Tensors and Matrices

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Overview

Ranks of 3-tensors
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1. Basic facts.
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2. Complexity.
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3. Matrix multiplication
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1. Rank one approximation.
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1. Rank one approximation.
2. Perron-Frobenius theorem
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2. Perron-Frobenius theorem
3. Rank $(R_1, R_2, R_3)$ approximations
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Diagonal scaling of nonnegative tensors to tensors with given rows, columns and depth sums
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Diagonal scaling of nonnegative tensors to tensors with given rows, columns and depth sums

Characterization of tensor in \(\mathbb{C}^{4 \times 4 \times 4}\) of border rank 4
Basic notions

Scalar $a \in F$, vector $x = (x_1, \ldots, x_n)^\top \in F^n$, matrix $A = [a_{ij}] \in F^{m \times n}$, 3-tensor $T = [t_{i,j,k}] \in F^{m \times n \times l}$, p-tensor $T = [t_{i_1, \ldots, i_p}] \in F^{n_1 \times \ldots \times n_p}$.
Basic notions

scalar $a \in \mathbb{F}$, vector $\mathbf{x} = (x_1, \ldots, x_n) \mathbf{^T} \in \mathbb{F}^n$, matrix $A = [a_{ij}] \in \mathbb{F}^{m \times n}$,

3-tensor $T = [t_{i,j,k}] \in \mathbb{F}^{m \times n \times l}$, p-tensor $T = [t_{i_1, \ldots, i_p}] \in \mathbb{F}^{n_1 \times \ldots \times n_p}$

Abstractly $\mathbb{U} := \mathbb{U}_1 \otimes \mathbb{U}_2 \otimes \mathbb{U}_3$ \ \dim \mathbb{U}_i = m_i, i = 1, 2, 3, \ \dim \mathbb{U} = m_1 m_2 m_3$
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HISTORY: Tensors-as now W. Voigt 1898
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Abstractly $U := U_1 \otimes U_2 \otimes U_3$ dim $U_i = m_i$, $i = 1, 2, 3$, dim $U = m_1 m_2 m_3$
Tensor $\tau \in U_1 \otimes U_2 \otimes U_3$

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Rank one tensor $t_{i,j,k} = x_i y_j z_k$, $(i, j, k) = (1, 1, 1), \ldots, (m_1, m_2, m_3)$ or decomposable tensor $x \otimes y \otimes z$
Basic notions

Scalar \( a \in \mathbb{F} \), vector \( \mathbf{x} = (x_1, \ldots, x_n)^\top \in \mathbb{F}^n \), matrix \( A = [a_{ij}] \in \mathbb{F}^{m \times n} \),
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or decomposable tensor \( \mathbf{x} \otimes \mathbf{y} \otimes \mathbf{z} \)
basis of \( \mathbb{U}_j \): \( [u_{1,j}, \ldots, u_{m_j,j}] \), \( j = 1, 2, 3 \)
basis of \( \mathbb{U} \): \( u_{i_1,1} \otimes u_{i_2,2} \otimes u_{i_3,3}, i_j = 1, \ldots, m_j, j = 1, 2, 3, \ldots \)
Basic notions

**scalar** \( a \in \mathbb{F} \), **vector** \( \mathbf{x} = (x_1, \ldots, x_n)^\top \in \mathbb{F}^n \), **matrix** \( A = [a_{ij}] \in \mathbb{F}^{m \times n} \),

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**basis of** \( \mathbb{U}_j \): \( [u_{1,j}, \ldots, u_{m_j,j}] j = 1, 2, 3 \)
**basis of** \( \mathbb{U} \): \( u_{i_1,1} \otimes u_{i_2,2} \otimes u_{i_3,3}, i_j = 1, \ldots, m_j, j = 1, 2, 3, \)
\( \tau = \sum_{i_1=i_2=i_3=1}^{m_1 m_2 m_3} t_{i_1,i_2,i_2} u_{i_1,1} \otimes u_{i_2,2} \otimes u_{i_3,3} \)
Ranks of tensors

Unfolding tensor: in direction 1:
\[ T = [t_{i, j, k}] \]
view as a matrix
\[ A_1 = [t_i, (j, k)] \in F^{m_1 \times (m_2 \cdot m_3)} \]
\[ R_1 := \text{rank} A_1 : \text{dimension of row or column subspace spanned in direction 1} \]

\[ T_i, 1 := [t_{i, j, k}]_{m_2, m_3} j, k = 1 \in F^{m_2 \times m_3}, i = 1, \ldots, m_1 \]
\[ T = \sum_{m_1 i = 1} T_i, 1 e_i, 1 \] (convenient notation)
\[ R_1 := \text{dim} \text{span} (T_1, 1, \ldots, T_{m_1}, 1) \].

Similarly, unfolding in directions 2, 3:
\[ \text{rank} T \text{ minimal} r :\]
\[ T = f_r (x_1, y_1, z_1, \ldots, x_r, y_r, z_r) := \sum_{r i = 1} x_i \otimes y_i \otimes z_i, \text{(CANDEC, PARFAC)} \]

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\( \text{rank } \mathcal{T} \text{ minimal } r : \)

\( \mathcal{T} = f_r(\mathbf{x}_1, \mathbf{y}_1, \mathbf{z}_1, \ldots, \mathbf{x}_r, \mathbf{y}_r, \mathbf{z}_r) := \sum_{i=1}^{r} \mathbf{x}_i \otimes \mathbf{y}_i \otimes \mathbf{z}_i, \)
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rank \( T \) minimal \( r \):

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\( \text{(CANDEC, PARFAC)} \)
Basic facts

FACT I:
\[ T \geq \max(R_1, R_2, R_3) \]

Reason:
\[ U_2 \otimes U_3 \sim F_{m_2 \times m_3} \equiv F_{m_2 \times m_3} \]

Note: \( R_1, R_2, R_3 \) are easily computable.

It is possible that \( R_1 \neq R_2 \neq R_3 \).

FACT II: For \( \tau = T = [t_i, j, k] \) let
\[ T_k = [t_i, j, k] \]
\[ m_1, m_2 \equiv 1 \in F_{m_1 \times m_2}, k = 1, \ldots, m_3. \]

Then \( \text{rank} T = \text{minimal dimension of subspace} \ L \subset F_{m_1 \times m_2} \text{spanned by rank one matrices containing} \ T_1, T_3, \ldots, T_m, T_3, \ldots, T_m. \)

\[ \text{COR} \quad \text{rank} T \leq \min(mn, ml, nl) \]
FACT I: $\text{rank } T \geq \max(R_1, R_2, R_3)$
Basic facts

**FACT I:** \( \text{rank} \ T \geq \max(R_1, R_2, R_3) \)

**Reason** \( U_2 \otimes U_3 \sim F^{m_2 \times m_3} \equiv F^{m_2 m_3} \)
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FACT II: For \( \tau = T = [t_{i,j,k}] \) let
\( T_{k,3} := [t_{i,j,k}]_{i,j=1}^{m_1,m_2} \in \mathbb{F}^{m_1 \times m_2}, k = 1, \ldots, m_3 \). Then rank \( T \) = minimal dimension of subspace \( L \subset \mathbb{F}^{m_1 \times m_2} \) spanned by rank one matrices containing \( T_{1,3}, \ldots, T_{m_3,3} \).
Basic facts

FACT I: \( \text{rank } \mathcal{T} \geq \max(R_1, R_2, R_3) \)
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COR \( \text{rank } \mathcal{T} \leq \min(mn, ml, nl) \)
Complexity of rank of 3-tensor

Hastad 1990: Tensor rank is NP-complete for any finite field and NP-hard for rational numbers.

PRF: 3-sat with $n$ variables, $m$ clauses, satisfiable iff $\text{rank} \ T = 4n + 2m$, $T \in \mathbb{F}(2^n + 3m) \times (3^n) \times (3^n + m)$.

otherwise rank is larger.
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Generic rank: $F = \mathbb{R}, \mathbb{C}$
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Border rank of $\mathcal{T}$ the minimum $k$ s.t. $\mathcal{T}$ is a limit of $\mathcal{T}_j, j \in \mathbb{N}$, rank $\mathcal{T}_j = k$. 

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generic rank=border rank=typical rank of $\mathbb{F}^{m \times n \times l}$: $\text{grank}_\mathbb{F}(m, n, l)$ - the rank of a random tensor $\mathcal{T} \in \mathbb{F}^{m \times n \times l}$
Generic rank of $\mathbb{F}^{m \times n \times l}$

**THM:** $\text{grank}_C(m, n, l) = \min(l, mn)$ for $(m - 1)(n - 1) + 1 \leq l$. 

**COR:** $\text{grank}_C(2, n, l) = \min(l, 2n)$ for $2 \leq n \leq l$. 

**Dimension count for** $F = \mathbb{C}$ and $2 \leq m \leq n \leq l \leq (m - 1)(n - 1) + 1$:

For $(m, n, l)$, $\text{rank}((a \otimes b)(ab)^{-1}z) = mnl$. 

**Fact:** $\text{grank}_C(3, 2p + 1, 2p + 1) = \lceil \frac{3(2p + 1)}{2} \rceil + 1$. 

**Conjecture** $\text{grank}_C(m, n, l) = \lceil \frac{mnl}{m + n + l - 2} \rceil$ for $2 \leq m \leq n \leq l < (m - 1)(n - 1)$ and $(3, 2p + 1, 2p + 1) \neq (3, 2p + 1, 2p + 1)$. 

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THM: $\text{grank}_\mathbb{C}(m, n, l) = \min(l, mn)$ for $(m - 1)(n - 1) + 1 \leq l$.

Reason: For $l = (m - 1)(n - 1) + 1$ a generic subspace of matrices of dimension $l$ in $\mathbb{C}^{m \times n}$ intersect the variety of rank one matrices in $\mathbb{C}^{m \times n}$ at least at $l$ lines which contain $l$ linearly independent matrices.
Generic rank of $\mathbb{F}^{m \times n \times l}$

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**Dimension count for $F = \mathbb{C}$ and $2 \leq m \leq n \leq l \leq (m - 1)(n - 1) + 1$:**

$$f_r : (\mathbb{C}^m \times \mathbb{C}^n \times \mathbb{C}^l)^r \rightarrow \mathbb{C}^{m \times n \times l}, \ x \otimes y \otimes z = (ax) \otimes (by) \otimes ((ab)^{-1}z)$$
Generic rank of $\mathbb{F}^{m \times n \times l}$

**THM:** $\text{grank}_{\mathbb{C}}(m, n, l) = \min(l, mn)$ for $(m - 1)(n - 1) + 1 \leq l$.

**Reason:** For $l = (m - 1)(n - 1) + 1$ a generic subspace of matrices of dimension $l$ in $\mathbb{C}^{m \times n}$ intersect the variety of rank one matrices in $\mathbb{C}^{m \times n}$ at least at $l$ lines which contain $l$ linearly independent matrices.

**COR:** $\text{grank}_{\mathbb{C}}(2, n, l) = \min(l, 2n)$ for $2 \leq n \leq l$.

**Dimension count for $\mathbb{F} = \mathbb{C}$ and $2 \leq m \leq n \leq l \leq (m - 1)(n - 1) + 1$:**

$$f_r : (\mathbb{C}^m \times \mathbb{C}^n \times \mathbb{C}^l)^r \to \mathbb{C}^{m \times n \times l}, \quad x \otimes y \otimes z = (ax) \otimes (by) \otimes ((ab)^{-1}z)$$

$$\text{grank}_{\mathbb{C}}(m, n, l)(m + n + l - 2) \geq mnl \Rightarrow \text{grank}_{\mathbb{C}}(m, n, l) \geq \left\lceil \frac{mnl}{(m+n+l-2)} \right\rceil$$
THM: $\text{grank}_\mathbb{C}(m, n, l) = \min(l, mn)$ for $(m - 1)(n - 1) + 1 \leq l$.

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$$\text{grank}_\mathbb{C}(m, n, l)(m + n + l - 2) \geq mnl \Rightarrow \text{grank}_\mathbb{C}(m, n, l) \geq \left\lceil \frac{mn}{r(m+n+l-2)} \right\rceil$$

Conjecture $\text{grank}_\mathbb{C}(m, n, l) = \left\lceil \frac{mn}{r(m+n+l-2)} \right\rceil$ for $2 \leq m \leq n \leq l < (m - 1)(n - 1)$ and $(3, n, l) \neq (3, 2p + 1, 2p + 1)$.
THM: $\text{grank}_C(m, n, l) = \min(l, mn)$ for $(m - 1)(n - 1) + 1 \leq l$.

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Conjecture $\text{grank}_C(m, n, l) = \left\lceil \frac{mnl}{m + n + l - 2} \right\rceil$ for $2 \leq m \leq n \leq l < (m - 1)(n - 1)$ and $(3, n, l) \neq (3, 2p + 1, 2p + 1)$.

Fact: $\text{grank}_C(3, 2p + 1, 2p + 1) = \left\lceil \frac{3(2p+1)^2}{4p+3} \right\rceil + 1$.
Bilinear maps and product of matrices

bilinear map: $\phi : U \times V \rightarrow W$
Bilinear maps and product of matrices

bilinear map: $\phi : U \times V \to W$

$[u_1, \ldots, u_m], [v_1, \ldots, v_n], [w_1, \ldots, w_l]$ bases in $U, V, W$
Bilinear maps and product of matrices

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\[ \phi(u_i, v_j) = \sum_{k=1} t_{i,j,k} w_k, \mathcal{T} := [t_{i,j,k}] \in \mathbb{F}^{m \times n \times l} \]
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\( \phi(c, d) = \sum_{a=1}^{r} (c^\top x)(d^\top y) z_a, \ c = \sum_{i=1}^{m} c_i u_i, \ d = \sum_{j=1}^{n} d_j v_j \)
Bilinear maps and product of matrices

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Complexity: \( r \)-products
Bilinear maps and product of matrices

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Complexity: \(r\)-products

Matrix product \( \tau : \mathbb{F}^{M \times N} \times \mathbb{F}^{N \times L} \to \mathbb{F}^{M \times L}, (A, B) \mapsto AB \)
Bilinear maps and product of matrices

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\( M = N = L = 2, \quad \text{grank}_C(4, 4, 4) = \lceil \frac{4 \times 4 \times 4}{4+4+4-2} \rceil = \lceil 6.4 \rceil = 7 \)
Bilinear maps and product of matrices

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Complexity: $r$-products

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Product of two $2 \times 2$ matrices is done by 7 multiplications
Known cases of rank conjecture

\[
g\frac{\text{rank}}{2}(3, 2^p, 2^q) = \lceil \frac{12p^2}{4} + 1 \rceil \quad \text{and} \quad g\frac{\text{rank}}{2}(3, 2^p - 1, 2^q - 1) = \lceil 3(2^p - 1)^2 \rceil + 1
\]

\[
\text{if} \quad n \not\equiv 2 \pmod{3}, \quad (n - 1, n, n) \quad \text{if} \quad n \equiv 0 \pmod{3}, \quad (4, m, m) \quad \text{if} \quad m \geq 4, \quad (n, n, n) \quad \text{if} \quad n \geq 4
\]

\[
(l, 2^p, 2^q) \quad \text{if} \quad l \leq 2^p \leq 2^q \quad \text{and} \quad \frac{lp}{2} + 2^p + 2^q - 2 \quad \text{is integer}
\]

Easy to compute \( g\frac{\text{rank}}{2}(C; m, n, l) \):

Pick at random \( w_r := (x_1, y_1, z_1, \ldots, x_r, y_r, z_r) \in (\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^l)^r \)

The minimal \( r \geq \lceil mnl(m + n + l - 2) \rceil \) s.t.

\[
\text{rank}(J(f)(w_r)) = mnl
\]

is \( g\frac{\text{rank}}{2}(C; m, n, l) \) (Terracini Lemma 1915)

Avoid round-off error:

\( w_r \in (\mathbb{Z}^m \times \mathbb{Z}^n \times \mathbb{Z}^l)^r \) find \( \text{rank}(J(f)(w_r)) \) exact arithmetic

I checked the conjecture up to \( m, n, l \leq 14 \).
Known cases of rank conjecture

\[ \text{grank}(3, 2p, 2p) = \left\lceil \frac{12p^2}{4p+1} \right\rceil \text{ and } \text{grank}(3, 2p - 1, 2p - 1) = \left\lceil \frac{3(2p-1)^2}{4p-1} \right\rceil + 1 \]
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\((n, n, n + 2) \text{ if } n \neq 2 \pmod{3},\)
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\((n, n, n + 2)\) if \(n \neq 2 \pmod{3}\),
\((n - 1, n, n)\) if \(n = 0 \pmod{3}\),
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\((l, 2p, 2q)\) if \(l \leq 2p \leq 2q\) and \(\frac{2lp}{l+2p+2q-2}\) is integer.
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\( (n, n, n + 2) \) if \( n \neq 2 \) (mod 3),

\( (n - 1, n, n) \) if \( n = 0 \) (mod 3),

\( (4, m, m) \) if \( m \geq 4 \),

\( (n, n, n) \) if \( n \geq 4 \)

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Easy to compute \( \text{grank}_C(m, n, l) \):
Known cases of rank conjecture

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Pick at random \( w_r := (x_1, y_1, z_1, \ldots, x_r, y_r, z_r) \in (\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^l)^r \)
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The minimal \( r \geq \left\lceil mnl \over (m+n+l-2) \right\rceil \) s.t. \( \text{rank} \ J(f_r)(\mathbf{w}_r) = mnl \)

is \( \text{grank}_\mathbb{C}(m, n, l) \) (Terracini Lemma 1915)
Known cases of rank conjecture

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Avoid round-off error:
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I checked the conjecture up to \( m, n, l \leq 14 \)
Generic rank III - the real case

For $m \leq l$,

$$\text{rank } R(m, n, l) = mn.$$  

For $2 \leq m \leq n \leq l < mn - 1$, there exist $V_1, \ldots, V_c(m, n, l) \subset R^{m \times n \times l}$ pairwise distinct open connected semi-algebraic sets s.t.

$$\text{Closure } (\bigcup_{i=1}^c V_i(m, n, l)) = R^{m \times n \times l}$$

for each $T \in V_1$,

$$\text{rank } T = \rho_i$$

for each $T \in V_i$.

For $l = (m - 1)(n - 1) + 1$,

$$\exists m, n : c(m, n, l) > 1, \rho_c(m, n, l) \geq \text{rank } C(m, n, l) + 1.$$  

Examples [3]

$m = n \geq 2$, $l = (m - 1)(n - 1) + 1$.

$m = n = 4$, $l = 11, 12$. 

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Tensors and Matrices"
For $mn \leq l$ $\text{rank}_\mathbb{R}(m, n, l) = mn$. 

Examples [3]

$m = n \geq 2, l = (m - 1)(n - 1) + 1$.
Generic rank III - the real case

For $mn \leq l$ \( \text{grank}_\mathbb{R}(m, n, l) = mn \).

For $2 \leq m \leq n \leq l < mn - 1$, there exist $V_1, \ldots, V_{c(m, n, l)} \subset \mathbb{R}^{m \times n \times l}$ pairwise distinct open connected semi-algebraic sets s.t.
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\[
\text{Closure}\left( \bigcup_{i=1}^{c(m, n, l)} V_i \right) = \mathbb{R}^{m \times n \times l}
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For $mn \leq l$ \( \text{grank}_\mathbb{R}(m, n, l) = mn \).

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\[
\text{Closure}(\bigcup_{i=1}^{c(m,n,l)}) = \mathbb{R}^{m \times n \times l}
\]

\[
\text{rank } \mathcal{T} = \text{grank}_\mathbb{C}(m, n, l) \text{ for each } \mathcal{T} \in V_1
\]
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\]

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\[\text{rank } \mathcal{T} = \rho_i \text{ for each } \mathcal{T} \in V_i\]

\[\rho_i \geq \text{grank}_\mathbb{C}(m, n, l) \text{ for } i = 2, \ldots, c(m, n, l)\]
For \( mn \leq l \) \( \text{grank}_\mathbb{R}(m, n, l) = mn \).

For \( 2 \leq m \leq n \leq l < mn - 1 \), there exist \( V_1, \ldots, V_{c(m,n,l)} \subset \mathbb{R}^{m \times n \times l} \) pairwise distinct open connected semi-algebraic sets s.t.

\[
\text{Closure}\left( \bigcup_{i=1}^{c(m,n,l)} \right) = \mathbb{R}^{m \times n \times l} \\
\text{rank } \mathcal{T} = \text{grank}_\mathbb{C}(m, n, l) \text{ for each } \mathcal{T} \in V_1 \\
\text{rank } \mathcal{T} = \rho_i \text{ for each } \mathcal{T} \in V_i \\
\rho_i \geq \text{grank}_\mathbb{C}(m, n, l) \text{ for } i = 2, \ldots, c(m, n, l)
\]

For \( l = (m - 1)(n - 1) + 1 \) \( \exists m, n: \)

\( c(m, n, l) > 1, \rho_{c(m,n,l)} \geq \text{grank}_\mathbb{C}(m, n, l) + 1 \)
Generic rank III - the real case

For $mn \leq l \text{ grank}_\mathbb{R}(m, n, l) = mn$.

For $2 \leq m \leq n \leq l < mn-1$, there exist $V_1, \ldots, V_{c(m,n,l)} \subset \mathbb{R}^{m \times n \times l}$ pairwise distinct open connected semi-algebraic sets s.t.

$\text{Closure}(\bigcup_{i=1}^{c(m,n,l)}) = \mathbb{R}^{m \times n \times l}$

$\text{rank } \mathcal{T} = \text{grank}_\mathbb{C}(m, n, l)$ for each $\mathcal{T} \in V_1$

$\text{rank } \mathcal{T} = \rho_i$ for each $\mathcal{T} \in V_i$

$\rho_i \geq \text{grank}_\mathbb{C}(m, n, l)$ for $i = 2, \ldots, c(m, n, l)$

For $l = (m-1)(n-1) + 1 \exists m, n:
\quad c(m, n, l) > 1, \rho_{c(m,n,l)} \geq \text{grank}_\mathbb{C}(m, n, l) + 1$

Examples [3]
$m = n \geq 2, \ l = (m-1)(n-1) + 1$.
$m = n = 4, \ l = 11, 12$
Rank one approximations
Rank one approximations

\[ \mathbb{R}^{m \times n \times l} \text{ IPS: } \langle A, B \rangle = \sum_{i=j=k}^{m,n,l} a_{i,j,k} b_{i,j,k}, \quad \|T\| = \sqrt{\langle T, T \rangle} \]
Rank one approximations

$\mathbb{R}^{m \times n \times l}$ IPS: $\langle A, B \rangle = \sum_{i=j=k}^{m,n,l} a_{i,j,k} b_{i,j,k}$, $\|T\| = \sqrt{\langle T, T \rangle}$

$\langle x \otimes y \otimes z, u \otimes v \otimes w \rangle = (u^T x)(v^T y)(w^T z)$
Rank one approximations

\[ \mathbb{R}^{m \times n \times l} \text{ IPS: } \langle A, B \rangle = \sum_{i=j=k}^{m,n,l} a_{i,j,k} b_{i,j,k}, \quad \| T \| = \sqrt{\langle T, T \rangle} \]

\[ \langle x \otimes y \otimes z, u \otimes v \otimes w \rangle = (u^\top x)(v^\top y)(w^\top z) \]

\(X\) subspace of \(\mathbb{R}^{m \times n \times l}\), \(X_1, \ldots, X_d\) an orthonormal basis of \(X\)
Rank one approximations

\( \mathbb{R}^{m \times n \times l} \) IPS: \( \langle A, B \rangle = \sum_{i=j=k}^{m,n,l} a_{i,j,k} b_{i,j,k}, \quad \|T\| = \sqrt{\langle T, T \rangle} \)

\( \langle x \otimes y \otimes z, u \otimes v \otimes w \rangle = (u^\top x)(v^\top y)(w^\top z) \)

**X** subspace of \( \mathbb{R}^{m \times n \times l} \), \( \mathcal{X}_1, \ldots, \mathcal{X}_d \) an orthonormal basis of **X**

\( P_{\mathcal{X}}(T) = \sum_{i=1}^{d} \langle T, \mathcal{X}_i \rangle \mathcal{X}_i, \quad \|P_{\mathcal{X}}(T)\|^2 = \sum_{i=1}^{d} \langle T, \mathcal{X}_i \rangle^2 \)
Rank one approximations

\[ \mathbb{R}^{m \times n \times l} \text{ IPS: } \langle A, B \rangle = \sum_{i=j=k}^{m,n,l} a_{i,j,k} b_{i,j,k}, \|T\| = \sqrt{\langle T, T \rangle} \]

\[ \langle x \otimes y \otimes z, u \otimes v \otimes w \rangle = (u^T x)(v^T y)(w^T z) \]

\( X \) subspace of \( \mathbb{R}^{m \times n \times l} \), \( \mathcal{X}_1, \ldots, \mathcal{X}_d \) an orthonormal basis of \( X \)

\[ P_X(T) = \sum_{i=1}^{d} \langle T, \mathcal{X}_i \rangle \mathcal{X}_i, \quad \|P_X(T)\|^2 = \sum_{i=1}^{d} \langle T, \mathcal{X}_i \rangle^2 \]

\[ \|T\|^2 = \|P_X(T)\|^2 + \|T - P_X(T)\|^2 \]
Rank one approximations

**IPs:** \( \langle A, B \rangle = \sum_{i=j=k} a_{i,j,k} b_{i,j,k}, \| T \| = \sqrt{\langle T, T \rangle} \)

\( \langle x \otimes y \otimes z, u \otimes v \otimes w \rangle = (u^T x)(v^T y)(w^T z) \)

\( X \) subspace of \( \mathbb{R}^{m \times n \times l} \), \( \mathcal{X}_1, \ldots, \mathcal{X}_d \) an orthonormal basis of \( X \)

\( P_X(T) = \sum_{i=1}^d \langle T, \mathcal{X}_i \rangle \mathcal{X}_i, \quad \| P_X(T) \|^2 = \sum_{i=1}^d \langle T, \mathcal{X}_i \rangle^2 \)

\( \| T \|^2 = \| P_X(T) \|^2 + \| T - P_X(T) \|^2 \)

Best rank one approximation of \( T \):
**Rank one approximations**

\[ \mathbb{R}^{m \times n \times l} \text{ IPS: } \langle A, B \rangle = \sum_{i=j=k}^{m,n,l} a_{i,j,k} b_{i,j,k}, \quad \|T\| = \sqrt{\langle T, T \rangle} \]

\[ \langle x \otimes y \otimes z, u \otimes v \otimes w \rangle = (u^T x)(v^T y)(w^T z) \]

**X** subspace of \( \mathbb{R}^{m \times n \times l} \), \( \mathcal{X}_1, \ldots, \mathcal{X}_d \) an orthonormal basis of **X**

\[ P_\mathcal{X}(T) = \sum_{i=1}^{d} \langle T, \mathcal{X}_i \rangle \mathcal{X}_i, \quad \|P_\mathcal{X}(T)\|^2 = \sum_{i=1}^{d} \langle T, \mathcal{X}_i \rangle^2 \]

\[ \|T\|^2 = \|P_\mathcal{X}(T)\|^2 + \|T - P_\mathcal{X}(T)\|^2 \]

Best rank one approximation of \( T \):

\[ \min_{x, y, z} \| T - x \otimes y \otimes z \| = \min_{\|x\| = \|y\| = \|z\| = 1, a} \| T - a \cdot x \otimes y \otimes z \| \]
Rank one approximations

\[ \mathbb{R}^{m \times n \times l} \text{ IPS: } \langle A, B \rangle = \sum_{i=j=k} a_{i,j,k} b_{i,j,k}, \quad \|T\| = \sqrt{\langle T, T \rangle} \]

\[ \langle x \otimes y \otimes z, u \otimes v \otimes w \rangle = (u^T x)(v^T y)(w^T z) \]

**X** subspace of \( \mathbb{R}^{m \times n \times l} \), \( X_1, \ldots, X_d \) an orthonormal basis of \( X \)

\[ P_X(T) = \sum_{i=1}^d \langle T, \mathcal{X}_i \rangle \mathcal{X}_i, \quad \|P_X(T)\|^2 = \sum_{i=1}^d \langle T, \mathcal{X}_i \rangle^2 \]

\[ \|T\|^2 = \|P_X(T)\|^2 + \|T - P_X(T)\|^2 \]

Best rank one approximation of \( T \):

\[ \min_{x,y,z} \|T - x \otimes y \otimes z\| = \min_{\|x\| = \|y\| = \|z\| = 1} \|T - a \times x \otimes y \otimes z\| \]

Equivalent: \[ \max_{\|x\| = \|y\| = \|z\| = 1} \sum_{i=j=k} t_{i,j,k} x_i y_j z_k \]
Rank one approximations

\[ \mathbb{R}^{m \times n \times l} \text{ IPS: } \langle A, B \rangle = \sum_{i=j=k}^{m,n,l} a_{i,j,k} b_{i,j,k}, \quad \| T \| = \sqrt{\langle T, T \rangle} \]

\[ \langle x \otimes y \otimes z, u \otimes v \otimes w \rangle = (u^\top x)(v^\top y)(w^\top z) \]

\( X \) subspace of \( \mathbb{R}^{m \times n \times l} \), \( \mathcal{X}_1, \ldots, \mathcal{X}_d \) an orthonormal basis of \( X \)

\[ P_X(T) = \sum_{i=1}^{d} \langle T, \mathcal{X}_i \rangle \mathcal{X}_i, \quad \| P_X(T) \|^2 = \sum_{i=1}^{d} \langle T, \mathcal{X}_i \rangle^2 \]

\[ \| T \|^2 = \| P_X(T) \|^2 + \| T - P_X(T) \|^2 \]

Best rank one approximation of \( T \):

\[ \min_{x,y,z} \| T - x \otimes y \otimes z \| = \min_{\| x \|=\| y \|=\| z \|=1,a} \| T - a x \otimes y \otimes z \| \]

Equivalent: \( \max_{\| x \|=\| y \|=\| z \|=1} \sum_{i=j=k}^{m,n,l} t_{i,j,k} x_i y_j z_k \)

Lagrange multipliers: \( T \times y \otimes z := \sum_{j=k=1}^{m,n,l} t_{i,j,k} y_j z_k = \lambda x \)

\[ T \times x \otimes z = \lambda y, \quad T \times x \otimes y = \lambda z \]
Rank one approximations

\[ \mathbb{R}^{m \times n \times l} \text{ IPS: } \langle A, B \rangle = \sum_{i=j=k}^{m,n,l} a_{i,j,k} b_{i,j,k}, \quad \| T \| = \sqrt{\langle T, T \rangle} \]

\[ \langle x \otimes y \otimes z, u \otimes v \otimes w \rangle = (u^T x)(v^T y)(w^T z) \]

\( X \) subspace of \( \mathbb{R}^{m \times n \times l} \), \( X_1, \ldots, X_d \) an orthonormal basis of \( X \)

\[ P_X(T) = \sum_{i=1}^{d} \langle T, X_i \rangle X_i, \quad \| P_X(T) \|^2 = \sum_{i=1}^{d} \langle T, X_i \rangle^2 \]

\[ \| T \|^2 = \| P_X(T) \|^2 + \| T - P_X(T) \|^2 \]

Best rank one approximation of \( T \):

\[ \min_{x,y,z} \| T - x \otimes y \otimes z \| = \min_{\| x \|=\| y \|=\| z \|=1,a} \| T - a \, x \otimes y \otimes z \| \]

Equivalent: \( \max_{\| x \|=\| y \|=\| z \|=1} \sum_{i=j=k}^{m,n,l} t_{i,j,k} x_i y_j z_k \)

Lagrange multipliers: \( T \times y \otimes z := \sum_{j=k}^{1} t_{i,j,k} y_j z_k = \lambda x \)

\( T \times x \otimes z = \lambda y, \ T \times x \otimes y = \lambda z \)

\( \lambda \) singular value, \( x, y, z \) singular vectors
Rank one approximations

$$\mathbb{R}^{m \times n \times l} \text{ IPS: } \langle A, B \rangle = \sum_{i=j=k}^{m,n,l} a_{i,j,k} b_{i,j,k}, \quad \|T\| = \sqrt{\langle T, T \rangle}$$

$$\langle x \otimes y \otimes z, u \otimes v \otimes w \rangle = (u^\top x)(v^\top y)(w^\top z)$$

$X$ subspace of $\mathbb{R}^{m \times n \times l}$, $X_1, \ldots, X_d$ an orthonormal basis of $X$

$$P_X(T) = \sum_{i=1}^{d} \langle T, X_i \rangle X_i, \quad \|P_X(T)\|^2 = \sum_{i=1}^{d} \langle T, X_i \rangle^2$$

$$\|T\|^2 = \|P_X(T)\|^2 + \|T - P_X(T)\|^2$$

Best rank one approximation of $T$:

$$\min_{x,y,z} \|T - x \otimes y \otimes z\| = \min_{\|x\|=\|y\|=\|z\|=1} \|T - a x \otimes y \otimes z\|$$

Equivalent: $$\max_{\|x\|=\|y\|=\|z\|=1} \sum_{i=j=k}^{m,n,l} t_{i,j,k} x_i y_j z_k$$

Lagrange multipliers: $$T \times y \otimes z := \sum_{j=k=1}^{l} t_{i,j,k} y_j z_k = \lambda x$$

$$T \times x \otimes z = \lambda y, \quad T \times x \otimes y = \lambda z$$

$\lambda$ singular value, $x, y, z$ singular vectors

How many distinct singular values are for a generic tensor?
\( \ell_p \) maximal problem and Perron-Frobenius
\( \ell_p \) maximal problem and Perron-Frobenius

\[ \|(x_1, \ldots, x_n)^\top\|_p := \left( \sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \]
$\ell_p$ maximal problem and Perron-Frobenius

$\| (x_1, \ldots, x_n) ^\top \|_p := \left( \sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}}$

**Problem:** \( \max \| x \|_p = \| y \|_p = \| z \|_p = 1 \sum_{i=j=k}^{m,n,l} t_{i,j,k} x_i y_j z_k \)

Assume that \( T \geq 0 \). Then \( x, y, z \geq 0 \)

For which values of \( p \) we have an analog of Perron-Frobenius theorem?

Yes, for \( p \geq 3 \), No, for \( p < 3 \),

Friedland-Gauber-Han [1]
\( \ell_p \) maximal problem and Perron-Frobenius

\[ \| (x_1, \ldots, x_n)^T \|_p \equiv \left( \sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} \]

**Problem:** \( \max \| x \|_p = \| y \|_p = \| z \|_p = 1 \sum_{i=j=k} t_{i,j,k} x_i y_j z_k \)

**Lagrange multipliers:** \( T \times y \otimes z := \sum_{j=k=1} t_{i,j,k} y_j z_k = \lambda x^{p-1} \)
\( \ell_p \) maximal problem and Perron-Frobenius

\[ \| (x_1, \ldots, x_n) \|_p := \left( \sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \]

**Problem:** \( \max \| x \|_p = \| y \|_p = \| z \|_p = 1 \sum_{i=j=k} t_{i,j,k} x_i y_j z_k \)

**Lagrange multipliers:** \( \mathcal{T} \times y \otimes z := \sum_{j=k=1}^{m,n,l} t_{i,j,k} y_j z_k = \lambda x^{p-1} \)

\( \mathcal{T} \times x \otimes z = \lambda y^{p-1}, \mathcal{T} \times x \otimes y = \lambda z^{p-1} \) \((p = \frac{2t}{2s-1}, t, s \in \mathbb{N})\)

Yes, for \( p \geq 3 \), No, for \( p < 3 \), Friedland-Gauber-Han [1]
\( \ell_p \) maximal problem and Perron-Frobenius

\[
\| (x_1, \ldots, x_n)^\top \|_p := \left( \sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}
\]

Problem: \( \max_{\|x\|_p = \|y\|_p = \|z\|_p = 1} \sum_{i=j=k}^{m,n,l} t_{i,j,k} x_i y_j z_k \)

Lagrange multipliers: \( T \times y \otimes z := \sum_{j=k=1}^{t,s} t_{i,j,k} y_j z_k = \lambda x^{p-1} \)
\( T \times x \otimes z = \lambda y^{p-1}, \ T \times x \otimes y = \lambda z^{p-1} \) \( (p = \frac{2t}{2s-1}, t, s \in \mathbb{N}) \)

\( p = 3 \) is most natural in view of homogeneity
\( \ell_p \) maximal problem and Perron-Frobenius

\[
\| (x_1, \ldots, x_n)^\top \|_p := \left( \sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}}
\]

Problem: \( \max \|x\|_p = \|y\|_p = \|z\|_p = 1 \sum_{i=j=k}^{m,n,l} t_{i,j,k} x_i y_j z_k \)

Lagrange multipliers: \( T \times y \otimes z := \sum_{j=k=1}^{t_{i,j,k}} y_j z_k = \lambda x^{p-1} \)
\( T \times x \otimes z = \lambda y^{p-1}, \ T \times x \otimes y = \lambda z^{p-1} \) (\( p = \frac{2t}{2s-1}, t, s \in \mathbb{N} \))

\( p = 3 \) is most natural in view of homogeneity

Assume that \( T \geq 0 \). Then \( x, y, z \geq 0 \)

For which values of \( p \) we have an analog of Perron-Frobenius theorem?

Yes, for \( p \geq 3 \), No, for \( p < 3 \),
Friedland-Gauber-Han [1]
$(R_1, R_2, R_3)$-rank approximation of 3-tensors
Fundamental problem in applications:
Approximate well and fast \( \mathcal{T} \in \mathbb{R}^{m_1 \times m_2 \times m_3} \) by rank \((R_1, R_2, R_3)\) 3-tensor.
$(R_1, R_2, R_3)$-rank approximation of 3-tensors

Fundamental problem in applications:
Approximate well and fast $T \in \mathbb{R}^{m_1 \times m_2 \times m_3}$ by rank $(R_1, R_2, R_3)$ 3-tensor.

Best $(R_1, R_2, R_3)$ approximation problem:
Find $U_i \subset \mathbb{F}^{m_i}$ of dimension $R_i$ for $i = 1, 2, 3$ with maximal
\[ \| P_{U_1 \otimes U_2 \otimes U_3}(T) \| . \]
Fundamental problem in applications:
Approximate well and fast $T \in \mathbb{R}^{m_1 \times m_2 \times m_3}$ by rank $(R_1, R_2, R_3)$ 3-tensor.

Best $(R_1, R_2, R_3)$ approximation problem:
Find $U_i \subset \mathbb{F}^{m_i}$ of dimension $R_i$ for $i = 1, 2, 3$ with maximal
$\|P_{U_1 \otimes U_2 \otimes U_3} (T)\|$.

Relaxation method:
Optimize on $U_1, U_2, U_3$ by fixing all variables except one at a time.
\((R_1, R_2, R_3)\)-rank approximation of 3-tensors

Fundamental problem in applications:
Approximate well and fast \( T \in \mathbb{R}^{m_1 \times m_2 \times m_3} \) by rank \((R_1, R_2, R_3)\) 3-tensor.

Best \((R_1, R_2, R_3)\) approximation problem:
Find \( U_i \subset F^{m_i} \) of dimension \( R_i \) for \( i = 1, 2, 3 \) with maximal
\[ \| P_{U_1 \otimes U_2 \otimes U_3}(T) \|. \]

Relaxation method:
Optimize on \( U_1, U_2, U_3 \) by fixing all variables except one at a time
This amounts to SVD (Singular Value Decomposition) of matrices:
$$(R_1, R_2, R_3)$$-rank approximation of 3-tensors

Fundamental problem in applications:
Approximate well and fast $T \in \mathbb{R}^{m_1 \times m_2 \times m_3}$ by rank $(R_1, R_2, R_3)$ 3-tensor.

Best $(R_1, R_2, R_3)$ approximation problem:
Find $U_i \subset \mathbb{F}^{m_i}$ of dimension $R_i$ for $i = 1, 2, 3$ with maximal $\|P_{U_1 \otimes U_2 \otimes U_3}(T)\|$. 

Relaxation method:
Optimize on $U_1, U_2, U_3$ by fixing all variables except one at a time
This amounts to SVD (Singular Value Decomposition) of matrices:
Fix $U_2, U_3$. Then $V = U_1 \otimes (U_2 \otimes U_3) \subset \mathbb{R}^{m_1 \times (m_2 \cdot m_3)}$
Fundamental problem in applications:
Approximate well and fast \( T \in \mathbb{R}^{m_1 \times m_2 \times m_3} \) by rank \((R_1, R_2, R_3)\) 3-tensor.

Best \((R_1, R_2, R_3)\) approximation problem:
Find \( U_i \subset \mathbb{F}^{m_i} \) of dimension \( R_i \) for \( i = 1, 2, 3 \) with maximal
\[ |\langle PU_1 \otimes U_2 \otimes U_3 (T)\rangle| \]

Relaxation method:
Optimize on \( U_1, U_2, U_3 \) by fixing all variables except one at a time
This amounts to SVD (Singular Value Decomposition) of matrices:
Fix \( U_2, U_3 \). Then \( V = U_1 \otimes (U_2 \otimes U_3) \subset \mathbb{R}^{m_1 \times (m_2 \cdot m_3)} \)

\[ \max_{U_1} |\langle PV (T)\rangle| \] is an approximation in 2-tensors=matrices
Fundamental problem in applications:
Approximate well and fast $\mathcal{T} \in \mathbb{R}^{m_1 \times m_2 \times m_3}$ by rank $(R_1, R_2, R_3)$ 3-tensor.

Best $(R_1, R_2, R_3)$ approximation problem:
Find $\mathbb{U}_i \subset \mathbb{F}^{m_i}$ of dimension $R_i$ for $i = 1, 2, 3$ with maximal 
\[ \| P_{\mathbb{U}_1 \otimes \mathbb{U}_2 \otimes \mathbb{U}_3}(\mathcal{T}) \| . \]

Relaxation method:
Optimize on $\mathbb{U}_1, \mathbb{U}_2, \mathbb{U}_3$ by fixing all variables except one at a time
This amounts to SVD (Singular Value Decomposition) of matrices:
Fix $\mathbb{U}_2, \mathbb{U}_3$. Then $\mathbb{V} = \mathbb{U}_1 \otimes (\mathbb{U}_2 \otimes \mathbb{U}_3) \subset \mathbb{R}^{m_1 \times (m_2 \cdot m_3)}$

\[ \max_{\mathbb{U}_1} \| P_{\mathbb{V}}(\mathcal{T}) \| \text{ is an approximation in 2-tensors=matrices} \]

Use Newton method on Grassmannians - Eldén-Savas 2009 [1]
Fast low rank approximation I:
Fast low rank approximations II:

Approximate $A \in \mathbb{R}^{m \times n}$ by $CUR$ where $C \in \mathbb{R}^{m \times p}$, $R \in \mathbb{R}^{q \times n}$ for some submatrices of $A$.

$$\min_{U \in C^{p \times q}} \|A - CUR\|_F$$

Achieved for $U = C^\dagger A R^\dagger$ (corresponds to best $CUR$ approximation on the entries read).


For given $A \in \mathbb{R}^{m \times n \times l}$, $F \in \mathbb{R}^{m \times p}$, $E \in \mathbb{R}^{n \times q}$, $G \in \mathbb{R}^{l \times r}$, where $\langle p \rangle \subset \langle n \rangle \times \langle l \rangle$, $\langle q \rangle \subset \langle m \rangle \times \langle l \rangle$, $\langle r \rangle \subset \langle m \rangle \times \langle l \rangle$,

$$\min_{U \in C^{p \times q \times r}} \|A - U \times F \times E \times G\|_F$$

Achieved for $U = A \times E^\dagger \times F^\dagger \times G^\dagger$ (CUR approximation of $A$ obtained by choosing $E$, $F$, $G$ submatrices of unfolded $A$ in the mode 1, 2, 3).
Approximate $A \in \mathbb{R}^{m \times n}$ by $CUR$ where $C \in \mathbb{R}^{m \times p}$, $R \in \mathbb{R}^{q \times n}$ for some submatrices of $A$. 
Approximate $A \in \mathbb{R}^{m \times n}$ by $CUR$ where $C \in \mathbb{R}^{m \times p}, R \in \mathbb{R}^{q \times n}$ for some submatrices of $A$.

$$\min_{U \in \mathbb{C}^{p \times q}} \| A - CUR \|_F \text{ achieved for } U = C^\dagger AR^\dagger$$
Approximate $A \in \mathbb{R}^{m \times n}$ by $CUR$ where $C \in \mathbb{R}^{m \times p}$, $R \in \mathbb{R}^{q \times n}$ for some submatrices of $A$.

$$\min_{U \in \mathbb{C}^{p \times q}} \| A - CUR \|_F$$ achieved for $U = C^\dagger AR^\dagger$

Faster choice: $U = A[I, J]^\dagger$
Approximate $A \in \mathbb{R}^{m \times n}$ by $CUR$ where $C \in \mathbb{R}^{m \times p}$, $R \in \mathbb{R}^{q \times n}$ for some submatrices of $A$.

$$\min_{U \in \mathbb{C}^{p \times q}} \|A - CUR\|_F$$ achieved for $U = C^\dagger AR^\dagger$

**Faster choice:** $U = A[I, J]^\dagger$

(corresponds to best $CUR$ approximation on the entries read)
Approximate $A \in \mathbb{R}^{m \times n}$ by $CUR$ where $C \in \mathbb{R}^{m \times p}$, $R \in \mathbb{R}^{q \times n}$ for some submatrices of $A$.

$$\min_{U \in \mathbb{C}^{p \times q}} \| A - CUR \|_F$$ achieved for $U = C^\dagger A R^\dagger$

Faster choice: $U = A[I, J]^\dagger$
(corresponds to best $CUR$ approximation on the entries read)

For given $A \in \mathbb{R}^{m \times n \times l}$, $F \in \mathbb{R}^{m \times p}$, $E \in \mathbb{R}^{n \times q}$, $G \in \mathbb{R}^{l \times r}$,
Approximate $A \in \mathbb{R}^{m \times n}$ by $CUR$ where $C \in \mathbb{R}^{m \times p}$, $R \in \mathbb{R}^{q \times n}$ for some submatrices of $A$.

$$\min_{U \in \mathbb{C}^{p \times q}} \| A - CUR \|_F$$

achieved for $U = C^\dagger AR^\dagger$

Faster choice: $U = A[I, J]^\dagger$

(corresponds to best $CUR$ approximation on the entries read)

For given $A \in \mathbb{R}^{m \times n \times l}$, $F \in \mathbb{R}^{m \times p}$, $E \in \mathbb{R}^{n \times q}$, $G \in \mathbb{R}^{l \times r}$, where $\langle p \rangle \subset \langle n \rangle \times \langle l \rangle$, $\langle q \rangle \subset \langle m \rangle \times \langle l \rangle$, $\langle r \rangle \subset \langle m \rangle \times \langle l \rangle$
Approximate $A \in \mathbb{R}^{m \times n}$ by $CUR$ where $C \in \mathbb{R}^{m \times p}$, $R \in \mathbb{R}^{q \times n}$ for some submatrices of $A$.

$$\min_{U \in \mathbb{C}^{p \times q}} \| A - CUR \|_F$$ achieved for $U = C^\dagger A R^\dagger$

Faster choice: $U = A[I, J]^\dagger$ (corresponds to best $CUR$ approximation on the entries read)

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$$\min_{U \in \mathbb{C}^{p \times q \times r}} \| A - U \times F \times E \times G \|_F$$ achieved for $U = A \times E^\dagger \times F^\dagger \times G^\dagger$
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$$\min_{U \in \mathbb{C}^{p \times q \times r}} \| A - U \times F \times E \times G \|_F$$

achieved for $U = A \times E^\dagger \times F^\dagger \times G^\dagger$

$CUR$ approximation of $A$ obtained by choosing $E, F, G$ submatrices of unfolded $A$ in the mode 1, 2, 3.
List of applications

- Face recognition
- Video tracking
- Factor analysis
List of applications

Face recognition
List of applications

Face recognition

Video tracking
List of applications

Face recognition

Video tracking

Factor analysis
Scaling of nonnegative tensors to tensors with given row, column and depth sums

$0 \leq T = [t_{i,j,k}] \in \mathbb{R}^{m \times n \times l}$ has given row, column and depth sums:

$r = (r_1, \ldots, r_m)^\top$, $c = (c_1, \ldots, c_n)^\top$, $d = (d_1, \ldots, d_l)^\top > 0$:
Scaling of nonnegative tensors to tensors with given row, column and depth sums

\[ 0 \leq \mathcal{T} = [t_{i,j,k}] \in \mathbb{R}^{m \times n \times l} \] has given row, column and depth sums:

\[ \mathbf{r} = (r_1, \ldots, r_m)^	op, \mathbf{c} = (c_1, \ldots, c_n)^	op, \mathbf{d} = (d_1, \ldots, d_l)^	op > \mathbf{0}: \]

\[ \sum_{j,k} t_{i,j,k} = r_i > 0, \sum_{i,k} t_{i,j,k} = c_j > 0, \sum_{i,j} t_{i,j,k} = d_k > 0 \]

\[ \sum_{i=1}^m r_i = \sum_{j=1}^n c_j = \sum_{k=1}^l d_k \]
Scaling of nonnegative tensors to tensors with given row, column and depth sums

Let \( 0 \leq \mathcal{T} = [t_{i,j,k}] \in \mathbb{R}^{m \times n \times l} \) have given row, column and depth sums: 

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\mathbf{r} = (r_1, \ldots, r_m) \mathbf{c} = (c_1, \ldots, c_n) \mathbf{d} = (d_1, \ldots, d_l) > \mathbf{0}:
\]

\[
\sum_{j,k} t_{i,j,k} = r_i > 0, \quad \sum_{i,k} t_{i,j,k} = c_j > 0, \quad \sum_{i,j} t_{i,j,k} = d_k > 0
\]

\[
\sum_{i=1}^m r_i = \sum_{j=1}^n c_j = \sum_{k=1}^l d_k
\]

Find necessary and sufficient conditions for scaling:

\( \mathcal{T}' = [t_{i,j,k} e^{x_i+y_j+z_k}] \), \( \mathbf{x}, \mathbf{y}, \mathbf{z} \) such that \( \mathcal{T}' \) has given row, column and depth sum.
Scaling of nonnegative tensors to tensors with given row, column and depth sums

\[ 0 \leq T = [t_{i,j,k}] \in \mathbb{R}^{m \times n \times l} \] has given row, column and depth sums:
\[ r = (r_1, \ldots, r_m)^\top, \ c = (c_1, \ldots, c_n)^\top, \ d = (d_1, \ldots, d_l)^\top > 0: \]
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\[ \sum_{i=1}^m r_i = \sum_{j=1}^n c_j = \sum_{k=1}^l d_k \]

Find nec. and suf. conditions for scaling:
\[ T' = [t_{i,j,k} e^{x_i+y_j+z_k}], \ x, y, z \] such that \( T' \) has given row, column and depth sum

Solution: Convert to the minimal problem:
\[ \min_{r^\top x = c^\top y = d^\top z = 0} f_T(x, y, z), \quad f_T(x, y, z) = \sum_{i,j,k} t_{i,j,k} e^{x_i+y_j+z_k} \]
Scaling of nonnegative tensors to tensors with given row, column and depth sums

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\[ \sum_{j,k} t_{i,j,k} = r_i > 0, \quad \sum_{i,k} t_{i,j,k} = c_j > 0, \quad \sum_{i,j} t_{i,j,k} = d_k > 0 \]

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Find nec. and suf. conditions for scaling:

\[ T' = [t_{i,j,k} e^{x_i+y_j+z_k}], \quad x, y, z \text{ such that } T' \text{ has given row, column and depth sum} \]

Solution: Convert to the minimal problem:

\[ \min r^\top x = c^\top y = d^\top z = 0 \quad f_T(x, y, z), \quad f_T(x, y, z) = \sum_{i,j,k} t_{i,j,k} e^{x_i+y_j+z_k} \]

Any critical point of \( f_T \) on \( S := \{ r^\top x = c^\top y = d^\top z = 0 \} \) gives rise to a solution of the scaling problem (Lagrange multipliers)
Scaling of nonnegative tensors to tensors with given row, column and depth sums

\(0 \leq \mathcal{T} = [t_{i,j,k}] \in \mathbb{R}^{m \times n \times l}\) has given row, column and depth sums:

\[
\begin{align*}
\mathbf{r} &= (r_1, \ldots, r_m)^\top, \quad \mathbf{c} = (c_1, \ldots, c_n)^\top, \quad \mathbf{d} = (d_1, \ldots, d_l)^\top > 0: \\
\sum_{j,k} t_{i,j,k} &= r_i > 0, \quad \sum_{i,k} t_{i,j,k} = c_j > 0, \quad \sum_{i,j} t_{i,j,k} = d_k > 0 \\
\sum_{i=1}^m r_i &= \sum_{j=1}^n c_j = \sum_{k=1}^l d_k
\end{align*}
\]

Find nec. and suf. conditions for scaling:

\(\mathcal{T}' = [t_{i,j,k}e^{x_i+y_j+z_k}], \mathbf{x}, \mathbf{y}, \mathbf{z}\) such that \(\mathcal{T}'\) has given row, column and depth sum

Solution: Convert to the minimal problem:

\[
\min_{\mathbf{r}^\top \mathbf{x} = \mathbf{c}^\top \mathbf{y} = \mathbf{d}^\top \mathbf{z} = 0} f_\mathcal{T}(\mathbf{x}, \mathbf{y}, \mathbf{z}), \quad f_\mathcal{T}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \sum_{i,j,k} t_{i,j,k}e^{x_i+y_j+z_k}
\]

Any critical point of \(f_\mathcal{T}\) on \(S := \{\mathbf{r}^\top \mathbf{x} = \mathbf{c}^\top \mathbf{y} = \mathbf{d}^\top \mathbf{z} = 0\}\) gives rise to a solution of the scaling problem (Lagrange multipliers)

\(f_\mathcal{T}\) is convex
Scaling of nonnegative tensors to tensors with given row, column and depth sums

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\[ r = (r_1, \ldots, r_m)^\top, \quad c = (c_1, \ldots, c_n)^\top, \quad d = (d_1, \ldots, d_l)^\top > 0: \]
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Find nec. and suf. conditions for scaling:
\[ \mathcal{T}' = [t_{i,j,k} e^{x_i+y_j+z_k}], \quad x, y, z \] such that \( \mathcal{T}' \) has given row, column and depth sum

Solution: Convert to the minimal problem:
\[ \min_{r^\top x = c^\top y = d^\top z = 0} f_{\mathcal{T}}(x, y, z), \quad f_{\mathcal{T}}(x, y, z) = \sum_{i,j,k} t_{i,j,k} e^{x_i+y_j+z_k} \]

Any critical point of \( f_{\mathcal{T}} \) on \( S := \{ r^\top x = c^\top y = d^\top z = 0 \} \) gives rise to a solution of the scaling problem (Lagrange multipliers)
\( f_{\mathcal{T}} \) is convex
\( f_{\mathcal{T}} \) is strictly convex implies \( \mathcal{T} \) is not decomposable: \( \mathcal{T} \neq \mathcal{T}_1 \oplus \mathcal{T}_2 \).
Scaling of nonnegative tensors II

if $f_T$ is strictly convex and is $\infty$ on $\partial S$, $f_T$ achieves its unique minimum

Equivalent to: the inequalities $x_i + y_j + z_k \leq 0$ if $t_i, j, k > 0$ and equalities $r^\top x = c^\top y = d^\top z = 0$ imply $x = 0^m, y = 0^n, z = 0^l$.

Fact: For $r = 1^m, c = 1^n, d = 1^l$ Sinkhorn scaling algorithm works. Newton method works, since the scaling problem is equivalent finding the unique minimum of strict convex function

Hence Newton method has a quadratic convergence versus linear convergence of Sinkhorn algorithm

True for matrices too

Are variants of Menon and Brualdi theorems hold in the tensor case? Yes for Menon, unknown for Brualdi
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Characterization of tensor in $\mathbb{C}^{4\times4\times4}$ of border rank 4

Major problem in algebraic statistics: phylogenetic trees and their invariants [1]
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Major problem in algebraic statistics:
phylogenetic trees and their invariants \cite{1}

\( W \subset \mathbb{C}^{4 \times 4} \) subspace spanned by four sections of \( T \in \mathbb{C}^{4 \times 4 \times 4} \)
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If \( \mathbf{W} \) contains an invertible matrix \( \mathbf{Z} \) then any other \( \mathbf{X}, \mathbf{Y} \in \mathbf{W} \) satisfy
\[ \mathbf{X}(\text{adj}\mathbf{Z})\mathbf{Y} = \mathbf{Y}(\text{adj}\mathbf{Z})\mathbf{X} \] - equations of degree 5

Strassen's condition hold for any 3 \( \times 3 \times 3 \) subtensor of \( \mathbf{T} \):
\[ \det(\mathbf{U}(\text{adj}\mathbf{W})\mathbf{V} - \mathbf{V}(\text{adj}\mathbf{W})\mathbf{U}) = 0, \mathbf{U}, \mathbf{V}, \mathbf{W} \in \mathbb{C}^{3 \times 3 \times 3} \]

- equations of degree 9

Friedland [5] one needs a equations of degree 16
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$X(adj Z)Y = Y(adj Z)X$ - equations of degree 5

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Strassen’s condition hold for any $3 \times 3 \times 3$ subtensor of $T$:

$\det(U(\text{adj} W)V - V(\text{adj} W)U) = 0, \quad U, V, W \in \mathbb{C}^{3 \times 3 \times 3}$

equations of degree 9
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References I


S. Friedland, On tensors of border rank $l$ in $\mathbb{C}^{m\times n\times l}$, arXiv:1003.1968v1.


