On the entropy of Z^d subshifts of finite type

Shmuel Friedland

University of Illinois at Chicago

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§0. Introduction

Let n be a positive integer and denote < n > = \{1, ..., n\}. We view < n > as an alphabet on n letters. Denote by < n > Z^d the set of all mappings of Z^d to the set < n >. By extending the Hamming metric on < n > × < n > to < n > Z^d one obtains that < n > Z^d is a compact metric space. The group Z^d acts as a group of (translation) automorphisms on < n > Z^d. A set S ⊂ Z^d which is closed and invariant under the action of Z^d is called a subshift. S is called a subshift of finite type (SFT) if there is a finite set of finite admissible configurations which generates S under the action of Z^d. More precisely, let F ⊂ Z^d be a finite set. Assume that P ⊂ < n > F. Then (F, P) defines a following Z^d-SFT S. For each a ∈ Z^d let F + a ∈ Z^d be the corresponding translation of F. Then x ∈ S iff for each a ∈ Z^d, π_{F+a}(x), the projection of x on the set F + a, is in P. See for example [Sch, Ch. 5].

The case of Z action, i.e. d = 1, is well understood. In that case, it is relatively easily to decide whether S is empty or not. Moreover, the topological entropy h(S) of the restriction of the standard shift to S is a logarithm of an algebraic integer ρ(F, P). The number ρ(F, P) is a the spectral radius of certain 0−1 square matrix induced by (F, P). Furthermore, the topological entropy h(S) is equal to the rate of growth of the number of periodic points. The case d > 1 is much more complicated. First, the problem wether S is an empty set or not is undecidable. This result for d = 2 goes back to Berger [Ber]. See also [K-M-W] and [Rob]. Second, there exists a SFT S ≠ ∅ which does not have periodic points. Moreover, in the case where S ≠ ∅ the topological entropy may be uncomputable, see [H-K-C] and [Gab].

The object of this paper to show that contrary to these results one has a natural and a simple criterion which either determines that S = ∅ or calculates the topological entropy of S ≠ ∅. There is no contradiction to the uncomputability of h(S) because we can not estimate the rate of convergence of our sequence. However, if we introduce a symmetry in Z^2 we can estimate the rate of convergence of our sequence. Moreover, in this case h(S) is the rate of growth the number of periodic points. Our main tool is to view a Z^d-SFT as a matrix SFT. See [M-P1, M-P2]. In fact our methods are very close to the methods of [M-P1, M-P2].

We now describe briefly the content of the paper. In §1 we define combinatorial entropy of Z^d-SFT. It can is computed by finite configurations. We then observe, using König’s method, that Z^d-SFT is nonempty iff every finite configuration is nonempty. In §2 we show that the combinatorial entropy is equal to the topological entropy of Z^d-SFT.
In §3 we show that simple symmetricity conditions yield that the topological entropy of $\mathbb{Z}^d$-SFT is equal to the periodic entropy. (The periodic entropy is the rate growth of the periodic points.) In the case $d = 2$ combined with the symmetricity assumption we obtain an algorithm for computing the entropy at any given precision. This result is due to [M-P2] under stricter conditions. The last section is devoted to various remarks.

§1. Preliminary results

Let $\Gamma \subset < n > \times < n >$. Set
\[
\Gamma^N = \{ x = (x_i)_1^N, (x_i, x_{i+1}) \in \Gamma, i = 1, \ldots, N - 1 \}, \\
\Gamma^\infty = \{ x = (x_i)_{i \in \mathbb{Z}} : (x_i, x_{i+1}) \in \Gamma, i \in \mathbb{Z} \}.
\]
Assume that $\Gamma_i \subset < n > \times < n >, i = 1, \ldots, d$. Set $\Gamma = (\Gamma_1, \ldots, \Gamma_d)$ and let
\[
\Gamma^\infty = \{ f : f \in < n > \mathbb{Z}^d, (f_{(i_1, \ldots, i_d)})_{i_1 \in \mathbb{Z}} \in \Gamma_k^\infty, \\
(i_1, \ldots, i_{k-1}, i_{k+1}, \ldots, i_d) \in \mathbb{Z}^{d-1}, k = 1, \ldots, d \}
\]
to be a $\mathbb{Z}^d$-SFT induced by $\Gamma$. We now show that a standard $\mathbb{Z}^d$-SFT is equivalent the $\mathbb{Z}^d$-SFT induced by $\Gamma$. For the case $d = 2$ this can be deduced from [Moz] who proved that every $\mathbb{Z}^2$-SFT is equivariant to Wang-tiling-space. It is easy to see that Wang-tiling-space is $\mathbb{Z}^2$-SFT induced by some special $\Gamma = (\Gamma_1, \Gamma_2)$.

Let $S$ be a SFT is given by the pair $(F, P)$ as in §0. Let $N = (N_1, \ldots, N_d) \in \mathbb{Z}^d, N_i \geq 1, i = 1, \ldots, d$. By $B(N)$ we denote the box $< N_1 > \times \cdots \times < N_d > \subset \mathbb{Z}^d$. Let $f = (f_{(i_1, \ldots, i_d)})_{(i_1, \ldots, i_d)}^N \in < n > B(N)$. Then $f$ is called $(F, P)$ admissible if for all $a \in \mathbb{Z}^d$ such that $F+a \subset B(N)$ we have the condition that $\pi_{F+a}(f)$ - the projection of $f$ on the set $F+a$ is $P$-admissible, i.e. $\pi_{F+a}(f) \in P$. Let $(1, \ldots, 1) \leq M(F) = (M_1(F), \ldots, M_d(F)) \in \mathbb{Z}^d$ be the dimension of the smallest box containing $F$. That is, $B(M(F)) \supset F + a$ for some $a \in \mathbb{Z}^d$ and $B(M(F))$ is minimal with respect to this property. For $M(F) \leq N \in \mathbb{Z}^d$ let $w(N, F, P)$ be the number of $(F, P)$ admissible words in $B(N).$ We then let
\[
h_{com}(F, P) = \limsup_{N_1, \ldots, N_d \to \infty} \frac{\log w(N, F, P)}{N_1 \cdots N_d}
\]
to be the combinatorial entropy of induced by $(F, P)$. We agree that $\log 0 = -\infty$. That is $h_{com}(F, P) \geq 0$ iff every box $B(N)$ has at least one $(P, F)$ admissible configuration $f \in < n > B(N)$. Observe next that if $M_i(F) = 1$ for some some $i$ then we effectively can consider the corresponding $\mathbb{Z}^{d-1}$-SFT. For
\[
N = (N_1, \ldots, N_d) \in \mathbb{Z}^d, N_k > 1, k = 1, \ldots, d
\]
let
\[
\Gamma^N = \{ f = (f_{(i_1, \ldots, i_d)})_{i_1=1}^{N_1, \ldots, i_d=1} : (f_{(i_1, \ldots, i_d)})_{i_k=1}^{N_k, i_k = 1} \in \Gamma_k^{N_k}, k = 1, \ldots, d \},
\]
for every \(i_1, ..., i_{k-1}, i_{k+1}, ..., i_d\).

Set

\[
F = \{1, 2\}^d = B(2, ..., 2), P = \Gamma^{(2, ..., 2)}, w(N, \Gamma) = w(N, F, P).
\]

Then for any \(N = (N_1, ..., N_d) > (1, ..., 1)\) the set \(\Gamma^N\) consists of all \((F, P)\) admissible words in \(< n >^B(N)\). Define \(h_{\text{com}}(\Gamma) = h_{\text{com}}(F, P)\).

\((1.1)\) Theorem. Let \(F \subset \mathbb{Z}^d\) be a finite set such that \(1 < M_i(F), i = 1, ..., d\). Assume that \(P \subset < n >^F\). Denote by \(T \subset < n >^{B(M(F))}\) the set of all \((F, P)\) admissible words in \(< n >^{B(M(F))}\). For each \(i = 1, ..., d\), and \(u \in T\) let \(\pi_{i, -}(u), \pi_{i, +}(u)\) be the projection of \(u\) on the sets

\[
B(M_i(F), ..., M_{i-1}(F), M_i(F) - 1, M_{i+1}(F), ..., M_d(F)),
\]

\[
B(M_i(F), ..., M_{i-1}(F), M_i(F) - 1, M_{i+1}(F), ..., M_d(F)) + (\delta_{i1}, ..., \delta_{id}).
\]

Set

\[
\Gamma_i = \{(u, v) : u, v \in T, \pi_{i, +}(u) = \pi_{i, -}(v)\} \subset T \times T, i = 1, ..., d, \quad \Gamma = (\Gamma_1, ..., \Gamma_d).
\]

Then for any \(N = (k_1 + M_1(F), ..., k_d + M_d(F)), k_i \geq 1, i = 1, ..., d\), the set of all \((F, P)\) admissible words in \(B(N)\) is in one to one correspondence with \(\Gamma^{(k_1+1, ..., k_d+1)}\) on the alphabet \(T\). In particular the set of all admissible \((F, P)\) words in \(< n >^{\mathbb{Z}^d}\) is in one to one correspondence with \(\Gamma^\infty\). Furthermore \(h_{\text{com}}(F, P) = h_{\text{com}}(\Gamma)\).

Proof. Let \(N = (k_1 + M_1(F), ..., k_d + M_d(F)), k_i \geq 1, i = 1, ..., d\). Assume that \(f \in < n >^{B(N)}\) be an \((F, P)\) admissible word. For \((l_1, ..., l_d), 1 \leq l_j \leq k_j + 1, j = 1, ..., d\), let \(g(l_1, ..., l_d)\) be the word in \(T\) which has the following coordinates in \(f\):

\[
l_i \leq j_i \leq l_i + M_i(F) - 1, i = 1, ..., d.
\]

It is straightforward to check that \(g = (g(l_1, ..., l_d))^{(k_1+1, ..., k_d+1)} \in \Gamma^{(k_1+1, ..., k_d+1)}\). Assume that \(g \in \Gamma^{(k_1+1, ..., k_d+1)}\). Use the above formula to find a unique \(f \in < n >^{B(N)}\) so that \(g\) is constructed from \(f\) as above. We claim that \(f\) is a \((F, P)\) admissible word in \(< n >^{B(N)}\). Assume that \(F + a \subset B(N)\). Then there exists \(l_1, ..., l_d, 1 \leq l_i \leq k_i + 1, i = 1, ..., d\), so that the coordinates of \(F + a\) satisfy the inequalities \((1.2)\). That is, \(\pi_{F+a}(f)\) lies in the word \(u\) generated by the projection of \(f\) on the coordinates specified by \((1.2)\). By the construction, \(u \in T\). In particular, \(\pi_{F+a}(f) = \pi_{F+a}(u) \in P\). Hence, \(f\) is a \((F, P)\) admissible word. Therefore, \(w(N, F, P)\) is equal to \(\omega(k_1 + 1, ..., k_d + 1) = \text{card}(\Gamma^{(k_1+1, ..., k_d+1)})\). All other assertions of the Theorem follow straightforward. \(\diamond\)

Let \((1, ..., 1) \leq N \in \mathbb{Z}^d\). Partition the box \(B(N)\) to \(p\) nontrivial boxes of dimensions \(N^i \in \mathbb{Z}^d, i = 1, ..., p\). It then follows that \(w(N, F, P) \leq \prod_{1}^{p} w(N^i, F, P)\). We thus deduce

\[
h_{\text{com}}(F, P) = \lim_{N_1, ..., N_d \to \infty} \log \frac{w((N_1, ..., N_d), F, P)}{N_1 \cdot \cdots \cdot N_d} = \lim_{m \to \infty} \frac{\log w((m, ..., m), F, P)}{m^d}.
\]
(1.3) Theorem. Let $S$ be a $\mathbb{Z}^d$-SFT given by $(F, P)$. Then

$$S \neq \emptyset \iff w((m, ..., m), F, P) \geq 1, m = 2, ..., .$$

That is, $S = \emptyset \iff h_{\text{com}} = -\infty$.

Proof. Clearly, if $S \neq \emptyset$ then $h_{\text{com}}(F, P) \geq 0$. In particular, $w((m, ... , m), \Gamma) \geq 1, m = 2, ...$. Assume now that $w((m, ... , m), \Gamma) \geq 1, m = 2, ...$. Consider the box $B((2m, ... , 2m)) - B_{2m}$ in $\mathbb{R}^d$ whose center is at the origin $(0, ..., 0)$. Let $\Theta_m \in \Gamma^{(2m, ..., 2m)}$ be an admissible filling of $B_{2m}$ by the alphabet $\{1, ..., n\}$. Consider the sequence $\{\Theta_m\}_{i=1}^\infty$. Look at the projection of this sequence on $B_2$. Pick up an infinite subsequence $\{\Theta_n\}_{i=1}^\infty$ whose so that the projection of each $\Theta_n$ on $B_2$ is the same element $\Psi_1 \in \Gamma^{(2, ..., 2)}$. From the sequence $\Theta_n$ pick a subsequence $\Theta_n'$ so that the projection of each element $\Theta_n'$ on $B_4$ is the same element $\Psi_2 \in \Gamma^{(4, ..., 4)}$. Continue this construction to obtain that the sequence $\Psi_k \in \Gamma^{(2m, ..., 2m)}, k = 1, ...,$ which are $2m \times \cdots \times 2m$ sections of an element $\Psi \in \Gamma^\infty$. The above argument is due to König [Kön].

Introduce on $< n >$ the Hamming metric $d(i, i) = 0, d(i, j) = 1, i \neq j \in < n >$. For $i = (i_1, ..., i_d) \in \mathbb{Z}^d$ we let $|i| = \sum_1^d |i_p|$. On $< n >^{\mathbb{Z}^d}$ define the following metric

$$d(f, g) = \frac{1}{2^{2d}} \sum_{i = (i_1, ..., i_d) \in \mathbb{Z}^d} \frac{d(f_i, g_i)}{2^{|i|}}, f = (f_i), g = (g_i) \in < n >^{\mathbb{Z}^d}.$$ 

It then follows that $< n >^{\mathbb{Z}^d}$ is a compact metric space. Let $e_i = (\delta_{i1}, ..., \delta_{id}), i = 1, ..., d,$ be the standard basis in $\mathbb{Z}^d$. Denote by $T_i : < n >^{\mathbb{Z}^d} \to < n >^{\mathbb{Z}^d}$ the following automorphism of $< n >^{\mathbb{Z}^d}$:

$$T_i(f_j) = (f_{j + e_i}), j \in \mathbb{Z}^d, f = (f_j) \in < n >^{\mathbb{Z}^d}.$$ 

$S \subseteq < n >^{\mathbb{Z}^d}$ is called a subshift (SF) if $S$ is closed and $T_iS = S, i = 1, ..., d$. In that case one defines a topological entropy $h(S)$ as follows. For $(1, ..., 1) \leq N = (N_1, ..., N_k)$ introduce the following new metric on $< n >^{\mathbb{Z}^d}$:

$$d_N(f, g) = \max_{0 \leq p < N_p, p = 1, ..., d} d(T_i^{1} \cdots T_d^{N}, f, T_i^{1} \cdots T_d^{N} g), f, g \in < n >^{\mathbb{Z}^d}.$$ 

Fix a positive $\epsilon > 0$ and let $K(S, N, \epsilon)$ be the maximal number of $\epsilon$ separated points in $S$ in the metric $d_N(\cdot, \cdot)$. We then let

$$h(S) = \lim_{\epsilon \to \infty} \limsup_{N_1, ..., N_d \to \infty} \log K(S, N, \epsilon) / N_1 \cdots N_d. \quad (1.4)$$

(1.5) Theorem. Let $\Gamma_i \subseteq < n > \times < n >, i = 1, ..., d$, and set $\Gamma = \Gamma_1 \times \cdots \times \Gamma_d$. Assume that $\Gamma^\infty \neq \emptyset$. Define $h(\Gamma) = h(\Gamma^\infty)$. For $(1, ..., 1) \leq N \in \mathbb{Z}^d$ let $w(N, \Gamma^\infty)$ be the number of all possible projections of $f \in \Gamma^\infty$ on a fixed box $B(N)$. Then

$$h(\Gamma) = \limsup_{N_1, ..., N_d \to \infty} \log w(N, \Gamma^\infty) / N_1 \cdots N_d.$$
In particular, $h(\Gamma) \leq h_{\text{com}}(\Gamma)$.

**Proof.** It is quite straightforward to see from the definition of $K(\Gamma^\infty, N, \epsilon)$ that for a small enough $\epsilon > 0$ there exist some constants $1 \leq a(\epsilon), 1 \leq b(\epsilon) \in \mathbb{Z}$ so that

$$w(N, \Gamma^\infty) \leq K(\Gamma^\infty, N, \epsilon) \leq a(\epsilon)w(N + (b(\epsilon), ..., b(\epsilon)) \in \Gamma^\infty).$$

Now the characterization of $h(\Gamma)$ follows straightforward from (1.4). As $w(N, \Gamma^\infty) \leq w(N, \Gamma)$ we deduce that $h(\Gamma) \leq h_{\text{com}}(\Gamma)$. \(\diamondsuit\)

§2. The equality of topological and combinatorial entropy for SFT

Let $\Gamma < n > \times < n >$. Denote by $A = A(\Gamma)$ the $0 - 1$ matrix induced by the graph $\Gamma$. Let $\rho(A)$ be the spectral radius of $A$. Set

$$\text{per}(\Gamma^N) = \{(x_i)_1^N : (x_i)_1^N \in \Gamma^N, x_1 = x_N \}.$$ 

Assume that $\Gamma_i < n > \times < n >, i = 1, ..., d$. Set

$$\Gamma = (\Gamma_1, ..., \Gamma_d), \Gamma^\downarrow = (\Gamma_1, ..., \Gamma_{i-1}, \Gamma_{i+1}, ..., \Gamma_d), i = 1, ..., d.$$ 

For

$$N = (N_1, ..., N_d) \in \mathbb{Z}^d, N_k > 1, k = 1, ..., d,$$

$$M = (M_1, ..., M_{d-1}) \in \mathbb{Z}^{d-1}, M_j > 1, j = 1, ..., d - 1,$$

let

$$\text{per}(\Gamma^N) = \{f = (f_{(i_1, ..., i_d)})_{i_1 = ..., i_d = 1}^{N_1, ..., N_d} : (f_{(i_1, ..., i_d)})_{i_k = 1}^{N_k} \in \text{per}(\Gamma_k^{N_k}), k = 1, ..., d\},$$

$$w_p(N, \Gamma) = \text{card}(\text{per}(\Gamma^N)),$$

$$\Gamma(k, M) = \{(a, b) : a = (a_{(i_1, ..., i_{d-1})}), b = (b_{(i_1, ..., i_{d-1})}) \in (\Gamma^\downarrow)^M,$$

$$(a_{(i_1, ..., i_{d-1})}, b_{(i_1, ..., i_{d-1})}) \in \Gamma_k, i_j = 1, ..., M_j, j = 1, ..., d - 1, \}, k = 1, ..., d,$$

$$p(\Gamma(k, M)) = \{(a, b) : a = (a_{(i_1, ..., i_{d-1})}), b = (b_{(i_1, ..., i_{d-1})}) \in \text{per}(\Gamma(\downarrow)^M),$$

$$(a_{(i_1, ..., i_{d-1})}, b_{(i_1, ..., i_{d-1})}) \in \Gamma_k, i_j = 1, ..., M_j, j = 1, ..., d - 1, \}, k = 1, ..., d,$$

$$A(k, M) = A(\Gamma(k, M)), \rho(k, M) = \rho(A(k, M)),$$

$$A(p(\Gamma(k, M))), \rho p(k, M) = \rho(A(p(\Gamma(k, M))), k = 1, ..., d.$$

Note that any $f \in \text{per}(\Gamma^N)$ has a unique minimal periodic extension to $\Gamma^\infty$. Set

$$h_{\text{p}}(\Gamma) = \limsup_{N_1, ..., N_d \to \infty} \frac{\log wp((N_1, ..., N_d), \Gamma)}{N_1 \cdots N_d}.$$
to be the periodic entropy of $\Gamma^\infty$.

**Theorem (2.1).** Let $d \geq 2$ and assume that $\Gamma_i \subset \subset n > \times < n >, i = 1, \ldots, d$. Consider $\mathbb{Z}^d$-SFT given by $\Gamma = (\Gamma_1, \ldots, \Gamma_d)$. Then

$$h_{com}(\Gamma) = -\infty \iff \forall M = (M_1, \ldots, M_{d-1}) \gg (1, \ldots, 1) \rho(k, M) = 0, k = 1, \ldots, d,$$

$$hp(\Gamma) = -\infty \iff \forall M = (M_1, \ldots, M_{d-1}) \rho p(k, M) = 0, k = 1, \ldots, d.$$

Furthermore

$$\lim \frac{\log \rho(k, (M_1, \ldots, M_{d-1}))}{M_1 \cdots M_{d-1}} = h_{com}(\Gamma), k = 1, \ldots, d,$$

$$\frac{\log \rho(k, (M_1, \ldots, M_{d-1}))}{M_1 \cdots M_{d-1}} \geq h_{com}(\Gamma), M_i > 1, i = 1, \ldots, d - 1, k = 1, \ldots, d,$$

$$\limsup \frac{\log \rho p(k, (M_1, \ldots, M_{d-1}))}{M_1 \cdots M_{d-1}} \leq hp(\Gamma), k = 1, \ldots, d.$$

**Proof.** We first prove the theorem for $d = 2$. In that case $M = (m)$ and we let $\Gamma(k, M) = \Gamma(k, m), \rho(k, M) = \rho(k, m)$ for $k = 1, 2$. Suppose first that there exists $N = (N_1, N_2)$ so that $\Gamma^N = \emptyset$. We then claim that $\rho(1, m) = 0$ for $m \geq N_2$. Suppose to the contrary that $\rho(1, m) \geq 1$. That is, $A(1, m)$ is not a nilpotent matrix. That is, $\Gamma(1, m)^l \neq \emptyset, l = 2, \ldots, \ldots$. Clearly,

$$\Gamma(1, m)^l = \Gamma(l, m). \quad (2.2)$$

Set $l = N_1$ to obtain a contradiction. Similarly, $\rho(2, m) = 0$ for $m \geq N_2$. Assume now that $\rho(1, m) = 0$ for some $m \geq 1$. Let $L_2(m) = card(\Gamma_2^m)$. Then $\Gamma(1, m)^{L_2(m)} = \emptyset$. Use (2.2) to deduce that $\Gamma(l^{L_2(m)}, m) = \emptyset$. Similar results hold if $\rho(2, m) = 0$.

Assume now $h_{com}(\Gamma) \geq 0$, i.e. $\rho(1, m) \geq 1, \rho(2, m) \geq 1, m = 1, \ldots$. We now prove the conditions related to the characterization of $h_{com}(\Gamma)$ in terms of $\rho(1, m)$. We claim that

$$\log \rho(1, p + q) \leq \log \rho(1, p) + \log \rho(1, q), p, q \geq 1. \quad (2.3)$$

Indeed, let $w((l, p), \Gamma), w((l, q), \Gamma), w((l, p + q), \Gamma)$ be the total number of words of length $l$ corresponding to the subshifts $\Gamma(1, p), \Gamma(1, q), \Gamma(1, p + q)$ respectively. Clearly, every word of length $l$ in $\Gamma(1, p + q)$ splits (from bottom to top) as a word in $\Gamma(1, p)$ followed by a word in $\Gamma(1, q)$. That is $w((l, p + q), \Gamma) \leq w((l, p), \Gamma)w((l, q), \Gamma)$. Take the logarithm of this inequality, divide by $l$ and take the limsup to deduce (2.3). It is a well known fact that (2.3) implies that the sequence $\left\{\frac{\log \rho(1, m)}{m}\right\}_1^\infty$ converges to a (nonnegative) limit $h$. Furthermore, $h \leq \frac{\log \rho(1, m)}{m}, m = 1, \ldots$. We now show that $h = h_{com}(\Gamma)$. Let $\{\epsilon_m\}_1^\infty$ be a positive sequence which converges to zero. Clearly, there exists a sequence of positive integers $\{l_m\}_1^\infty$ converging to $\infty$ so that

$$\frac{\log w((l_m, m), \Gamma)}{l_m} > \log \rho(1, m) - \epsilon_m, m = 1, \ldots.$$


Hence,
\[ h_{com}(\Gamma) \geq \lim \sup \frac{\log w((l_m, m), \Gamma)}{l_m m} \geq h. \]

We now show the reversed inequality. Let \( \{m_i\}_1^\infty, \{n_i\}_1^\infty \) be two sequences of positive integers which converge to \( \infty \). We claim that

\[ \lim \sup \frac{\log w((n_i, m_i), \Gamma)}{n_i m_i} \leq h. \]

Pick a positive \( \delta > 0 \). Pick a positive integer \( m \) so that \( \frac{\log \rho(1, m)}{m} < h + \delta \). Let \( K > 1 \) so that
\[ \forall n > K \max_{1 \leq k \leq m} \left( \frac{\log w((n, k), \Gamma)}{n} - \log \rho(1, k) \right) < \delta. \]

Assume that \( m_i, n_i > K \). Set \( m_i = p_i m + q_i, 1 \leq q_i \leq m \). Consider a word of length \( n_i \) corresponding to SFT induced by \( \Gamma(1, m_i) \). This word splits (from bottom to top) as \( p_i \) words induced by \( \Gamma(1, m) \) and a word induced by \( \Gamma(1, q) \) of length \( n_i \) respectively. Hence, \( w((n_i, m_i), \Gamma) \leq w((n_i, m), \Gamma) \rho^{p_i} w((n_i, q_i), \Gamma) \). That is

\[ \frac{\log w((n_i, m_i), \Gamma)}{n_i m_i} \leq \frac{\log w((n_i, m), \Gamma)}{n_i m} + \frac{\log w((n_i, q_i), \Gamma)}{n_i m_i} \leq \frac{\log \rho(1, m)}{m} + \frac{\delta}{m} + \max_{1 \leq k \leq m} \rho(1, k) + \delta, m_i, n_i > K. \]

Thus, \( \lim \sup_{m_i, n_i \to \infty} \frac{\log w((n_i, m_i), \Gamma)}{m_i n_i} < h + 2\delta \). These arguments prove the theorem for \( \rho(1, m) \). Similar arguments verify the theorem for \( \rho(2, m) \).

We now consider the periodic solutions. Assume first that \( \text{per}(\Gamma^N) \neq \emptyset \) for some \( N = (N_1, N_2), N_1 > 1, N_2 > 1 \). It then follows that

\[ \text{per}(\Gamma^M) \neq \emptyset, M = (N_1 + i(N_1 - 1), N_2 + j(N_2 - 1)), i, j = 0, \ldots. \quad (2.4) \]

We then claim that \( \rho p(1, N_2) \geq 1, \rho p(2, N_1) \geq 1 \). Consider first the matrix \( Ap(1, N_2) \). If \( \rho p(1, N_2) = 0 \), i.e. \( Ap(1, N_2) \) is nilpotent, we could not have arbitrary long words in the SFT induced by \( p(\Gamma(1, N_2)) \). This contradicts (2.4) for \( j = 0 \). Similarly, \( \rho p(2, N_1) \geq 1 \). Assume now that \( \rho p(1, N_2) \geq 1 \) for some \( N_2 > 1 \). Then the SFT induced by \( p(\Gamma(1, N_2)) \) has at least one periodic word of length \( N_1 > 1 \), i.e. \( \text{per}(p(\Gamma(1, N_2)))^{N_1} \neq \emptyset \). As every periodic word of length \( N_1 \) in the SFT corresponding to \( p(\Gamma(1, N_2)) \) is an element of \( \text{per}(\Gamma^{(N_1, N_2)}) \) we deduce in particular \( \text{per}(\Gamma^{(N_1, N_2)}) \neq \emptyset \). That is,

\[ hp(\Gamma) = -\infty \iff \rho p(1, m) = \rho p(2, m) = 0, m = 2, \ldots. \]

Assume now that \( hp(\Gamma) \geq 0 \). We now prove the theorem for \( \rho p(1, m) \). Consider the SFT induced by \( p(\Gamma(1, m)) \). Then \( \text{w}(l, m, \Gamma) \) is the number of periodic words of length \( l \) of this SFT. As \( \rho p(1, m) \geq 1 \) we know that for any \( \delta > 0 \) there exists \( l = l(\delta) \) so that
\[
\log \frac{wp((l,m),\Gamma)}{l} \geq \log \rho p(1,m) - \delta.
\]
Assume that \(\{m_i\}_{i=1}^{\infty}\) is a strictly increasing sequence of positive integers so that

\[
\limsup_{m \to \infty} \frac{\log \rho p(1,m)}{m} = \lim_{i \to \infty} \frac{\log \rho p(1,m_i)}{m_i}.
\]

Let \(\{l_i\}_{i=1}^{\infty}\) be a strictly increasing sequence so that \(\log \frac{wp((l_i,m_i),\Gamma)}{l_i} \geq \log \rho p(1,m_i) - 1, i = 2,\ldots\). We then deduce \(\limsup_{m \to \infty} \frac{\log \rho p(1,m)}{m} \leq hp(\Gamma)\). The analogous result for \(\rho p(2,m)\) is proved similarly.

Let \(d > 2\). Assume that \((1,\ldots,1) < M \in \mathbb{Z}^d_{-1}\). Partition the box \(B(M)\) to \(p\) nontrivial boxes of dimensions \(M_i \in \mathbb{Z}^d_{-1}, i = 1,\ldots,p\). We denote this fact by \(M = \bigcup_1^p M_i\). We then have the following generalization of (2.3).

\[
\log \rho(k,M) \leq \sum_1^p \log \rho(k,M_i), k = 1,\ldots,d. \tag{2.3}'
\]

Similarly, all assertions of the theorem for \(d > 2\) are derived in an analogous way. ◇

**Theorem.** Let \(d \geq 2\) and assume that \(\Gamma_i \subset < n > \times < n >, i = 1,\ldots,d\). Consider the \(\mathbb{Z}^d\)-SFT given by \(\Gamma = (\Gamma_1,\ldots,\Gamma_d)\). Then

\[
h_{com}(\Gamma) = h(\Gamma).
\]

To prove the theorem we need the following result.

**Lemma.** Let the assumptions of Theorem 2.5 hold. Assume furthermore that \(\Gamma^\infty \neq \emptyset\). Let \(M, N_1, N_2 \in \mathbb{Z}^d\) and assume that \((1,\ldots,1) \leq M \leq N_1 \leq N_2\). Then

\[
\pi_{B(2M)+N_1-M}(\Gamma^{2N_1}) \supset \pi_{B(2M)+N_2-M}(\Gamma^{2N_2}).
\]

Assume that \(f \in \Gamma^{2M}\). Then

\[
\exists g \in \Gamma^\infty \pi_{B(2M)}g = f \iff \forall N f \in \pi_{B(2M)+N-M}(\Gamma^{2N}).
\]

**Proof.** The first claim of the lemma is trivial. Assume that \(g \in \Gamma^\infty\). Let \(f = \pi_{B(2M)}g\). Clearly, \(\forall N f \in \pi_{B(2M)+N-M}(\Gamma^{2N})\). The reverse implication is proved by using König’s argument as in the proof of Theorem 1.3. ◇

**Proof of Theorem 2.5** By Theorem 1.3 \(h_{com}(\Gamma) = -\infty \iff h(\Gamma) = -\infty\). Thus, it is enough to consider the case \(h_{com}(\Gamma) \geq 0\). As \(w(N,\Gamma) \geq w(N,\Gamma^\infty)\) Theorem 1.5 implies that \(h_{com}(\Gamma) \geq h(\Gamma)\). Thus \(h_{com}(\Gamma) = 0 \Rightarrow h(\Gamma) = 0\). Hence, it is left to prove the theorem in the case \(h_{com}(\Gamma) > 0\). For simplicity of the exposition we consider the case \(d = 2\).
Fix $k \geq 1$ and let $m \geq k$. Consider the graph $\Gamma(1, 2m)$. It represents a SFT induced by an infinite horizontal strip of width $2m$ in the vertical direction. Erase from the above infinite horizontal strip $m - k$ first and last infinite rows. We then obtain a $S(2m)(2k)$ a SFT induced by the graph $\Gamma(1, 2m)$. Furthermore, $S(2m)(2k)$ is a subshift of $S(2k)$ induced by $\Gamma(1, 2m)$. Clearly, we have the inclusions

$$S(2k) \supset S(2(k + 1))(2k) \supset \cdots \supset S(2m)(2k) \supset \cdots.$$ 

Fix a box of dimension $(2l, 2k)$ and let $w_{2m}(2l, 2k)$ be the projection of $S(2m)(2k)$ on this box. Clearly

$$w((2l, 2k), \Gamma) > w_{2(k + 1)}(2l, 2k) > \cdots > w_{2m(k)}(2l, 2k) = w_{2(m(k) + 1)}(2l, 2k) = \ldots.$$ 

König’s argument yield that

$$w((2l, 2k), \Gamma^\infty) = w_{2m(k)}(2l, 2k).$$

We claim that

$$w((2l, 2k), \Gamma^\infty)^{p - 2m(k)} \geq \frac{\rho(1, p2k)^{2l}}{\rho(1, 2k)^{2l2m(k)}}, p >> 1. \quad (2.7)$$

To prove this inequality consider the infinite horizontal strip of width $p2k$ where $p > 2m(k)$. In this strip pick up a box of dimension $(rl, p2k)$ where $r >> 1$. It then follows that

$$w((rl, p2k), \Gamma) \approx K_1 (rl)^{s_1} \rho(1, p2k)^{rl}$$

for some fixed integer $s_1$. We view the above strip as composed of $p$ infinite strips of width $2k$. For $m(k)$ the most upper strips and for $m(k)$ the most lower strips the number of words in the box $(rl, 2k)$ does not exceed

$$w((rl, 2k), \Gamma) \approx K_2 (rl)^{s_2} \rho(1, 2k)^{rl}.$$ 

We now consider all the other infinite horizontal strip of width $2k$. Observe that they are all SFT contained in $S(2m(k))(2k)$. Denote by $C(l, 2k)$ all distinct projections of $\Gamma^\infty$ on a box $B(l, 2k)$. Denote by $\Delta(l, 2k) \subset C(l, 2k) \times C(l, 2k)$ the following graph induced by all distinct projections of $\Gamma^\infty$ on the box $B((2l, 2k))$. That is $(x, y) \in \Delta(l, 2k)$ iff $(x, y)$ is obtained by the projection on $B((2l, 2k))$ of some possible configuration in $\Gamma^\infty$. Let $w(t, \Delta(l, 2k))$ be the number of words of length $t$ in the SFT induced by $\Delta(l, 2k)$. Set $\tilde{\rho}(l, 2k) = \rho(A(\Delta(l, 2k))$ It then follows that for $r >> 1$

$$w(r, \Delta(l, 2k)) \approx K_3 r^{s_3} \tilde{\rho}(l, 2k)^r.$$ 

We next claim that

$$w((2l, 2k), \Gamma^\infty) \geq \tilde{\rho}(l, 2k)^2. \quad (2.8)$$
Indeed, we trivially have that \( w(2r, \Delta(l, 2k)) \leq w((2l, 2k), \Gamma^\infty) \). Use the asymptotic value of \( w(2r, 2k), \Delta(l, 2k) \) for \( r \gg 1 \) to deduce (2.8). From the definitions of \( m(k) \) and \( \rho(l, 2k) \), it follows that for \( p > 2m(k) \)

\[
w((2rl, p2k), \Gamma) \leq w((2rl, 2k), \Gamma)^{2m(k)} w(2r, \Delta(l, 2k))^{p-2m(k)}.
\]

Use the asymptotic equalites for the above words and the inequality (2.8) to deduce (2.7). Take the \( 2lp - th \) root of (2.7) and let \( p \to \infty \). Use Theorem 2.1 to deduce that

\[
\lim \inf_{l \to \infty} \frac{\log w((2l, 2k), \Gamma^\infty)}{2l} \geq 2kh_{com}(\Gamma).
\]

Hence,

\[
h(\Gamma) = \lim \sup_{k,l \to \infty} \frac{\log w((l, k), \Gamma^\infty)}{kl} \geq \lim \inf_{k \to \infty} \frac{1}{2k} \lim \inf_{l \to \infty} \frac{\log w((2l, 2k), \Gamma^\infty)}{2l} \geq h_{com}(\Gamma).
\]

Thus, \( h(\Gamma) = h_{com}(\Gamma) \) and the proof of the theorem is completed. \( \diamond \)

§3. A symmetricity condition

(3.1) **Theorem.** Let \( \Gamma_i \subset < n > \times < n >, i = 1, \ldots, d \), and consider \( \mathbb{Z}^d \)-SFT given by \( \Gamma = (\Gamma_1, \ldots, \Gamma_d) \). Assume that \( \Gamma_1, \ldots, \Gamma_{d-1} \) are symmetric. Then \( hp(\Gamma) = h(\Gamma) \).

**Proof.** We prove the theorem by the induction on \( d \). Assume first that \( d = 2 \). From Theorem 1.3 we deduce that \( \rho(2, 2) = 0 \Rightarrow h(\Gamma) = -\infty \). Assume that \( \rho(2, 2) \geq 1 \). We now show that \( hp(\Gamma) \geq 0 \). Observe first that \( \text{per}(\Gamma(2, 2)^l) \neq \emptyset \) for some \( l > 1 \). In particular, \( \text{per}(\Gamma_2) \neq \emptyset \), i.e. \( \rho(A(\Gamma_2)) = \rho(2, 1) \geq 1 \). Consider \( \rho(\Gamma(1, l)) \). The above assumption means that \( \rho(\Gamma(1, l)) \) has at least one edge. As \( \Gamma_1 \) is symmetric we deduce that \( \rho(\Gamma(1, l)) \) is also a symmetric matrix. Hence, \( \rho\rho(1, l) \geq 1 \). Theorem 2.1 implies that \( hp(\Gamma) \geq 0 \). Thus \( hp(\Gamma) = h(\Gamma) = -\infty \iff \rho(2, 2) = 0 \).

In what follows we assume that \( \rho(2, 2) \geq 1 \). We now prove that \( hp(\Gamma) = h(\Gamma) \). Clearly, \( hp(\Gamma) \leq h(\Gamma) \). As we showed that \( hp(\Gamma) \geq 0 \) it is enough to consider the case \( h(\Gamma) > 0 \). Note that Theorems 2.1 and Theorem 2.5 yield that \( \rho(2, m) > 1, m = 2, \ldots, \). Fix \( m \geq 1 \). Let \( wp(l) \) be the number of periodic words in the SFT induced by \( \Gamma(2, 2m + 1) \) of length \( l \). Set

\[
L_{p2}(l) = \text{card}(\text{per}(\Gamma_{2}^m)), B = \{b_{ij}\}_{1}^{L_{p2}(l)} = A(p(\Gamma(1, l))), B^{2m} = \{b_{ij}^{(2m)}\}_{1}^{L_{p2}(l)}.
\]

It then follows that \( wp(l) = \sum_{i=j=1}^{L_{p2}(l)} b_{ij}^{(2m)} \). Recall that \( B \) is a nonnegative symmetric matrix. Hence, its spectral norm is equal to its spectral radius \( \rho p(1, l) \). As \( wp(l) = eB^{2m}e^T, e = (1, \ldots, 1) \) we deduce that \( wp(l) \leq \rho p(1, l)^{2m} L_{p2}(l) \). Observe next that

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trace($B^{2m}$) = $wp(2m + 1, l)$. As $B^{2m}$ is a symmetric matrix with nonnegative eigenvalues it follows that trace($B^{2m}$) $\geq \rho p(1, l)^{2m}$. Combine the above inequalities to deduce

$$wp(l) \leq wp(2m + 1, l)Lp_2(l) \leq wp(2m + 1, l)n^{l-1}.$$ 

Fix $\delta, 0 < \delta$. Choose a strictly increasing sequence $\{l_m\}_1^\infty$ so that $\frac{\log wp(l_m)}{l_m} > \log \rho(2, 2m + 1) - \delta$. Use Theorem 2.1 and the above inequalities to deduce

$$h(\Gamma) = \lim_{m \to \infty} \frac{\log \rho(2, 2m + 1)}{2m + 1} \leq \liminf_{m \to \infty} \frac{\log wp(l_m)}{(2m + 1)l_m} \leq \liminf_{m \to \infty} \frac{\log wp(2m + 1, l_m)}{(2m + 1)l_m} \leq hp(\Gamma).$$

This proves the equality $hp(\Gamma) = h(\Gamma)$ for $d = 2$.

Assume that the result holds for $d \geq l \geq 2$ and let $d = l + 1$. Choose $\delta > 0$ and $M = (M_1, ..., M_l) >> (1, ..., 1)$ so that $\frac{\log \rho(1, 2, M)}{M_1, ..., M_l} < h(\Gamma) + \delta$. (We are assuming the nontrivial case $\rho(l + 1, M) \geq 1 \iff h(\Gamma) \geq 0$.) Furthermore, we assume that $M_1, ..., M_l$ are odd numbers. Choose $N_{l+1} >> 1$ so that $w(M_1, ..., M_l, N_{l+1})$- the total number of words in $(\Gamma(l + 1, M))^{N_{l+1}}$ is not greater then $(1 + \delta)^{N_{l+1}}$ times $wp_{l+1}(M_1, ..., M_l, N_{l+1}) = \text{card}(\text{per}(\Gamma(l+1, M)^{N_{l+1}}))$. Let $p_{l+1}(\Gamma(1, (M_2, ..., M_l, N_{l+1}))) \subset \Gamma(1, (M_2, ..., M_l, N_{l+1}))$ be the subgraph generated by all the words of length $(M_2, ..., M_l, N_{l+1})$ in the SFT induced by $(\Gamma_2, ..., \Gamma_{l+1})$ which are periodic with respect to the last coordinate. Note that this graph is symmetric. Moreover,

$$(p_{l+1}(\Gamma(1, (M_2, ..., M_l, N_{l+1}))))^{M_1} = \text{per}(\Gamma(l + 1, M)^{N_{l+1}}) \neq \emptyset.$$ 

The arguments of the proof for $d = 2$ show that $h(\Gamma)$ - the density of words of length $(N_1, ..., N_{l+1})$ is equal to the density of the words periodic in the last and the first coordinates. Let $p_{l, l+1}(\Gamma(2, (M_1, M_3, ..., M_l, N_{l+1}))) \subset \Gamma(2, (M_1, M_3, ..., M_l, N_{l+1}))$ be the subgraph generated by all the words of length $(M_1, M_3, ..., M_l, N_{l+1})$ in SFT induced by $(\Gamma_1, \Gamma_3, ..., \Gamma_{l+1})$ which are periodic in the first and the last coordinate. As $\Gamma_2$ is symmetric it follows that $p_{l, l+1}(\Gamma(2, (M_1, M_3, ..., M_l, N_{l+1})))$ is also symmetric. Use the previous arguments to deduce that $h(\Gamma)$ is the density of words periodic in $1, 2, l + 1$ coordinates. Continue in this manner to deduce that $h(\Gamma) = hp(\Gamma)$. ⊗

Our results yield a new proof that the periodic entropy $hp(\Gamma)$ computed by Lieb [Lie] is equal to the standard entropy $h(\Gamma)$. See [B-K-W] for a specific proof of the above equality for the ice rule model in zero field.

Under the assumptions of Theorem 3.1 it is possible to give lower estimates for $h(\Gamma)$. To do that we need to introduce the following notation. Let $U \subset d > b$ be a set of cardinality $p$. We then agree that $U = \{i_1, ..., i_p\}, 1 \leq i_1 < \cdots < i_p \leq d$. For $N = (N_1, ..., N_d)$ set $N^U = (N_{i_1}, ..., N_{i_p})$. In particular, $N^{(k)} = (N_1, ..., N_{k-1}, N_{k+1}, ..., N_d), k = 1, ..., d$. Assume the assumptions of Theorem 3.1. For any nontrivial set $U \subset d > b$ we consider the SFT on $\mathbb{Z}^{\text{card}(U)}$ induced on $\Gamma^U = (\Gamma_{i_1}, ..., \Gamma_{i_p})$. Suppose that $k \in U, V = U \{k\}, \text{card}(V) \geq 1$. Then $\Gamma(k, N^V)$ is graph induced by the SFT corresponding to $\Gamma^U$. Let $\rho(k, N^V)$ be the spectral radius of this graph. Given three pairwise disjoint sets $V, \{k\}, W \subset d > b$ we consider the following contraction of $\rho(k, N^{V \cup W})$ on $V$ indices

$$\rho_V(k, N^W) = \lim_{N_i \to \infty, i \in V} \rho(k, N^{V \cup W}) \prod_{i \in V} N_i.$$

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Let $U = \{k\} \cup V$. Observe that $\log \rho_V(k, N^V) = h(\Gamma^U)$.

(3.2) Theorem. Let $\Gamma_i \subset \langle n \rangle \times \langle n \rangle$, $i = 1, \ldots, d$, and consider the $\mathbb{Z}^d$-SFT given by $\Gamma = (\Gamma_1, \ldots, \Gamma_d)$. Assume that $\Gamma_k$ is symmetric. Then

$$
\rho(i, N^{\{i\}}) \leq \rho(\{i\}, N^{\{i,k\}})^{N_k-1} \rho(i, N^{\{i,k\}}),
$$

$N = (N_1, \ldots, N_d) \geq (1, 1, \ldots, 1), i = 1, \ldots, k-1, k+1, \ldots, d$.

Proof. Fix $N_i \geq 1$, $j = 1, \ldots, i-1, i+1, \ldots, d$. For a small positive $\delta > 0$ choose $N_i >> 1$ so that

$$(1 - h)^{N_i} \rho(i, N^{\{i\}})^{N_i} \leq w(N) = \text{card}((\Gamma(i, N^{\{i\}}))^{N_i}),$$

$$
\omega(N^{\{k\}}) = \text{card}((\Gamma(i, N^{\{i,k\}}))^{N_i}) \leq (1 + \delta)^{N_i} \rho(i, N^{\{i,k\}})^{N_i}.
$$

Let $C = A(\Gamma(k, N^{\{k\}}))$. Then $C$ is $\omega(N^{\{k\}}) \times \omega(N^{\{k\}})$ symmetric matrix with the spectral norm equal to $\rho(k, N^{\{k\}})$. Set $e = (1, \ldots, 1)$. The maximal characterization of the maximal eigenvalue of $C^{N_k-1}$ yields

$$w(N, \Gamma) = e C^{N_k-1} e^T \leq \rho(k, N^{\{k\}})^{N_k-1} \omega(N^{\{k\}}).$$

Taking the $N_i-$th root in the above inequality and letting $N_i \to \infty$ we deduce the theorem.

\hfill \Box.

Combine Theorems 3.1-3.2 for $d = 2, k = 1$ with Theorems 2.1 and 2.5 to obtain.

(3.3) Corollary. Let $\Gamma_1, \Gamma_2 \subset \langle n \rangle \times \langle n \rangle$. Assume that $\Gamma_1$ is symmetric and consider the $\mathbb{Z}^2$-SFT induced by $\Gamma = (\Gamma_1, \Gamma_2)$. Then

$$
\frac{\log \rho(2, k)}{k-1} - \frac{\log \rho(2, 1)}{k-1} \leq h_p(\Gamma) = h(\Gamma) \leq \frac{\log \rho(2, k)}{k}, \ k = 2, \ldots.
$$

The above Corollary under stronger assumptions is due to [M-P2]. Note that Corollary 3.3 enables one to calculate effectively the entropy $h(\Gamma)$ up to an arbitrary precision.

We now apply Theorem 3.2 for $d = 3$ assuming that $\Gamma_2$ is symmetric. Let $N_1 = p \geq 1, N_2 = q \geq 2, k = 2, i = 3$ to deduce

$$
\frac{\log \rho(3, (p, q))}{p(q-1)} - \frac{\log \rho(3, p)}{p(q-1)} \leq \frac{\log \rho(3, (p, q))}{p}.
$$

Let $p \to \infty$. We then get the inequalities

$$
\frac{\log \rho_{\{1\}}(3, q)}{q-1} - \frac{h(\Gamma^{\{1\}})}{q-1} \leq h(\Gamma).
$$

(3.4)
This yields a lower bound for $h(\Gamma)$ which converges to $h(\Gamma)$ as $q \to \infty$. To obtain computable lower bounds for $h(\Gamma)$ in terms of various $\rho(k, M)$ we assume that $\Gamma_3$ is symmetric. First observe that Theorem 2.1 gives an upper bound on $h(\Gamma^{(1.3)})$. Use Theorem 3.2 with $k = 3, i = 1, M_2 = q, M_3 = r$ to deduce

$$\frac{\log \rho(1, (q, r))}{r - 1} - \frac{\log \rho(1, q)}{r - 1} \leq \log \rho(1)(3, q).$$

Use the above inequalities in (3.4) to obtain a lower bound for $h(\Gamma)$ which in principle can be arbitrary close to $h(\Gamma)$. (Choose all the numbers entering in this inequality to be big enough.)

§4. Observations

Let $\Gamma \subset < n > \times < n >$ be a directed graph on $n$ vertices. For any nontrivial set $V \subset < n >$ set $\Gamma(V) = \Gamma \cap V \times V$. $\Gamma$ is called a strongly connected graph if any two vertices $i, j \in < n >$ are connected by a path in a graph. This is equivalent to the statement that $A(\Gamma)$ is an irreducible matrix. If $\Gamma$ is not strongly connected then $< n >$ is decomposed to a disjoint union

$\begin{align*}
\langle n \rangle &= \bigcup_{i=0}^{p} U_i, U_i \cap U_j = \emptyset, 0 \leq i < j \leq p, \text{card}(U_i) \geq 1, i = 1, \ldots, p, \\
A(\Gamma(U_0))^n &= 0, (A(\Gamma(U_i)) + I)^n > 0, i = 1, \ldots, p.
\end{align*}
$ (4.1)

Here $I$ stands for the identity matrix and $B > 0$ denote a real valued matrix whose all entries are positive. The set $U_0$ is called a transient set. That is, if we consider any path with edges in our graph $\Gamma$ each transient vertex will appear at most once. Equivalently, any closed path will not contain any transient vertex, while for each vertex in $\bigcup_{i=0}^{p} U_i$ there exists a closed path which contains this vertex. The set $\bigcup_{i=0}^{p} U_i$ is the set of nontransient vertices. Moreover, each graph $\Gamma(U_i)$ is a strongly connected for $i = 1, \ldots, p$. Furthermore, $U_1, \ldots, U_p$ are maximal sets with this property. That is, for $1 \leq i < j \leq p$ either there is no path of $\Gamma$ connecting $U_i$ to $U_j$ or $U_j$ to $U_i$ (or both). The reduced graph $\text{red}(\Gamma)$ is defined as follows. The states (vertices) of the reduced graph are the transient vertices $U_0$ and the new states $[U_1], \ldots, [U_q]$. Let $\text{red}(n) = \text{card}(U_0) + p$. Then $\text{red}(\Gamma) \subset< \text{red}(n) > \times < \text{red}(n) >$ does not have self loops, i.e. $(i, i) \notin \text{red}(\Gamma), i \in < \text{red}(n) >$. Furthermore $(i, j) \in \text{red}(\Gamma)$ iff there is at least one edge in $\Gamma$ which goes from one vertex represented by the state $i$ to one vertex represented by the state $j$. It then follows that $A(\text{red}(\Gamma))$ is a nilpotent matrix. Let $x = (x_{ji})^n_1 \in \Gamma^m, m >> 1$. The generic picture dictated by the reduced graph is as follows. First we may have a couple of transient vertices $x_1, \ldots, x_t \in U_0, (x_i, x_{i+1}) \in \text{red}(\Gamma), i = 1, \ldots, t_1 - 1$. (It may happen that we do not have transient vertices, i.e. $t_1 = 0$.) Then we have a sequence of an arbitrary length $k_1, x_{t_1+1}, \ldots, x_{t_1+k_1} \in U_{j_1}, (x_{t_1}, [U_{ji}]) \in \text{red}(\Gamma)$. Then we may have another few transient states $x_{t_1+k_1+1}, \ldots, x_{t_1+k_1+t_2} \in U_0, ([U_{ji}, x_{t_1+k_1+1}), (x_i, x_{i+1}) \in \text{red}(\Gamma), i = t_1+k_1+1, \ldots, t_1+k_1+t_2-1, (t_2 \geq 0)$. This sequence may be followed by another arbitrary
long sequence $x_{t_1+k_1+t_2+1}, \ldots, x_{t_1+k_1+t_2+k_2} \in U_{j_2}$, $(x_{t_1+k_1+t_2}, [U_{j_2}]) \in \text{red}(\Gamma)$. If $t_2 = 0$ we then have the condition $([U_{j_1}], [U_{j_2}]) \in \text{red}(\Gamma)$. This process may continue until we reach the final state of the reduced graph. In particular, the arbitrary long sequences belong to pairwise distinct components $U_{j_1}, \ldots, U_{j_l}$ whose order depends on the structure of the reduced graph. In particular, $1 \leq l \leq n$.

These properties can be deduced straightforward from the Frobenius normal form of a nonnegative matrix, e.g. [Gan]. Consult for example with [F-S]. In particular, $\rho(A(\Gamma)) = \max_{1 \leq i \leq p} \rho(A(\Gamma^{(i)}))$. A graph $\Gamma \subset < n > \times < n >$ is called nontransient if it does not have a transient set, i.e. $U_0 = \emptyset$. For a general graph we let $\Gamma' = \Gamma(\bigcup_i U_i)$ to be the nontransient part of $\Gamma$. As $h(\Gamma) = \log \rho(A(\Gamma))$ we deduce that $h(\Gamma) = \max_{1 \leq i \leq p} h(\Gamma(U_i)) = h(\Gamma')$.

Finally observe that the periodic orbits under the shift correspond to closed paths in the graph $\Gamma$. Hence, any periodic orbit has vertices only in one per $((\Gamma(U_i))^N)$. We now show that some these results can be generalised to SFT in higher dimension.

(4.2) Lemma. Let $\Gamma_i \subset < n > \times < n >, i = 1, \ldots, m$. Then one of the following mutually exclusive conditions hold:
(i) For any nontrivial subset $V \subset < n >$ there exists $k \in < m >$ such that $\Gamma_k(V)$ has a nontrivial transient set of vertices in $V$.
(ii) There exist a maximal (nontrivial) subset $V \subset < n >$, so that $\Gamma_k(V)$ is a nontransient graph on $V$ for $k = 1, \ldots, m$.

Proof. Let $U_{0,i} \subset < n >$ be the set of transient vertices of the graph $\Gamma_i$, $i = 1, \ldots, m$. If $U_{0,i} = \emptyset, i = 1, \ldots, m$, then we have the condition (ii) with $V = < n >$. Let $V_1 = < n > \setminus \bigcup_i m U_{0,i}$. If $V_1 = \emptyset$ then the condition (i) holds. Assume that $< n > \neq V_1 \neq \emptyset$. Repeat the above process for $\Gamma_i(V_1), i = 1, \ldots, m$ to deduce either (i) or (ii). $\diamond$

(4.3) Theorem. $\Gamma_i \subset < n > \times < n >, i = 1, \ldots, d$, and consider $\mathbb{Z}^d$-SFT given by $\Gamma = (\Gamma_1, \ldots, \Gamma_d)$. Assume first that condition (i) of Lemma 4.2 holds. Then $h(\Gamma) = -\infty$. Assume now that $V$ is the maximal (nontrivial) subset of $< d >$ so that $\Gamma_k(V)$ is nontransient for $k = 1, \ldots, d$. Set $\Gamma(V) = (\Gamma_1(V), \ldots, \Gamma_d(V))$. Then $h(\Gamma) = h(\Gamma(V))$.

Proof. Clearly, the theorem trivially holds if $h(\Gamma) = -\infty$. Assume that $h(\Gamma) \geq 0$. That is for each $N = (N_1, \ldots, N_d)$, $N_i \geq 1, i = 1, \ldots, d$, $\rho(k, N^{(k)}) \geq 1, k = 1, \ldots, d$. As in the proof of Lemma 4.2 consider the transient set $U_{0,k}$ for the graph $\Gamma_k$ for $k = 1, \ldots, d$. If all $U_{0,k} = \emptyset$ then $V = < n >$ and the theorem is trivial in this case. Suppose that $U_{0,k} \neq \emptyset$. Fix $N^{(k)}$. As $\rho(k, N^{(k)}) \geq 1$ we know that $h(k, N^{(k)})$ is given by the density of the periodic words $\text{per}(\Gamma(k, N^{(k)}))$. Observe next that every periodic word in $\text{per}(\Gamma(k, N^{(k)}))$ is induced by a word $f = (f(j_1, \ldots, j_d))_{j_1=1}^{N_1} \cdots_{j_d=1}^{N_d}$ such that

$$(f_{(j_1, \ldots, j_d)})_{j_1=1}^{N_1} \in \text{per}(\Gamma_k)^{N_k}, j_l = 1, \ldots, N_l, l = 1, \ldots, k - 1, k + 1, \ldots, d.$$ 

Hence, the coordinates of each vector $(f_{(j_1, \ldots, j_d)})_{j_1=1}^{N_1}$ belong to some set $U_{k,i}$ appearing in the decomposition (4.1) of the nontransient set for $\Gamma_k$. Note that the value of $i$ may depend on $(j_1, \ldots, j_{k-1}, j_{k+1}, \ldots, j_d)$. In particular, all the coordinates of $f$ are in the set

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$V_1 = \{ n > \mid U_{0,k} \}$. Let $\Gamma(V_1) = (\Gamma_1(V_1), \ldots, \Gamma_d(V_1))$. Theorems 2.1 and 2.5 yield that $h(\Gamma) = h(\Gamma(V_1))$. Repeat this process as in the proof of Lemma 4.2. If we obtain the condition (i) of Lemma 4.2 we deduce that $h(\Gamma) = -\infty$ which contradicts our assumption that $h(\Gamma) \geq 0$. Hence, the second condition of Lemma 4.2 holds. By the above arguments $h(\Gamma) = h(\Gamma(V_1))$ and the proof of the theorem is concluded. \end{proof}

Let $\Gamma_1, \Gamma_2 \subset \{ n \}$. Set $X = (\Gamma_2)^\infty$. Then $X$ is a closed compact space in the Tychonoff topology. (More precisely, $X$ is a Cantor set.) Set $\Delta = \Delta(\Gamma_1, \Gamma_2) \subset X \times X$ be the following closed graph

$$\Delta = \{(x, y) : x = (x_i)_{i \in \mathbb{Z}}, (y_i)_{i \in \mathbb{Z}} \in X, (x_i, y_i) \in \Gamma_1, i \in \mathbb{Z}\}.$$ 

Define $\Delta^m, \Delta^\infty$ as in the introduction. Note that

$$\Delta^m = \emptyset \iff \rho(2, m) = 0, m = 2, \ldots,$$

$$\Delta^\infty = \emptyset \iff \Gamma^\infty = \emptyset, \Gamma = (\Gamma_1, \Gamma_2).$$

Observe that if $\Gamma_1$ is symmetric then $\Delta$ is also symmetric.

In [Fri1-2] we studied the entropy $h(\Delta)$ of the shift $\sigma$ restricted to $\Delta^\infty$. Here $\sigma((x_i)_{i \in \mathbb{Z}}) = (x_{i+1})_{i \in \mathbb{Z}}$. It is not difficult to show that if $h(\Gamma) > 0$ then $h(\Delta) = \infty$. Thus, $h(\Gamma)$ can be considered as the renormalization of the entropy $h(\Delta)$. More precisely if $N(k, \epsilon)$ is the number of $k-\epsilon$ separated sets then one can show that up to a multiplicative constant that the right renormalization is:

$$h(\Gamma) = \lim_{\epsilon \to 0} \limsup_{k \to \infty} \log \frac{N(k, \epsilon)}{k \log \frac{1}{\epsilon}}.$$ 

Moreover, the dynamics of $\mathbb{Z}^2$ shift restricted to $\Gamma^\infty$ is related to the dynamics of the standard shift restricted to $\Delta^\infty$. It would be interesting to explore in more details this relation. Similar ideas apply to higher dimensional $\mathbb{Z}^d$-SFT.

References


[Gab] W.P. Gabriel, On dynamical systems with uncomputable topological entropy,
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