \to C^*(X), n = 1, ..., .$$

For simplicity of notation we identify $\hat{T}, \hat{T}^*$ with T and no ambiguity will arise. Note if μ is a probability measure then $P_n(T)\mu$ is also a probability measure. Hence $|P_n(T)| = 1$. Denote $\Pi(X,T) \subset \Pi(X)$ the set of all T invariant probability measures. Krylov-Bogolyubov theorem claims that $\Pi(X,T)$ is nonempty. Hence, $\Pi(X,T)$ is a convex w^* compact set. The Krein-Milman theorem yields that $\Pi(X,T)$ is the w^* closure of the convex hall of its extremal points $E(X,T)$. If X is a compact metric space then $E(X,T)$ is the set of all $\mu \in E(X,T)$ for which T is an ergodic measure-preserving transformation of $(X, \mathcal{B}, \mu)$ [**Wal**, Ch. 6].

Let $\mathcal{S}$ be a set of continuous transformations of X to itself. We say that $\mu \in \Pi(X)$ is an $\mathcal{S}$-invariant measure if $T\mu = \mu, \forall T \in \mathcal{S}$. Denote by $\Pi(X, \mathcal{S}) = \cap_{T \in \mathcal{S}} \Pi(X,T)$ the set of all $\mathcal{S}$-invariant probability measures. $\Pi(X, \mathcal{S})$ can be empty. Assume that $\Pi(X, \mathcal{S})$ is nonnempty. Then $\Pi(X, \mathcal{S})$ is a convex compact set in w^* topology. Denote by $E(X, \mathcal{S})$ the set of the extreme points of $\Pi(X, \mathcal{S})$. Let $\mu \in \Pi(X, \mathcal{S})$. We say that $\mathcal{S}$ is $\mu - ergodic$ if for any $\mathcal{S}$-invariant set $A \in \mathcal{B}$, i.e. $T^{-1}A = A, T \in \mathcal{S}$, the condition $\mu(A) = 0, 1$ is satisfied. The following lemma generalizes the the characterization of $\Pi(X, \mathcal{S})$.

Lemma 1.4. *Let X be a compact Hausdorff space and assume that $\mathcal{S}$ is a set of continuous transformations of X to itself. Assume that $\Pi(X, \mathcal{S})$ is nonempty. Then $E(X, \mathcal{S})$ consists of all $\mu \in \Pi(X, \mathcal{S})$ for which $\mathcal{S}$ is ergodic.*

Proof. Let $\mu \in \Pi(X, \mathcal{S})$. Assume that $\mathcal{S}$ is not μ-ergodic. Then there exists an invariant set $E \in \mathcal{B}$ so that $0 < \mu(E) < 1$. Let $\mu_1, \mu_2 \in \Pi(X)$ be the probability measures obtained by the restriction of μ to $E, X \backslash E$ respectively, i.e.

$$\mu_1(C) = \frac{\mu(C \cap E)}{\mu(E)}, \mu_2(C) = \frac{\mu(C \cap X \backslash E)}{\mu(X \backslash E)}, C \in \mathcal{B}.$$

As E is an $\mathcal{S}$-invariant set it follows that $\mu_1, \mu_2 \in \Pi(X, \mathcal{S})$. Thus $\mu = \mu(E)\mu_1 + (1-\mu(E))\mu_2$ and we deduce that $\mu \notin E(X, \mathcal{S})$.

Assume now that $\mathcal{S}$ is μ-ergodic. Assume to the contrary that $\mu \notin E(X, \mathcal{S})$. Let

$$\mu = p\mu_1 + (1-p)\mu_2, \mu_1 \neq \mu_2 \in \Pi(X, \mathcal{S}), 0 < p < 1.$$

We obtain a contradiction by showing that $\mu_1 = \mu$. As $p\mu_1 \leq \mu$, i.e. $p\mu_1(C) \leq \mu(C), \forall C \in \mathcal{B}$ it follows that μ_1 is absolutely continuous with respect to μ. Set $f = \frac{d\mu_1}{d\mu}$ to be the Radon-Nykodim derivative. As $\mu, \mu_1 \in \Pi(X, \mathcal{S})$ it follows that f is $\mathcal{S}$ invariant. That is, $f(Tx) = f(x)$ almost μ-everywhere for all $T \in \mathcal{S}$. As $\mathcal{S}$ is $\mu-$ergodic we deduce that $f = Constant$ almost $\mu-$everywhere. Finally, since μ, μ_1 are probability measures we get that $f = 1 \Rightarrow \mu_1 = \mu$ contrary to our assumptions. $\diamond$

Let $\mathcal{S}$ be a set of continuous maps of X to itself so that each pair $T_1, T_2 \in \mathcal{S}$ commutes. It is well known that $\mathcal{S}$ has an invariant probability measure μ on (X, μ). See for example [**D-S**, VI.9.41]. (This is the semigroup version property (I) of amenable groups.) We now give a generalization of this fact.

Lemma 1.5. *Let X be a compact Hausdorff space. Assume that $\mathcal{S}$ is a semigroup of continuous transformations of X to itself. ($T_1, T_2 \in \mathcal{S} \Rightarrow T_1T_2, T_2T_1 \in \mathcal{S}$.) Suppose that there exists a probability measure on $(X, \mathcal{B})$ so that*

$$T_1T_2\mu = T_2T_1\mu, \forall T_1, T_2 \in \mathcal{G}.$$

If either $\mathcal{S}$ is countable or X is a compact metric space then $\mathcal{S}$ has an invariant probability measure on $(X, \mathcal{B})$.

Proof. We first show that for any finite subset $\{T_1, ..., T_k\} \subset \mathcal{S}$ there exists a probability measure ν which is invariant under $\{T_1, ..., T_k\}$. Using the assumption that $\mathcal{S}$ is a semigroup and the "commutativity" of $\mathcal{S}$ with repect to μ we easily deduce

$$\begin{aligned} &T_1^{i_1}T_2^{i_2}...T_k^{i_k}\mu = T_2^{i_2}...T_k^{i_k}T_1^{i_1}\mu = ... = T_k^{i_k}T_1^{i_1}T_2^{i_2}...T_{k-1}^{i_{k-1}}\mu, \\ &1 \leq i_1, ..., i_k \in \mathbf{Z}. \end{aligned}$$

It then follows that

$$\begin{aligned} &P_n(T_1)P_n(T_2)...P_n(T_k)\mu = P_n(T_2)...P_n(T_k)P_n(T_1)\mu = ... \\ &= P(T_k)P_n(T_1)...P_n(T_{k-1})\mu = \mu_n. \end{aligned}$$

Note that μ_n is a probability measure. Let $\nu \in \Pi(X)$ lie in the w^* closure of the sequence $\{\mu_i\}_1^\infty$. That is, for a given $f \in C(X)$ there exists a subsequence $\mu_{n_i}, i = 1, ...,$ so that

$$\nu(f) = \lim_{i\to\infty} \mu_{n_i}(f), \nu(T_jf) = \lim_{i\to\infty} \mu_{n_i}(T_jf), j = 1, ..., k.$$

Observe next that

$$T_j\mu_n = T_jP_n(T_j)P_n(T_{j+1})...\mu = P_n(T_j)P_n(T_{j+1})...\mu + \frac{T_j^{n+1}-T_j}{n}P_n(T_{j+1})...\mu.$$

Hence

$$|T_j\mu_n - \mu_n| \le \frac{2}{n}.$$

We therefore deduce that $\nu(T_jf) = \nu(f), j = 1, ..., k$. As f was an arbitrary continuous function it follows that $T_j\nu = \nu, j = 1, ..., k$. Thus, if $\mathcal{S}$ is finite, we deduce the existence of invariant measure with respect to $\mathcal{S}$. Assume that $\mathcal{S}$ is infinite and let $\{T_i\}_1^\infty \subset \mathcal{S}$. Let ν_k be an invariant probability measure for the set $\{T_1, ..., T_k\}, k = 1, ...,$. Let $\nu \in \Pi(X)$ be in the w^* closure of the sequence $\{\nu_i\}_1^\infty$. It then follows that ν is an invariant measure for $\{T_i\}_1^\infty$. This argument proves the lemma in the case $\mathcal{S}$ is countable.

Assume that $\mathcal{S}$ is not countable and X is a compact metric space. Then $C(X)$ is separable, e.g. [**D-S**, V.7.12]. Let $\{f_i\}_1^\infty \subset C(X)$ be a dense set in the unit ball of $C(X)$. As $C(X)$ is separable it follows that for each i the family of functions $Tf_i, T \in S$ contains a sequence of of functions $\tilde{T}_jf_i, \tilde{T}_j \in \mathcal{S}, j = 1, ...,$ so that the closure of this sequence contains all the functions $Tf_i, T \in \mathcal{S}$. Hence, there exists a sequence $\{T_j\}_1^\infty \subset \mathcal{S}$ so that the closure of the sequence $T_1f_i, T_2f_i, ...,$ contains all the functions $Tf_i, T \in \mathcal{S}$ for each $i = 1, ...,$. Let ν be an invariant probability measure for the sequence $\{T_j\}_1^\infty$. Hence, $\nu(T_jf_i) = \nu(f_i), j = 1, 2, ...,$. As the closure of the sequence $T_1f_i, T_2f_i, ...,$ contains all the functions $Tf_i, T \in \mathcal{S}$ we deduce that $\nu(Tf_i) = \nu(f_i)$ for any $T \in \mathcal{S}$ and any i. As $f_1, f_2, ...,$ is dense in the unit ball of $C(X)$ it follows that $\nu(Tf) = \nu(f), \forall f \in C(X)$. Thus, ν is an invariant probability measure for $\mathcal{S}$. $\diamond$

We shall use the following result in the sequel. Assume that a group Γ has a faithful representation in $GL_n(\mathbf{k})$ for some field k of 0 characteristic. Then Tits alternative [**Tit**] yields that either Γ contains a free subgroup on two generators or Γ is virtually solvable. (Here we use the standard terminology that a group Γ has a virtual property (P) if it has a subgroup of a finite index which has the property (P).)

§2. Invariant measures of automorphisms of complex projective spaces

Let $\mathbf{F}$ be a field. The n dimensional projective space $\mathbf{FP}^n$ is given by a canonical projection

$$\pi : \mathbf{F}^{n+1}\backslash\{0\} \to \mathbf{FP}^n, \pi(\alpha x) = \pi(x), x = (x_1, ..., x_{n+1})^T \in \mathbf{F}^{n+1}\backslash\{0\}, \alpha \in \mathbf{F}^*.$$

Note that $\mathbf{F}^n$ is isomorphic to an open set in Zariski topology in $\mathbf{FP}^n$. (Set $x_{n+1} = 1$ in a canonical projection.) Thus, $\mathbf{FP}^n$ is a compactification of $\mathbf{F}^n$ respectively. Recall that $GLP_n(\mathbf{F}) : \mathbf{FP}^n \to \mathbf{FP}^n$ is the group of projective automorphisms induced by the action of $GL_{n+1}(\mathbf{F})$ on the covering space $\mathbf{F}^{n+1}$. Thus, $GLP_n(\mathbf{F}) \sim GL_{n+1}(\mathbf{F})/\mathbf{F}^*I$. Equivalently,

$$GLP_n(\mathbf{F}) \sim SGL_{n+1}(\mathbf{F})/C_{n+1}, C_{n+1} = \{\lambda I, \lambda^{n+1} = 1\}.$$

Here, $SGL_{n+1}(\mathbf{F})$ is the group of all $n+1 \times n+1$ matrices over F with determinant equal to 1. For $A \in GL_{n+1}(\mathbf{F})$ we let $T = \pi(A) \in Aut(\mathbf{FP}^n)$ be the automorphism T induced by a canonical projection $\pi : \mathbf{F}^{n+1}\backslash\{0\} \to \mathbf{FP}^n$. In what follows we let $\mathbf{F} = \mathbf{C}$ although some of the notions will be still valid for general fields. Denote by $U_{n+1} \subset GL_{n+1}(\mathbf{C}), SU_{n+1} \subset SGL_{n+1}(\mathbf{C})$ the groups of $n+1 \times n+1$ unitary and special unitary matrices. Let $UP_n \subset GLP_n(\mathbf{C})$ be the subgroup of all automorphisms of $\mathbf{CP}^n$ induced by U_{n+1} or SU_{n+1}. (Note that $UP_n \sim SU_{n+1}/C_{n+1}$.)

A subset $L \subset \mathbf{CP}^n$ is a called a linear subspace of $\mathbf{CP}^n$ of dimension k if its homogeneous coordinates describe the set $L'\backslash\{0\} = \pi^{-1}(L)$ where $L' \subset \mathbf{C}^{n+1}$ is $k+1$ dimensional subspace of $\mathbf{C}^{n+1}$. Vice versa, any $k+1$ dimensional subspace $L' \subset \mathbf{C}^{n+1}$ induces a unique k dimensional linear subspace $L = \pi(L'\backslash\{0\}) \subset \mathbf{CP}^n$. Note that $L \sim \mathbf{CP}^k, L' \sim \mathbf{C}^{k+1}$. Let $S^1 = \{\zeta, \zeta \in \mathbf{C}, |\zeta| = 1\}$ be the unit circle. We view S^1 and $S^1 \times ... \times S^1 \subset \mathbf{C}^m$ as compact abelian groups and tori of dimension 1 and m respectively. A compact set $\mathcal{A} \subset \mathbf{CP}^n$ is called a compact abelian group if there exists a linear k dimensional subspace $L \subset \mathbf{CP}^n$, a canonical projection $\phi : \mathbf{C}^{k+1}\backslash\{0\} \to L$, a compact abelian group $\mathcal{A}' \subset S^1 \times ... \times S^1 \subset \mathbf{C}^{k+1}$ so that $\mathcal{A} = \phi(\mathcal{A}')$. Let $\mathcal{A}''$ be obtained from $\mathcal{A}'$ by dividing any vector in $\mathcal{A}'$ by its last coordinate. It then follows that $\mathcal{A}'' \subset S^1 \times ... \times S^1 \subset \mathbf{C}^{k+1}$ is also a compact abelian group. Furthermore, $\phi(\mathcal{A}'') = \mathcal{A}$. The normalization that the last coordinate of any vector in $\mathcal{A}''$ is equal to 1 means that $\mathcal{A}'' \sim \mathcal{A}$ and we identify $\mathcal{A}''$ with $\mathcal{A}$ and no ambiguity will arise. Let $\mathcal{A}_0$ be the connected component of $\mathcal{A}$ containing the identity. Then $\mathcal{A}_0$ is an r dimensional torus. We call r the dimension of $\mathcal{A}$. Thus, $\mathcal{A}$ is a union of N r dimensional tori. By the Haar measure of $\mathcal{A}$ we mean the unique probability measure of $\mathcal{A}$ which is invariant under the action of $\mathcal{A}$ on itself.

Fix a canonical projection. For $z = (z_1, ..., z_{n+1})^T \in \mathbf{C}^{n+1}$ set $|z| = (\sum_1^{n+1} |z_i|^2)^{\frac{1}{2}}$. Let

$$\omega = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}|z|^2, z \in \mathbf{C}^{n+1}\backslash\{0\}$$

be the $(1,1)$ form corresponding to the above projection. Recall that for $1 \le k \le n$ the form $[\frac{\omega^k}{k!}]$ represents a generator of $H^{2k}(\mathbf{CP}^n, \mathbf{Z}) = \mathbf{Z}$. (The odd cohomology (homology) classes of $\mathbf{CP}^n$ over $\mathbf{Z}$ are trivial.) Note that $\frac{\omega^n}{n!} = d\mu, \mu \in \Pi(\mathbf{CP}^n)$. Let $Aut(\mathbf{CP}^n)$ be the group of (complex) automorphisms of n dimensional complex projective space $\mathbf{CP}^n$. It is well known that it is given by $GLP_n(\mathbf{C})$, e.g. [**G-H**]. Assume that $T \in Aut(\mathbf{CP}^n)$. Denote by $T^*\omega$ the pull back of ω by T. Clearly,

$$[\frac{\omega^k}{k!}] = [\frac{(T^*\omega)^k}{k!}], k = 1, ..., n.$$

Lemma 2.1. *Let $T \in Aut(\mathbf{CP}^n)$ be represented by $A \in GL_{n+1}(\mathbf{C})$ with respect to a fixed projection $\pi : \mathbf{C}^{n+1}\backslash\{0\} \to \mathbf{CP}^n$. Assume that ω is the $(1,1)$ form induced by the above projection. Then for a fixed $1 \le k \le n$*

$$\omega^k = (T^*\omega)^k \iff A = \lambda U, \lambda \in \mathbf{C}^*, U \in U_{n+1}.$$

In particular, UP_n is the subgroup of all automorphisms of $\mathbf{CP}^n$ which preserves the probability measure $\frac{\omega^n}{n!}$.

Proof. Assume that T is represented by $U \in U_{n+1}$. As $|Uz| = |z|, z \in \mathbf{C}^{n+1}$ it follows that $\omega = T^*\omega$. Recall that any $A \in GL_{n+1}(\mathbf{C})$ is of the form $A = UDV, U, V \in U_{n+1}, D = diag(d_1, ..., d_{n+1})$. Hence, it is enough to prove the lemma in the case T is represented by a diagonal matrix D. In that case a straightforward calculation shows that $\omega^k = (T^*\omega)^k$ iff $d_1 = ... = d_{n+1} = d$, i.e. $D = dI$. Choose $k = n$ to deduce the last claim of the theorem. $\diamond$

We note that UP_n is not virtually solvable. Indeed, assume to the contrary that UP_n is virtually solvable. It then follows that SU_{n+1} is virtually solvable. As SU_{n+1} connected we deduce that SU_{n+1} must be solvable. This is equivalent to the statement that the Lie algebra of SU_{n+1} is solvable which is false. Thus, UP_n, viewed as a discrete group, is not amenable but does have a fixed point in $\Pi(\mathbf{CP}^n)$.

The following theorem is a more precise version of some Furstenberg's results in [**Fur**].

Theorem 2.2. *Let $T \in Aut(\mathbf{CP}^n)$ be represented by $A \in GL_{n+1}(\mathbf{C})$ with respect to a fixed projection $\pi : \mathbf{C}^{n+1}\backslash\{0\} \to \mathbf{CP}^n$. Then the recurrent set $R(T)$ is equal to the union of k pairwise disjoint linear subspaces*

$$L_i = L_i(T) \subset \mathbf{CP}^n, dim(L_i) = n_i, i = 1, ..., k = k(T).$$

Let $L_i' \subset \mathbf{C}^{n+1}$ be the corresponding linear subspace of dimension $n_i + 1$. Then L_i' is spanned by all eigenvectors of A corresponding to all eigenvalues of A situated on some circle $|\zeta| = r > 0, \zeta \in \mathbf{C}$. In particular, $\sum_1^k n_i \le n + 1 - k$ and any T-invariant measure $\mu \in \Pi(\mathbf{CP}^n, T)$ is supported on $R(T)$. Any extreme measure $\mu \in E(\mathbf{CP}^n, T)$ is the Haar measure of a compact abelian group $\mathcal{A} \subset L_i$ which is a closure of $\{T^j x\}_{j=0}^{\infty}$ for some $x \in L_i$.

Proof. We first show that any point $x \in L_i$ is in $R(T)$. Let $\pi(z) = x, z = e_1 + ... + e_p$ where $e_1, ..., e_p$ are p linearly independent eigenvectors of A - $Ae_i = \lambda_i e_i, i = 1, ..., p, |\lambda_1| = ... = |\lambda_p| = r > 0$. Let $A' = \frac{1}{r}A$. Since A' represents also T we may assume that $r = 1$. Thus, $A^i z = \sum_{j=1}^p \lambda_j^i e_j$. That is, the coordinates of $A^i z$ are represented by a point $(\lambda_1^i, ..., \lambda_p^i) \subset S^1 \times \cdots \times S^1 \subset \mathbf{C}^p$. It then follows that all the limit points of $\{A^i z\}_1^{\infty}$ form a compact abelian group $\mathcal{A}'$ which contains the point z (corresponding to the identity element in $\mathcal{A}'$). Hence, the closure of $orb_T(x) = \{T^j x\}_0^{\infty}$ is the compact abelian group $\mathcal{A} \subset L_i$. In particular, $\cup_1^k L_i \subset R(T)$.

We now prove the containment $\cup_1^k L_i \supset R(T)$. Fix $0 \ne z \in \mathbf{C}^{n+1}$ and consider the sequence $\{A^j z\}_0^{\infty}$. Note that all the vectors in this sequence lie in the cyclic space $W = span\{z, Az, ..., A^n z\}$. Assume that $dimW = m$ and let $B = A_{|W}$. Choose a basis $e_1, ..., e_m$ so that so that B is represented in this basis as a Jordan matrix, i.e. is a basis composed of generalized eigenvectors of B. That is, each e_i satisifies the equality $(B - \lambda_i I)^{l_i} e_i = 0$. Assume that m_i is the minimal integer for which the above equality holds. Then m_i is called the index of e_i and denoted by $index(e_i)$ If $index(e_i) = 1$ then e_i is an eigenvector of B with corresponding eigenvalue λ_i. If $index(e_i) > 1$ then e_i is called a generalized eigenvalue corresponding to the eigenvalue λ_i. As usual, let $spec(B)$ denote the spectrum of B. Assume $\lambda \in spec(B)$, i.e. λ is an eigenvalue of B. Then

$j = index(\lambda)$ is the maximal index of all generalized eigenvectors cooresponding to λ. Let $z = \sum_1^m \xi_i e_i$. Assume that $index(e_i) = index(\lambda_i)$. As $e_1, ..., e_m$ is a Jordan basis and the dimension of the cyclic space generated by x is m it follows that $\xi_i \neq 0$, e.g. [**Gan**]. Let $\rho(B)$ be the spectral radius of B. Denote by $domspec(B) \subset spec(B)$ the dominant spectrum of B. That is, it is the set of all eigenvalues $\lambda \in spec(B)$ which lie on the maximal circle $|\zeta| = \rho(B)$ and which have the maximal index τ among all eigenvalues on the maximal circle. Equivalently, $domspec(B)$ is the set of all eigenvalues of B lying on the maximal circle to which correspond the maximal Jordan blocks of length τ. Assume that the number of these blocks is β. (Here, $domspec(B)$ is counted with multiplicites, according to the number of maximal Jordan blocks. That is, $domspec(B)$ has exactly $\tau\beta$ eigenvalues.) It is straightforward to show, e.g. use the explicit formulas for B^j in [**Gan**, Ch. 5], that the sequence $\frac{B^j x}{j^{\tau-1}\rho(B)^j}$ is bounded. Furthermore, all the accumulation points of this sequence correspond to a compact abelian group $\mathcal{A}' \subset \mathbf{C}^\beta$ in the subspace whose basis consists of β eigenvectors corresponding to β maximal Jordan blocks of the β eigenvalues in $domspec(B)$. (Note that this eigenvectors are determined uniquely.) This shows that the limit points any orbit $\{T^i x\}_0^\infty, x \in \mathbf{CP}^n$ is a compact abelian group $\mathcal{A}$ such that $\mathcal{A} \subset L_i$ for some L_i. $\mathcal{A}$ is generated in the way we described in the beginning of the proof. We thus showed that $R(T) = \cup_1^k L_i(T)$ and it is a closed set.

It is known that for any $\mu \in \Pi(\mathbf{CP}^n, T)$ one has the equality $\mu(R(T)) = 1$. See for example [**Wal**, §6.4] (the remark after Cor. 6.15.1). Hence all the invariant measures of T are supported on $\cup_1^k L_i(T)$. Let $\mu \in \Pi(\mathbf{CP}^n, T)$. Then μ is a convex combination of the restrictions of μ to those $L_i(T)$ for which $\mu(L_i(T)) > 0$. Thus any $\mu \in E(\mathbf{CP}^n, T)$ is supported exactly on one $L_i(T)$. As $TL_i(T) = L_i(T)$ w.l.o.g. we may assume that $L_i(T) = \mathbf{CP}^m$. To simplify our notation we assume that $m = n$. Choose a canonical projection so that the representation matrix A is a diagonal matrix $D = diag(d_1, ..., d_{n+1}), 1 = |d_i|, i = 1, ..., n+1$. Let $\mu \in E(\mathbf{CP}^n, T)$. Let $x \in supp(\mu)$. That is, for any open set $U \subset \mathbf{CP}^n$ so that $x \in U$ we deduce that $\mu(U) > 0$. Let

$$z = \sum_1^l \xi_i e_{n_i}, \xi_i \neq 0, i = 1, ..., l, 1 \leq n_1 < ... < n_l \leq n, \pi(z) = x.$$

By considering $D' = \frac{1}{d_{n_1}} D$ we may assume that $d_{n_1} = 1$. Then the closure of the group generated by $(d_{n_1}, ..., d_{n_l})$ correspond to the compact abelian group $\mathcal{A}(x) \subset \mathbf{CP}^n$ which we identify with $Closure(orb_T(x))$. Clearly, $\mathcal{A}(x) \subset supp(\mu)$. Note that the topology of $\mathcal{A}(x)$ depends only on the set $\{n_1, ..., n_l\}$. Hence, $\mathbf{CP}^n$ is foliated by $\mathcal{A}(x)$. This foliation correspond to a very simple stratification of $\mathbf{CP}^n$ to a tree of subspaces spanned by subset of $e_1, ..., e_{n+1}$. Hence, $supp(\mu)$ foliated also by $\mathcal{A}(x)$. We now point out how this foliation shows that all measures in $E(\mathbf{CP}^n, T)$ are supported on some $\mathcal{A}(x)$. Assume to the contrary that

$$supp(\mu) = \cup_{i \in \mathcal{I}} \mathcal{A}(x_i), |\mathcal{I}| > 1, \mathcal{A}(x_i) \cap \mathcal{A}(x_j) = \emptyset, i \neq j.$$

Let

$$\mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_2, \mathcal{I}_1 \cap \mathcal{I}_2 = \emptyset, 1 \leq |\mathcal{I}_1|, |\mathcal{I}_2|, \cup_{i \in \mathcal{I}_j} \mathcal{A}(x_i) \in \mathcal{B}, j = 1, 2.$$

We claim that

$$\mu(\cup_{i \in \mathcal{I}_1} \mathcal{A}(x_i)) \mu(\cup_{i \in \mathcal{I}_2} \mathcal{A}(x_i)) = 0. \tag{2.3}$$

Otherwise let

$$\mu_j(E) = \frac{\mu(E \cap (\cup_{i \in \mathcal{I}_j} \mathcal{A}(x_i))}{\mu(\cup_{i \in \mathcal{I}_j} \mathcal{A}(x_i))}, j = 1, 2.$$

By the construction $\mu_1, \mu_2 \in \Pi(\mathbf{CP}^n, T)$. Clearly, μ is a convex combination of μ_1, μ_2 which contradicts the assumption that $\mu \in E(\mathbf{CP}^n, T)$. We thus deduce (2.3). It is not difficult to show that (2.3) yields that $|\mathcal{I}| = 1$ contrary to our assumption. Hence, $supp(\mu) = \mathcal{A}(x)$. We showed that $\mathcal{A}(x)$ is isomorphic to the closure of the group $(d_{n_1}^i, ..., d_{n_l}^i), i \in \mathbf{Z}, d_{n_1} = 1$. Denote this closure by $\mathcal{A}$. The action of T of $\mathcal{A}(x)$ is equivalent to the multiplication by $(d_{n_1}, ..., d_{n_l})$ on $\mathcal{A}$. Hence μ is the Haar measure on $\mathcal{A}$. $\diamond$

Note that the proof of the above theorem yields that if $T \in Aut(\mathbf{CP}^n)$ preserves the measure $\mu, d\mu = \frac{\omega^n}{n!}$, the transformation T is not μ-ergodic.

Let $T \in Aut(\mathbf{CP}^n), \mu \in \Pi(\mathbf{CP}^n, T)$. We say that μ is strictly supported on $Y = \cup_1^{k(T,\mu)} L_i(T, \mu)$ if the following conditions hold. First, $supp(\mu) \subset Y \subset R(T)$. Second, $L_1(T, \mu), ..., L_k(T, \mu), k = k(T, \mu)$ are k pairwise disjoint linear subspaces of $\mathbf{CP}^n$. The last condition is that Y is minimal with respect to first two conditions. As the intersection of two linear subspaces in $\mathbf{CP}^n$ is either empty or linear subspace from Theorem 2.2 we deduce the existence of Y. We call Y the T-linear support of μ and denote it by $lsupp(\mu, T)$. Let $\mathcal{T} \subset Aut(\mathbf{CP}^n)$ be a nonempty set. Consider the set

$$\cap_{T \in \mathcal{T}} (\cup_1^{k(T)} L_i(T)).$$

Here, $k = k(T), L_1(T), ..., L_k(T)$ are defined as in Theorem 2.2. Suppose that the above intersection is nonempty. Hence

$$\emptyset \neq \cap_{T \in \mathcal{T}} (\cup_1^{k(T)} L_i(T)) = \cup_1^{k(\mathcal{T})} L_i(\mathcal{T}). \tag{2.4}$$

Here $L_1(\mathcal{T}), ..., L_k(\mathcal{T}), k = k(\mathcal{T})$ are k pairwise disjoint linear subspaces of $\mathbf{CP}^n$. That is, for each $1 \leq i \leq k(\mathcal{T})$ and each $T \in \mathcal{T}$ we have $L_i(\mathcal{T}) \subset L_j(T)$ for some $j, 1 \leq j \leq k(T)$. The following result follows straightforward from Theorem 2.2.

Corollary 2.5. *Let $\mathcal{G} \subset Aut(\mathbf{CP}^n)$ be a group having an invariant measure $\mu \in \Pi(\mathbf{CP}^n)$. Then (2.4) holds ($\mathcal{T} = \mathcal{G}$). Furthermore, $\mathcal{G}$ acts on $\{L_1(\mathcal{G}), ..., L_{k(\mathcal{G})}(\mathcal{G})\}$. ($TL_i(\mathcal{G}) = L_j(\mathcal{G}), j = j(T), T \in \mathcal{G}$.) More precisely*

$$\emptyset \neq \cap_{T \in \mathcal{G}} lsupp(\mu, T) = \cup_1^{k(\mathcal{G},\mu)} L_i(\mathcal{G}, \mu)$$

and $\mathcal{G}$ acts on $\{L_1(\mathcal{G}, \mu), ..., L_{k(\mathcal{G},\mu)}(\mathcal{G}, \mu)\}$.

In our recent paper [**Fri1**] we give necessary and sufficient conditions for $\mathcal{G} \subset Aut(\mathbf{CP}^n)$ to have an invariant measure $\mu \in \Pi(\mathbf{CP}^n)$. Let $G \subset GL_{n+1}(\mathbf{C})$ be a lifting of $\mathcal{G}$, i.e. $\pi(G) = \mathcal{G}$. Then $\mathcal{G}$ has an invariant measure $\mu \in \Pi(\mathbf{CP}^n)$ iff there a lifting G with the following property. There exists a nontrivial G-invariant subspace $L \subset \mathbf{C}^{n+1}$ such that $G\big|L$ is a bounded group. Equivalently, one can choose an inner product on L so that $\mathcal{G}\big|\pi(L) \subset UP(\pi(L))$.

For $A \in GL_{n+1}(\mathbf{C}), T = \pi(A)$ set $R(T) = \cup_1^{k(T)} L_i(T)$. We then let $L_i(A) \subset \mathbf{C}^{n+1}$ be the linear subspace so that $\pi(L_i(A) \backslash \{0\}) = L_i(T), i = 1, ..., k(T) = k(A)$. Assume that $\mathcal{G}' \subset GL_{n+1}(\mathbf{C})$ is a group. Clearly, $\mathcal{G}'$ is virtually solvable iff $\mathcal{G} = \pi(\mathcal{G}') \subset Aut(\mathbf{CP}^n)$ is virtually solvable.

Theorem 2.6. *Let*

$$\mathcal{T}_1 \subset GL_{n+1}(\mathbf{C}), \mathcal{T}_2 \subset Aut(\mathbf{CP}^n)$$

be finite sets. Denote by $\mathcal{G}_1, \mathcal{G}_2$ the groups generated by $\mathcal{T}_1, \mathcal{T}_2$ respectively. If

$$\{0\} = \cap_{A \in \mathcal{T}_1} (\cup_1^{k(A)} L_i(A)),$$
$$\emptyset = \cap_{T \in \mathcal{T}_2} (\cup_1^{k(T)} L_i(T))$$

then $\mathcal{G}_1$ and $\mathcal{G}_2$ contain a free subgroup on two generators.

Proof. Consider first $\mathcal{G}_1$. The Tits alternative [**Tit**] claims that either $\mathcal{G}_1$ is virtually solvable or $\mathcal{G}_1$ contains a free subgroup on two generators. Assume to the contrary that $\mathcal{G}_1$ does not contain a free subgroup on two generators. Then $\mathcal{G}_1$ is virtually solvable. Set

$$\mathcal{T}_1' = \pi(\mathcal{T}_1), \mathcal{G}_1' = \pi(\mathcal{G}_1), \mathcal{T}_1', \mathcal{G}_1' \in Aut(\mathbf{CP}^n).$$

Note that the intersection condition for $\mathcal{T}_1'$ holds. As $\mathcal{G}_1'$ is also virtually solvable $\mathcal{G}_1'$ have an invariant probability measure by Corollary 1.3. The intersection property for $\mathcal{T}_1'$ contradicts Corollary 2.5. Hence, $\mathcal{G}_1$ contains a free subgroup on two generators. In a similar way, $\mathcal{G}_2$ contains a free subgroup on two generators. $\diamond$

We remark that one can give similar conditions on the invariant mesures of $T \in GLP_n(\mathbf{R})$ and on $\mathcal{G} \subset GLP_n(\mathbf{R})$ to have an invariant probability measure. We just have to combine the pairs of conjugate eigenvalues and eigenvectors of $A \in GL_{n+1}(\mathbf{R})$. In some applications it is advantageous to consider the containments

$$\mathbf{R}^{n+1} \subset \mathbf{C}^{n+1}, \mathbf{RP}^n \subset \mathbf{CP}^n, GL_{n+1}(\mathbf{R}) \subset GL_{n+1}(\mathbf{C}), GLP_n(\mathbf{R}) \subset GLP_n(\mathbf{C}).$$

We now bring one application to a subgroup of Möbius transformations of n-hyperbolic space. Consult for example with [**Bea**] for good reference on this subject. Let $B_n \subset \mathbf{R}^n$ be the closed n-dimensional unit ball centered at the origin. Denote by H_n the interior of B_n. Then H_n has the standard Poincaré metric of n-hyperbolic space. Consider the group of Möbius transformation $\mathcal{M}_n$ of H_n. $\mathcal{M}_n$ are orientation preserving isometries of H_n. Also $\mathcal{M}_n$ can be viewed as a group of homeomorphisms of B_n which map S^{n-1}, the boundary of B_n, onto itself. We view $\mathcal{M}_n$ as a group of homeomorphisms of B_n. It is well known that $\mathcal{M}_n$ can be represented as subgroup of $GL_{n+1}(\mathbf{R})$. ($\mathcal{M}_n$ preserves the Lorentzian cone.) Let $T \in \mathcal{M}_n$. Then T is called loxodromic if T has exactly two fixed points located on S^{n-1}. T is called parabolic if T has exactly one fixed point on S^{n-1}. T is called elliptic if T has at least one fixed point in H_n. For a loxodromic or parabolic T the recurrence set $R(T)$ is equal to $Fix(T)$ the set of the fixed points of T, e.g. [**Wal**,

§6.4]. If T is loxodromic then $\Pi(B_n, T)$ is a convex combination of two Dirac measures concentrated on $Fix(T)$. If T is parabolic then $\Pi(B_n, T)$ consists of the Dirac measure concentrated at $Fix(T)$. Recall that a group $\Gamma \subset \mathcal{M}_n$ is called elementary if Γ has a finite invariant set $K \subset B_n$. The following lemma is well known and it can be deduced from the arguments above.

Lemma 2.7. *Let $\Gamma \subset \mathcal{M}_n$ act on B_n as a group of homeomorphisms. Assume that Γ does not consists entirely of elliptic elements. Then Γ has an invariant measure iff Γ is elementary. That is, either all elements of Γ have a common fixed point $\xi \in S^{n-1}$ (δ_ξ is the invariant measure) or all the elements of Γ fix some set of two points $\{\xi, \eta\} \subset S^{n-1}$ ($\frac{\delta_\xi + \delta_\eta}{2}$ is the invariant measure).*

It is not hard to show that if $\Gamma \subset \mathcal{M}_n$ consists entirely of elliptic elements then $T^{-1}\Gamma T$ is a subgroup of orthogonal rotations of B_n around the origin for some $T \in \mathcal{M}_n$. In that case Γ has an invariant measure. (The Lebesgue measure of S^{n-1} is an invariant measure for $T^{-1}\Gamma T$.)

Theorem 2.8. *Let $\Gamma \subset \mathcal{M}_n$ be a nonelementary group which does not consists entirely of elliptic elements. Then Γ contains a free subgroup on two generators.*

Proof. Pick up a nonelliptic element T_1. Since Γ is nonelementary it easy to show that there exists exists $T_2 \in \Gamma$ such that T_2^2 does not fix $Fix(T_1)$. Then Γ_1 generated by T_1, T_2 is nonelementary. Since Γ has a faithful representation in $GL_{n+1}(\mathbf{R})$ Tits alternative claims that either Γ_1 is virtually solvable or Γ_1 contains a free group on two generators. Use Theorem 1.2 and Lemma 2.7 to deduce that Γ_1 contains a free subgroup on two generators. $\diamond$

§3. Auslander's conjecture

Let $\mathbf{F}$ be a field. Then any affine invertible transformation on $\mathbf{F}^n$ is given by $u \to Au + a, A \in GL_n(\mathbf{F}), a, u = (u_1, ..., u_n)^T \in \mathbf{F}^n$. This induces a corresponding linear transformation $(u^T, u_{n+1})^T \to ((Au + au_{n+1})^T, u_{n+1})^T$. Denote by $Aff_n(\mathbf{F})$ the group of affine invertible transformations of $\mathbf{F}^n$. Thus, we have the monomorphism

$$\psi : Aff_n(\mathbf{F}) \to GL_{n+1}(\mathbf{F}), \psi((A, a)) = \hat{A} = (\hat{A}_{ij})_1^2, \hat{A}_{11} = A, \hat{A}_{12} = a, \hat{A}_{21} = 0, \hat{A}_{22} = 1.$$

Note that $\psi : Aff_n(\mathbf{F}) \to GL_{n+1}(\mathbf{F})$ induces the monomorphism $\hat{\psi} : Aff_n(\mathbf{F}) \to GLP_n(\mathbf{F})$. Finally we view $GLP_n(\mathbf{R})$ as a subgroup of $GLP_n(\mathbf{C}) = Aut(\mathbf{CP}^n)$.

A complete affinely flat manifold M of dimension n is equal to $\mathbf{R}^n/\Gamma$, where Γ is a discrete group of the group of affine transformation $Aff_n(\mathbf{R})$ which acts freely and properly discontinously on $\mathbf{R}^n$, e.g. [**Mil**]. (Recall that Γ acts freely if $\gamma(x) = x, \gamma \in \Gamma \Rightarrow \gamma = id$ for any $x \in R^n$. Γ acts properly discontinuously if for any compact $K \subset \mathbf{R}^n$ the set of $\gamma \in \Gamma$ with $\gamma(K) \cap K \neq \emptyset$ is finite.) In [**Aus**] Auslander "proved" that if M is compact then

$\pi_1(M) \sim \Gamma$ is virtually solvable. Unfortunately, Auslander's proof has a gap. The above statement is commonly referred as Auslander's conjecture. As $Aff_n(\mathbf{R})$ has a faithful representation in $GL_{n+1}(\mathbf{R})$ it follows that any discrete solvable $\Gamma \subset Aff_n(\mathbf{R})$ is polycyclic, e.g. [**Wol**]. Milnor [**Mil**] showed that if a countable group G is torsion-free and virtually polycyclic then G is isomorphic to $\pi_1(M)$ for some complete affinely flat manifold. He asked if $\pi_1(M)$ of any completely affinely flat manifold is virtually polycyclic. Auslander's conjecture was proved for $dim(M) = 2, 3$ by Fried and Goldman [**F-G**]. Goldman and Kamishima [**G-K**] proved Auslander's conjecture for any dimension on condition that the linear part of Γ preserves a Lorentzian form. Tomanov [**Tom**] extended the result of [**G-K**] to Γ whose linear part of Γ preserves a generalized Lorentzian form. Recently, Margulis [**Mar3**] proved Auslander's conjecture for $n = 4, 5$. On the negative side, Margulis [**Mar1-2**] constructed a noncompact 3 dimensional complete affinely flat manifold M whose fundamental group contains a free subgroup on two generators. See also [**D-G**].

Suppose that Γ is virtually solvable. Consider $L(\Gamma)$ - the linear of Γ. Clearly, $L(\Gamma)$ is also virtually solvalbe. Viewing $L(\Gamma)$ as a subgroup of $Aut(\mathbf{CP}^{n-1})$ we deduce that $L(\Gamma)$ has an invariant measure on $\mathbf{CP}^{n-1}$. Since Γ acts freely and properly discontinuously on $\mathbf{R}^n$ it follows that invariant measures of $\hat{\psi}(\Gamma)$ are supported on $\mathbf{RP^{n-1}}(\mathbf{CP}^{n-1})$. (Recall Malcev's theorem [**Mal**] that Γ is virtually similar to a subgroup of upper triangular matrices over complexes.) Thus, a weaker form of Auslander's conjecture is that if $M = \mathbf{R^n}/\Gamma$ is an affinely flat compact manifold then $L(\Gamma) \subset Aut(CP^{n-1})$ has an invariant probability measure. We call this conjecture the weak Auslander conjecture. We do not know if the weak Auslander conjecture is equivalent to Auslander's conjecture. We are convinced that the solution of the weak Auslander conjecture is a major step toward proving the original conjecture. We now discuss two approaches to the weak Auslander conjecture.

Compactify $\mathbf{R}^n$ to B_n - the closed unit ball in $\mathbf{R}^n$. Let B_n^o be the interior of B_n. Then

$$\phi : \mathbf{R}^n \to B_n^o, \phi(x) = \frac{x}{1 + |x|}, x \in \mathbf{R}^n, \psi : B_n^o \to \mathbf{R}^n, \psi(y) = \frac{y}{1 - |y|}, y \in B_n^o.$$

Thus, an invertible affine transformation $(A, a) \in Aff_n(\mathbf{R})$ acts on $S^{n-1} = \partial B_n$ by the rule $y \mapsto \frac{Ay}{|Ay|}$. That is, we lift the action of $GL_n(\mathbf{R})$ on $\mathbf{RP}^{n-1}$ to its double cover S^{n-1}. In what follows we assume that $\Gamma \subset Aff_n(\mathbf{R})$ acts freely and properly discontinuously on $\mathbf{R}^n$ such that $M = \mathbf{R}^n/\Gamma$ is a compact manifold. Let (M, g) be a Riemannian metric on M. Denote by $(\mathbf{R}^n, \hat{g})$ the lifting of (M, g) to $\mathbf{R}^n$. For $x, y \in \mathbf{R}^n$ denote by $dist_g(x, y)$ the distance between x, y in $(\mathbf{R}^n, g)$. (It is not hard to show that this distance is finite and realized by some geodesic from x to y.) Then Γ is a subgroup of the group of isometries $Iso(\mathbf{R}^n, \hat{g})$. ($dist_g(\gamma(x), \gamma(y)) = dist_g(x, y), x, y \in \mathbf{R}^n, \gamma \in \Gamma$.) Fix a point $\xi \in \mathbf{R}^n$. Let

$$D(\xi, g) = \{x \in \mathbf{R}^n : dist_g(x, \xi) < dist_g(x, \gamma(\xi)), \gamma \in \Gamma, \gamma \neq id\}$$

be the Dirichlet polygon centered at ξ. Since M is compact it follows that $\partial D(\xi, g)$ consists of a finite number of sides of the form

$$dist_g(x, \xi) = dist_g(x, \gamma(\xi)), \gamma \in S \subset \Gamma, S^{-1} = S, id \notin S, |S| < \infty.$$

Thus, the S is a generating set of Γ. Moreover, $Closure(D(\xi,g))$ induces a tessellation of $\mathbf{R}^n$ by the tiles $\gamma(Closure(D(\xi,g)),\gamma\in\Gamma$ so that the set of neighboring tiles of $Closure(D(\xi,g))$ is $\gamma(Closure(D(\xi,g)),\gamma\in S$. Consider the the Poincaré series

$$p(x,y,s)=\sum_{\gamma\in\Gamma}e^{-sdist_g(x,\gamma(y))},x,y\in D(\xi,g).$$

We claim that Poincaré series converge for $s>\omega$ for some $0<\omega<\infty$. This is a straightforward consequence of the fact that a sectional curvature of (M,g) and hence of $(\mathbf{R}^n,\hat{g})$ is bounded below by some c which may be negative. Consult for example with [**Nic**] for the hyperbolic case (sectional curvature -1) and with [**Gun**] for the volume growth on Riemannian manifolds with bounded sectional curvature. Let $\kappa, 0\le\kappa\le\omega$ be the critical exponent of Poincaré series. Assume for simplicity that the Poincaré series diverge for $s=\kappa$. (Note that the value of κ and the divergence of Poincaré series do not depend on choices of x,y.) Then one can construct the Patterson-Sullivan measure [**Pat**]-[**Sul**] as follows. Let

$$\mu(x,y,s)=\frac{1}{p(y,y,s)}\sum_{\gamma\in\Gamma}e^{-sdist_g(x,\gamma(y))}\delta_{\gamma(y)},s>\kappa.$$

Note that Γ acts on the above family of measures:

$$\hat{\alpha}(\mu(x,y,s))=\mu(\alpha^{-1}(x),y,s),\alpha\in\Gamma.$$

Denote by $\mathcal{M}(x,y)$ be the w^* closure of all $weak^*$ limit of the measures $\mu(x,y,s)$ when $s\to\kappa^+$. Thus, $\hat{\alpha}\mathcal{M}(x,y)=\mathcal{M}(\alpha^{-1}(x),y)$. As Γ acts properly discontinuously on $\mathbf{R}^n\sim B_n^o$ it follows that $\mathcal{M}(x,y)$ is supported on the limit set of $\Gamma(y)$ situated on $\partial B_n=S^{n-1}$. Fix a point y for all choices of $x\in D(\xi,g)$. Let $\mu_x\in\mathcal{M}(x,y)$. Set

$$\mu_{\alpha^{-1}(x)}=\hat{\alpha}(\mu_x).$$

It then follows that $\mu_{\alpha^{-1}(x)}\in\mathcal{M}(\alpha^{-1}(x),y)$. For discrete group of Möbius transformations acting on B_n^o equipped with the standard hyperbolic metric the works of Patterson [**Pat**] and Sullivan [**Sul**] show that the measure μ_x satisfies the remarkable identity

$$\frac{d\mu_{x'}}{d\mu_x}(\xi)=\Big(\frac{P(x,\xi)}{P(x',\xi)}\Big)^\kappa,P(x,\xi)=\frac{1-|x|^2}{|x-\xi|^2},x\in B_n^o,\xi\in S^{n-1}.$$

Here, $P(x,\xi)$ is the Poisson kernel and $\frac{d\mu_{x'}}{d\mu_x}(\xi)$ is the Radon-Nikodym derivative. Furthermore, if Γ is geometrically finite then μ_x is unique (up to a multiple by a positive constant). Finally, if Γ is convex cocompact then μ_x is unique and κ is the Hausdorff measure of the limit set $\Gamma(y)$ (which does not depend on the choice of y). Thus, μ_x is an invariant measure of Γ iff $\kappa=0$, i.e. Γ is elementary.

Thus, on could try to investigate the set $\mathcal{M}(x,y)$ in our case. The following simple example shows that we should expect a different behavior in our case. Assume that Γ

is a discrete group of translation of $\mathbf{R}^n$ such that M is a compact torus. In that case it is easy to show that the limit set of $\Gamma(y)$ is S^{n-1}. Choose $(\mathbf{R}^n, \hat{g})$ to be the standard metric on $\mathbf{R}^n$. In that case it follows that $\kappa = 0$. Furthermore, each $\mathcal{M}(x, y)$ consists of one measure $\mu(x, y)$ which is invariant under the action of Γ. In [**Fri2**] we construct the Patterson-Sullivan measure for any discrete torsion free group Γ of a given Lie group. However our construction differs from the above construction.

We now discuss the second approach to the weak Auslander conjecture. We will assume that by changing the metric g on M, if necessary, we can achieve the following conditions on $D(\xi, g)$ and the tesselation induced by $D(\xi, g)$.

Conditions 3.1. *The boundary of $D(\xi, g)$ is piecewise smooth.*

$$dim(\partial D(\xi, g) \cap \gamma(\partial D(\xi, g))) = n - 1, \forall \gamma \in S.$$

For any open interval $(a, b) \subset Closure(D(\xi, g) \cup_{\gamma \in S} \gamma(D(\xi, g)))$ the set $(a, b) \cap (D(\xi, g) \cup_{\gamma \in S} \gamma(D(\xi, g)))$ consists of a finite number of open intervals with the total length equal to the length of (a, b).

We believe that the above conditions can be always satisfied by a properly chosen Riemannian metric on M. Note that the last condition of Conditions 3.1 implies that the boundary of $D(\xi, g)$ does not contain a nontrivial interval (a geodesic of $\mathbf{R}^n$ in the standard metric).

Combine Condition 3.1. with the assumption that Γ acts discretely and properly discontinuously to deduce that any finite interval (a, b) intersects a finite number of tiles $\gamma(Closure(D(\xi, g))), \gamma \in T \subset \Gamma, |T| < \infty$. Furthermore, $(a, b) \cap (\cup_{\gamma \in T} \gamma(D(\xi, g)))$ consists of a finite number of open intervals whose length is equal to the length of the interval (a, b). Let $\mathbf{R}^n \times S^{n-1}$ be the trivial S^{n-1} fiber bundle over $\mathbf{R}^n$. We now construct a flow

$$F_t : \mathbf{R}^n \times S^{n-1} \to \mathbf{R}^n \times S^{n-1}, t \in \mathbf{R}.$$

For $(x, p) \in \mathbf{R}^n \times S^{n-1}$ we let $F_t(x, p) = (X(x, p, t), p), t \in \mathbf{R}$ where $X(x, p, t)$ is on the line $l(x, p) = \{y, y = x + \tau p, \tau \in \mathbf{R}\}$. For $t > 0$ the point $X(x, p, t)$ moves in the direction of p. For $t < 0$ the flow travels in the direction $-p$. The velocity of the flow $X(x, p, t)$ on a line $l(x, p)$ is constant on $l(x, p, \gamma) = l(x, p) \cap \gamma(D(\xi, g))$ for a fixed $\gamma \in \Gamma$. For $\gamma = id$ this velocity is equal to 1. For a general γ this velocity is determined uniquely by the condition that the $\gamma^{-1}(X(x, p, t)_{|l(x,p,\gamma)})$ gives the flow $X(x', p', t)$ on $D(\xi, g)$. Specifically, let $\gamma = (A_\gamma, a_\gamma) \in Aff_n(\mathbf{R})$. Then the velocity of the flow $X(x, p, t)$ while moving on $l(x, p, \gamma)$ is equal to $\frac{1}{|A_\gamma^{-1} p|}$. Note that the velocity of the flow $X(x, p, t)$ is not defined on $\gamma(\partial D(\xi, g))$. Since any compact interval $[a, b]$ meets $\cup_{\gamma \in \Gamma} \gamma(\partial D(\xi, g))$ at a finite number of points it follows that the flow F_t is a well defined continuous nondifferentiable flow.

We let $Aff_n(\mathbf{R})$ to act on $\mathbf{R}^n \times S^{n-1}$ as follows.

$$(x, p) \mapsto (Ax + a, \frac{Ap}{|Ap|}), (A, a) \in Aff_n(\mathbf{R}).$$

It then follows that Γ acts properly discontinuously on $\mathbf{R}^n \times S^{n-1}$. Let $\hat{M} = \mathbf{R}^n \times S^{n-1}/\Gamma$ be an S^{n-1} bundle over M. Our construction yields that the flow F_t induces a continuous flow $\hat{F}_t : \hat{M} \to \hat{M}$. The flow $\hat{F}_t$ corresponds to the following "billiard" flow on $D(\xi, g)$. Pick a point $x \in D(\xi, g)$ and a direction $p \in S^{n-1}$. Then the billiard ball is moving in the direction p with unit speed until the billiard hits $\partial D(\xi, g)$ at x'. Then the ball continues its way from the point $\gamma^{-1}(x') \in \partial D(\xi, g)$ into $D(\xi, g)$ with unit speed in the direction $\frac{A_\gamma^{-1} p}{|A_\gamma^{-1} p|}$ for a unique $\gamma \in S$. Krylov-Bogolyubov's theorem yields that $\hat{F}_t$ has an invariant measure $\nu \in \Pi(\hat{M})$. It seems that the existence of an invariant measure for $\hat{\psi}(\Gamma)$ could be constructed by using ν.

References

[**Aus**] L. Auslander, The structure of compact locally affine manifolds, *Topology* 3 (1964), 131-139.

[**B-T**] S. Banach and A. Tarski, Sur la decomposition des ensembles de points en partes respectivement congruentes, *Fund, Math.* 6 (1924), 244-277.

[**Bea**] A.F. Beardon, *The geometry of discrete groups*, Springer-Verlag, New York, 1982.

[**D-G**] T. Drumm and W. Goldman, Complete flat Lorentz 3-manifolds with free fundamental group, *Int. J. Math.* 1 (1990), 149-161.

[**D-S**] N. Dunford and J.T. Schwartz, *Linear opertors*, Part I, Interscience Pub., New York, 1967.

[**F-G**] D. Fried and W. Goldman, Three-dimensional affine crystallographic groups, *Advances in Math.* 48 (1983), 1-49.

[**Fri1**] S. Friedland, Properly discontinuous groups on certain matrix homogeneous spaces, preprint, July 1994.

[**Fri2**] S. Friedland, The Patterson-Sullivan measure for discrete groups of Lie groups, in preparation.

[**Fur**] H. Furstenberg, A note on Borel's density theorem, *Proc. AMS* 55 (1976), 209-212.

[**Gan**] F.R. Gantmacher, *The theory of matrices*, I, Chelsea Publ. Co., New York, 1959.

[**G-K**] W. Goldman and Y. Kamishima, The fundamental groups a compact flat space form is virtually polycyclic, *J. Differential Geometry* 19 (1984), 233-240.

[**Gre**] F.P. Greenleaf, *Invariant means on topological groups and their applications*, Van Nostrand Reinhold, New York, 1969.

[**G-H**] P. Griffiths and J. Harris, *Principles of algebraic geometry*, Wiley Interscience, 1978.

[**Gun**] P. Günther, Einige Sätze über das Volumenelement eines Riemannschen Raumes, *Publ. Math. Debrecen* 7 (1960), 78-92.

[**Mal**] A.I. Malcev, On certain classes of infinite solvable groups, *Mat. Sb.* 28 (70) (1951), 567-588.

[**Mar1**] G.A. Margulis, Free properly discontinuous groups of affine transformations, *Dokl. Akad. Nauk SSSR* 272 (1983), 937-940.

[**Mar2**] G.A. Margulis, Complete affine locally flat manifolds with a free fundamental group, J. Soviet Math. 134 (1987), 129-133.

[**Mar3**] On the Zariski closure of the linear part of a properly discontinuous group of affine transformations, preprint.

[**Mil**] J. Milnor, On fundamental groups of completely affine flat manifolds, *Advances in Math.* 25 (1977), 178-187.

[**Neu**] J. von Neumann, Zur allgemeinen theorie des masses, *Fund. Math.* 13 (1929), 73-116.

[**Nic**] P.J. Nicholls, *Ergodic theory of discrete groups*, Cambridge Univ. Press, Cambridge, 1989.

[**Pat**] S.J. Patterson, The limit set of a Fuchsian group, *Acta Math.* 136 (1976), 241-272.

[**Ric**] N. Rickert, Amenable groups and groups with the fixed point property, *Trans. AMS* 127 (1967), 221-232.

[**Sul**] D. Sullivan, The density at infinity of a discrete group of hyperbolic motions, *Publ. Math. I.H.E.S.* 50 (1979), 171-202.

[**Tit**] J. Tits, Free subgroups in linear groups, *J. Algebra* 20 (1972), 250-270.

[**Tom**] G. Tomanov, The virtual solvability of the fundamental group of a generalized Lorentz space form, *J. Differential Geometry*, 32 (1990), 539-547.

[**Wal**] P. Walters, *An introduction to ergodic theory*, Springer-Verlag, New York, 1982.

[**Wol**] J.A. Wolf, Growth of finitely generated solvable groups and curvature of riemannian manifolds, *J. Differential Geometry* 2 (1968) 421-446.

[**Zim**] R.J. Zimmer, *Ergodic theory and semisimple groups*, Birkhäuser, Basel 1984.

Acknowledgement. I would like to thank to C. McMullen, B. Weiss and R. Zimmer for their useful remarks.