On the entropy of $\mathbf{Z}^d$ subshifts of finite type

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§0. Introduction

Let n be a positive integer and denote $< n >= \{1, ..., n\}$. We view $< n >$ as an alphabet on n letters. Denote by $< n >^{\mathbf{Z}^d}$ the set of all mappings of $\mathbf{Z}^d$ to the set $< n >$. By extending the Hamming metric on $< n > \times < n >$ to $< n >^{\mathbf{Z}^d}$ one obtains that $< n >^{\mathbf{Z}^d}$ is a compact metric space. The group $\mathbf{Z}^d$ acts as a group of (translation) automorphisms on $< n >^{\mathbf{Z}^d}$. A set $S \subset \mathbf{Z}^d$ which is closed and invariant under the action of $\mathbf{Z}^d$ is called a subshift. S is called a subshift of finite type (SFT) if there is a finite set of finite admissible configurations which generates S under the action of $\mathbf{Z}^d$. More presicely, let $F \subset \mathbf{Z}^d$ be a finite set. Assume that $P \subset < n >^F$. Then (F, P) defines a folowing $\mathbf{Z}^d$-SFT S. For each $a \in \mathbf{Z}^d$ let $F + a \subset \mathbf{Z}^d$ be the corresponding translation of F. Then $x \in S$ iff for each $a \in \mathbf{Z}^d$, $\pi_{F+a}(x)$, the projection of x on the set $F + a$, is in P. See for example [**Sch**, Ch. 5].

The case of $\mathbf{Z}$ action, i.e. $d = 1$, is well understood. In that case, it is relatively easily to decide whether S is empty or not. Moreover, the topological entropy $h(S)$ of the restriction of the standard shift to S is a logarithm of an algebraic integer $\rho(F, P)$. The number $\rho(P, F)$ is a the spectral radius of certain $0 - 1$ square matrix induced by (F, P). Furthermore, the topological entropy $h(S)$ is equal to the rate of growth of the number of periodic points. The case $d > 1$ is much more complicated. First, the problem wether S is an empty set or not is undecidable. This result for $d = 2$ goes back to Berger [**Ber**]. See also [**K-M-W**] and [**Rob**]. Second, there exists a SFT $S \neq \emptyset$ which does not have periodic points. Moreover, in the case where $S \neq \emptyset$ the topological entropy may be uncomputable, see [**H-K-C**] and [**Gab**].

The object of this paper to show that contrary to these results one has a natural and a simple criterion which either determines that $S = \emptyset$ or calculates the topological entropy of $S \neq \emptyset$. There is no contradition to the uncomputability of $h(S)$ because we can not estimate the rate of convergence of our sequence. However, if we introduce a symmetry in $\mathbf{Z}^2$ we can estimate the rate of convergence of our sequence. Moreover, in this case $h(S)$ is the rate of growth the number of periodic points. Our main tool is to view a $\mathbf{Z}^d$-SFT as a matrix SFT. See [**M-P1,M-P2**]. In fact our methods are very close to the methods of [**M-P1,M-P2**].

We now describe briefly the content of the paper. In §1 we define combinatorial entropy of $\mathbf{Z}^d$-SFT. It can is computed by finite configurations. We then observe, using Köning's method, that $\mathbf{Z}^d$-SFT is nonempty iff every finite configuration is nonempty. In §2 we show that the combinatorial entropy is equal to the topological entropy of $\mathbf{Z}^d$-SFT.

In §3 we show that simple symmetricity conditions yield that the topological entropy of $\mathbf{Z}^d$-SFT is equal to the periodic entropy. (The periodic entropy is the rate growth of the periodic points.) In the case $d = 2$ combined with the symmetricity assumption we obtain an algorithm for computing the entropy at any given precision. This result is due to [**M-P2**] under stricter conditions. The last section is devoted to various remarks.

§1. Preliminary results

Let $\Gamma \subset< n > \times < n >$. Set

$$\Gamma^N = \{x = (x_i)_1^N, (x_i, x_{i+1}) \in \Gamma, i = 1, ..., N-1\},$$
$$\Gamma^\infty = \{x = (x_i)_{i\in\mathbf{Z}} : (x_i, x_{i+1}) \in \Gamma, i \in \mathbf{Z}\}.$$

Assume that $\Gamma_i \subset< n > \times < n >, i = 1, ..., d$. Set $\Gamma = (\Gamma_1, ..., \Gamma_d)$ and let

$$\Gamma^\infty = \{f : f \in< n >^{\mathbf{Z}^d}, (f_{(i_1,...,i_d)})_{i_k\in\mathbf{Z}} \in \Gamma_k^\infty,$$
$$(i_1, ..., i_{k-1}, i_{k+1}, ..., i_d) \in \mathbf{Z}^{d-1}, k = 1, ..., d, \}$$

to be a $\mathbf{Z}^d$-SFT induced by Γ. We now show that a standard $\mathbf{Z}^d$-SFT is a equivalent the $\mathbf{Z}^d$ -SFT induced by Γ. For the case $d = 2$ this can be deduced from [**Moz**] who proved that every $\mathbf{Z}^2$-SFT is equivariant to Wang-tiling-space. It is easy to see that Wang-tiling-space is $\mathbf{Z}^2$-SFT induced by some special $\Gamma = (\Gamma_1, \Gamma_2)$.

Let S be a SFT is given by the pair (F, P) as in §0. Let $N = (N_1, ..., N_d) \in \mathbf{Z}^d, N_i \geq 1, i = 1, ..., d$. By $B(N)$ we denote the box $< N_1 > \times \cdots \times < N_d >\subset \mathbf{Z}^d$. Let $f = (f_{(i_1,...,i_d)})_{(1,...,1)}^N \in< n >^{B(N)}$. Then f is called (F, P) admissible if for all $a \in \mathbf{Z}^d$ such that $F + a \subset B(N)$ we have the condition that $\pi_{F+a}(f)$ - the projection of f on the set $F+a$ is P-admissible, i.e. $\pi_{F+a}(f) \in P$. Let $(1, ..., 1) \leq M(F) = (M_1(F), ..., M_d(F)) \in \mathbf{Z}^d$ be the dimension of the smallest box containing F. That is, $B(M(F)) \supset F + a$ for some $a \in \mathbf{Z}^d$ and $B(M(F))$ is minimal with respect to this property. For $M(F) \leq N \in \mathbf{Z}^d$ let $w(N, F, P)$ be the number of (F, P) admissible words in $B(N)$. We then let

$$h_{com}(F, P) = \limsup_{N_1,...,N_d\to\infty} \frac{\log w(N, F, P)}{N_1 \cdots N_d}$$

to be the combinatorial entropy of induced by (F, P). We agree that $\log 0 = -\infty$. That is $h_{com}(F, P) \geq 0$ iff every box $B(N)$ has at least one (P, F) admissible configuration $f \in< n >^{B(N)}$. Observe next that if $M_i(F) = 1$ for some some i then we effectively can consider the corresponding $\mathbf{Z}^{d-1}$-SFT. For

$$N = (N_1, ..., N_d) \in \mathbf{Z}^d, N_k > 1, k = 1, ..., d$$

let

$$\Gamma^N = \{f = (f_{(i_1,...,i_d)})_{i_1=...=i_d=1}^{N_1,...,N_d} : (f_{(i_1,...,i_d)})_{i_k=1}^{N_k} \in \Gamma_k^{N_k}, k = 1, ..., d\},$$

for every $i_1, ..., i_{k-1}, i_{k+1}, ..., i_d$.

Set

$$F = \{1,2\}^d = B(2,...,2), P = \Gamma^{(2,...,2)}, w(N,\Gamma) = w(N,F,P).$$

Then for any $N = (N_1, ..., N_d) > (1, ..., 1)$ the set Γ^N consists of all (F,P) admissible words in $< n >^{B(N)}$. Define $h_{com}(\Gamma) = h_{com}(F,P)$.

(1.1) Theorem. *Let $F \subset \mathbf{Z}^d$ be a finite set such that $1 < M_i(F), i = 1, ..., d$. Assume that $P \subset < n >^F$. Denote by $T \subset < n >^{B(M(F))}$ the set of all F,P admissible words in $< n >^{B(M(F))}$. For each $i = 1, ..., d$, and $u \in T$ let $\pi_{i,-}(u), \pi_{i,+}(u)$ be the projection of u on the sets*

$$B(M_1(F), ..., M_{i-1}(F), M_i(F)-1, M_{i+1}(F), ..., M_d(F)),$$
$$B(M_1(F), ..., M_{i-1}(F), M_i(F)-1, M_{i+1}(F), ..., M_d(F)) + (\delta_{i1}, ..., \delta_{id}).$$

Set

$$\Gamma_i = \{(u,v) : u, v \in T, \pi_{i,+}(u) = \pi_{i,-}(v)\} \subset T \times T, i = 1, ..., d, \ \Gamma = (\Gamma_1, ..., \Gamma_d).$$

Then for any $N = (k_1 + M_1(F), ..., k_d + M_d(F)), k_i \geq 1, i = 1, ..., d,$ the set of all (F,P) admissible words in $B(N)$ is in one to one correspondence with $\Gamma^{(k_1+1,...,k_d+1)}$ on the alphabet T. In particular the set of all admissible (F,P) words in $< n >^{\mathbf{Z}^d}$ is in one to one correspondence with Γ^∞. Furthermore $h_{com}(F,P) = h_{com}(\Gamma)$.

Proof. Let $N = (k_1 + M_1(F), ..., k_d + M_d(F)), k_i \geq 1, i = 1, ..., d$. Assume that $f \in < n >^{B(N)}$ be an (F,P) admissible word. For $(l_1, ..., l_d), 1 \leq l_j \leq k_j + 1, j = 1, ..., d$, let $g_{(l_1,...,l_d)}$ be the word in T which has the following coordinates in f:

$$l_i \leq j_i \leq l_i + M_i(F) - 1, i = 1, ..., d. \tag{1.2}$$

It is straightforward to check that $g = (g_{(l_1,...,l_d)})_{(1,...,1)}^{(k_1+1,...,k_d+1)} \in \Gamma^{(k_1+1,...,k_d+1)}$. Assume that $g \in \Gamma^{(k_1+1,...,k_d+1)}$. Use the above formula to find a unique $f \in < n >^{B(N)}$ so that g is constructed from f as above. We claim that f is a (F,P) admissible word in $< n >^{B(N)}$. Assume that $F + a \subset B(N)$. Then there exists $l_1, ..., l_d, 1 \leq l_i \leq k_i + 1, i = 1, ..., d$, so that the coordinates of $F + a$ satisfy the inequalities (1.2). That is, $\pi_{F+a}(f)$ lies in the word u generated by the projection of f on the coordinates specified by (1.2). By the construction, $u \in T$. In particular, $\pi_{F+a}(f) = \pi_{F+a}(u) \in P$. Hence, f is a (F,P) admissible word. Therefore, $w(N,F,P)$ is equal to $\omega(k_1+1, ..., k_d+1) = card(\Gamma^{(k_1+1,...,k_d+1)})$. All other assertions of the Theorem follow straightforward. $\diamond$

Let $(1, ..., 1) \leq N \in \mathbf{Z}^d$. Partition the box $B(N)$ to p nontrivial boxes of dimensions $N^i \in \mathbf{Z}^d, i = 1, ..., p$. It then follows that $w(N,F,P) \leq \prod_1^p w(N^i, F, P)$. We thus deduce

$$h_{com}(F,P) = \lim_{N_1,...,N_d \to \infty} \frac{\log w((N_1, ..., N_d), F, P)}{N_1 \cdots N_d} = \lim_{m \to \infty} \frac{\log w((m, ..., m), F, P)}{m^d}.$$

(1.3) Theorem. *Let S be a $\mathbf{Z}^d$-SFT given by (F,P). Then*

$$S \neq \emptyset \iff w((m,...,m),F,P) \geq 1, m = 2,...,.$$

That is, $S = \emptyset \iff h_{com} = -\infty.$

Proof. Clearly, if $S \neq \emptyset$ then $h_{com}(F,P) \geq 0$. In particular, $w((m,...,m),\Gamma) \geq 1, m = 2,...,$. Assume now that $w((m,..,m),\Gamma) \geq 1, m = 2,...,$. Consider the box $B((2m,...,2m))$ - B_{2m} in $\mathbf{R}^d$ whose center is at the origin $(0,...,0)$. Let $\Theta_m \in \Gamma^{(2m,\cdots,2m)}$ be an admissible filling of B_{2m} by the alphabet $\{1,...,n\}$. Consider the sequence $\{\Theta_m\}_1^\infty$. Look at the projection of this sequence on B_2. Pick up an infinite subsequence $\{\Theta_{n_i^1}\}_{i=1}^\infty$ whose so that the projection of each $\Theta_{n_i^1}$ on B_2 is the same element $\Psi_1 \in \Gamma^{(2,\cdots,2)}$. From the sequence $\Theta_{n_i^1}$ pick a subsequence $\Theta_{n_i^2}$ so that the projection of each element $\Theta_{n_i^2}$ on B_4 is the same element $\Psi_2 \in \Gamma^{(4,\cdots,4)}$. Continue this construction to obtain that the sequence $\Psi_k \in \Gamma^{(2m,\cdots,2m)}, k = 1,...,$ which are $2m \times \cdots \times 2m$ sections of an element $\Psi \in \Gamma^\infty$. The above argument is due to Köning [**Kön**]. $\diamond$

Introduce on $<n>$ the Hamming metric $d(i,i) = 0, d(i,j) = 1, i \neq j \in <n>$. For $i = (i_1,...,i_d) \in \mathbf{Z}^d$ we let $|i| = \sum_1^d |i_p|$. On $<n>^{\mathbf{Z}^d}$ define the following metric

$$d(f,g) = \frac{1}{2^{2d}} \sum_{i=(i_1,\ldots,i_d)\in\mathbf{Z}^d} \frac{d(f_i,g_i)}{2^{|i|}}, f = (f_i), g = (g_i) \in <n>^{\mathbf{Z}^d}.$$

It then follows that $<n>^{\mathbf{Z}^d}$ is a compact metric space. Let $e_i = (\delta_{i1},...,\delta_{id}), i = 1,...,d$, be the standard basis in $\mathbf{Z}^d$. Denote by $T_i : <n>^{\mathbf{Z}^d} \to <n>^{\mathbf{Z}^d}$ the following automorphism of $<n>^{\mathbf{Z}^d}$:

$$T_i(f_j) = (f_{j+e_i}), j \in \mathbf{Z}^d, f = (f_j) \in <n>^{\mathbf{Z}^d}.$$

$S \subset <n>^{\mathbf{Z}^d}$ is called a subshift (SF) if S is closed and $T_iS = S, i = 1,...,d$. In that case one defines a topological entropy $h(S)$ as follows. For $(1,...,1) \leq N = (N_1,...,N_k)$ introduce the following new metric on $<n>^{\mathbf{Z}^d}$:

$$d_N(f,g) = \max_{0\leq i_p<N_p, p=1,\ldots,d} d(T_1^{i_1}\cdots T_d^{i_d}f, T_1^{i_1}\cdots T_d^{i_d}g), \ f,g \in <n>^{\mathbf{Z}^d}.$$

Fix a positive $\epsilon > 0$ and let $K(S,N,\epsilon)$ be the maximal number of ϵ separated points in S in the metric $d_N(\cdot,\cdot)$. We then let

$$h(S) = \lim_{\epsilon\to\infty} \limsup_{N_1,\ldots,N_d\to\infty} \frac{\log K(S,N,\epsilon)}{N_1\cdots N_d}. \tag{1.4}$$

(1.5) Theorem. *Let $\Gamma_i \subset <n> \times <n>, i = 1,...,d$, and set $\Gamma = (\Gamma_1,...,\Gamma_d)$. Assume that $\Gamma^\infty \neq \emptyset$. Define $h(\Gamma) = h(\Gamma^\infty)$. For $(1,...,1) \leq N \in \mathbf{Z}^d$ let $w(N,\Gamma^\infty)$ be the number of all possible projections of $f \in \Gamma^\infty$ on a fixed box $B(N)$. Then*

$$h(\Gamma) = \limsup_{N_1,\ldots,N_d\to\infty} \frac{\log w(N,\Gamma^\infty)}{N_1\cdots N_d}.$$

In particular, $h(\Gamma) \le h_{com}(\Gamma)$.

Proof. It is quite straightforward to see from the definition of $K(\Gamma^\infty, N, \epsilon)$ that for a small enough $\epsilon > 0$ there exist some constants $1 \le a(\epsilon), 1 \le b(\epsilon) \in \mathbf{Z}$ so that

$$w(N,\Gamma^\infty) \le K(\Gamma^\infty, N, \epsilon) \le a(\epsilon) w(N + (b(\epsilon), ..., b(\epsilon)), \Gamma^\infty).$$

Now the characterization of $h(\Gamma)$ follows straightforward from (1.4). As $w(N,\Gamma^\infty) \le w(N,\Gamma)$ we deduce that $h(\Gamma) \le h_{com}(\Gamma)$. $\diamond$.

§2. The equality of topological and combinatorial entropy for SFT

Let $\Gamma \subset < n > \times < n >$. Denote by $A = A(\Gamma)$ the $0-1$ matrix induced by the graph Γ. Let $\rho(A)$ be the spectral radius of A. Set

$$per(\Gamma^N) = \{(x_i)_1^N : (x_i)_1^N \in \Gamma^N, x_1 = x_N\}.$$

Assume that $\Gamma_i \subset < n > \times < n >, i = 1, ..., d$. Set

$$\Gamma = (\Gamma_1, ..., \Gamma_d), \Gamma^{\overline{\{i\}}} = (\Gamma_1, ..., \Gamma_{i-1}, \Gamma_{i+1}, ..., \Gamma_d), i = 1, ..., d.$$

For

$$N = (N_1, ..., N_d) \in \mathbf{Z}^d, N_k > 1, k = 1, ..., d,$$
$$M = (M_1, ..., M_{d-1}) \in \mathbf{Z}^{d-1}, M_j > 1, j = 1, ..., d-1,$$

let

$$per(\Gamma^N) = \{f = (f_{(i_1,...,i_d)})_{i_1=...=i_d=1}^{N_1,...,N_d} : (f_{(i_1,...,i_d)})_{i_k=1}^{N_k} \in per(\Gamma_k^{N_k}), k = 1, ..., d\},$$
$$wp(N,\Gamma) = card(per(\Gamma^N)),$$
$$\Gamma(k,M) = \{(a,b) : a = (a_{(i_1,...,i_{d-1})}), b = (b_{(i_1,...,i_{d-1})}) \in (\Gamma^{\overline{\{k\}}})^M,$$
$$(a_{(i_1,...,i_{d-1})}, b_{(i_1,...,i_{d-1})}) \in \Gamma_k, i_j = 1, ..., M_j, j = 1, ..., d-1, \}, k = 1, ..., d,$$
$$p(\Gamma(k,M)) = \{(a,b) : a = (a_{(i_1,...,i_{d-1})}), b = (b_{(i_1,...,i_{d-1})}) \in per((\Gamma^{\overline{\{k\}}})^M),$$
$$(a_{(i_1,...,i_{d-1})}, b_{(i_1,...,i_{d-1})}) \in \Gamma_k, i_j = 1, ..., M_j, j = 1, ..., d-1, \}, k = 1, ..., d,$$
$$A(k,M) = A(\Gamma(k,M)), \rho(k,M) = \rho(A(k,M)),$$
$$Ap(k,M) = A(p(\Gamma(k,M))), \rho p(k,M) = \rho(Ap(k,M)), k = 1, ..., d.$$

Note that any $f \in per(\Gamma^N)$ has a unique minimal periodic extension to Γ^∞. Set

$$hp(\Gamma) = \limsup_{N_1,...,N_d \to \infty} \frac{\log wp((N_1, ..., N_d), \Gamma)}{N_1 \cdots N_d}$$

to be the periodic entropy of Γ^∞.

(2.1) Theorem. *Let $d \geq 2$ and assume that $\Gamma_i \subset <n> \times <n>, i=1,...,d$. Consider $\mathbf{Z}^d$-SFT given by $\Gamma = (\Gamma_1, ..., \Gamma_d)$. Then*

$$h_{com}(\Gamma) = -\infty \iff \forall M = (M_1, ..., M_{d-1}) >> (1, ..., 1)\ \rho(k, M) = 0, k = 1, ..., d,$$
$$hp(\Gamma) = -\infty \iff \forall M = (M_1, ..., M_{d-1})\ \rho p(k, M) = 0, k = 1, ..., d.$$

Furthermore

$$\lim_{M_1,...,M_{d-1}\to\infty} \frac{\log \rho(k, (M_1, ..., M_{d-1}))}{M_1 \cdots M_{d-1}} = h_{com}(\Gamma), k = 1, ..., d,$$
$$\frac{\log \rho(k, (M_1, ..., M_{d-1}))}{M_1 \cdots M_{d-1}} \geq h_{com}(\Gamma), M_i > 1, i = 1, ..., d-1, k = 1, ..., d,$$
$$\limsup_{M_1,...,M_{d-1}\to\infty} \frac{\log \rho p(k, (M_1, ..., M_{d-1}))}{M_1 \cdots M_{d-1}} \leq hp(\Gamma), k = 1, ..., d.$$

Proof. We first prove the theorem for $d = 2$. In that case $M = (m)$ and we let $\Gamma(k, M) = \Gamma(k, m), \rho(k, M) = \rho(k, m)$ for $k = 1, 2$. Suppose first that there exists $N = (N_1, N_2)$ so that $\Gamma^N = \emptyset$. We then claim that $\rho(1, m) = 0$ for $m \geq N_2$. Suppose to the contrary that $\rho(1, m) \geq 1$. That is, $A(1, m)$ is not a nilpotent matrix. That is, $\Gamma(1, m)^l \neq \emptyset, l = 2, ...,$. Clearly,

$$\Gamma(1, m)^l = \Gamma^{(l,m)}. \tag{2.2}$$

Set $l = N_1$ to obtain a contradiction. Similarly, $\rho(2, m) = 0$ for $m \geq N_2$. Assume now that $\rho(1, m) = 0$ for some $m \geq 1$. Let $L_2(m) = card(\Gamma_2^m)$. Then $\Gamma(1, m)^{L_2(m)} = \emptyset$. Use (2.2) to deduce that $\Gamma^{(L_2(m), m)} = \emptyset$. Similar results hold if $\rho(2, m) = 0$.

Assume now $h_{com}(\Gamma) \geq 0$, i.e. $\rho(1, m) \geq 1, \rho(2, m) \geq 1, m = 1, ...,$. We now prove the conditions related to the characterization of $h_{com}(\Gamma)$ in terms of $\rho(1, m)$. We claim that

$$\log \rho(1, p + q) \leq \log \rho(1, p) + \log \rho(1, q), p, q \geq 1. \tag{2.3}$$

Indeed, let $w((l, p), \Gamma), w((l, q), \Gamma), w((l, p + q), \Gamma)$ be the total number of words of length l corresponding to the subshifts $\Gamma(1, p), \Gamma(1, q), \Gamma(1, p + q)$ respectively. Clearly, every word of length l in $\Gamma(1, p + q)$ splits (from bottom to top) as a word in $\Gamma(1, p)$ followed by a word in $\Gamma(1, q)$. That is $w((l, p + q), \Gamma) \leq w((l, p), \Gamma)w((l, q), \Gamma)$. Take the logarithm of this inequality, divide by l and take the lim sup to deduce (2.3). It is a well known fact that (2.3) implies that the sequence $\{\frac{\log \rho(1,m)}{m}\}_1^\infty$ converges to a (nonnegative) limit h. Furthermore, $h \leq \frac{\log \rho(1,m)}{m}, m = 1, ...,$. We now show that $h = h_{com}(\Gamma)$. Let $\{\epsilon_m\}_1^\infty$ be a positive sequence which converges to zero. Clearly, there exists a sequence of positive integers $\{l_m\}_1^\infty$ converging to ∞ so that

$$\frac{\log w((l_m, m), \Gamma)}{l_m} > \log \rho(1, m) - \epsilon_m, m = 1, ..., .$$

Hence,

$$h_{com}(\Gamma) \geq \limsup \frac{\log w((l_m, m), \Gamma)}{l_m m} \geq h.$$

We now show the reversed inequality. Let $\{m_i\}_1^\infty, \{n_i\}_1^\infty$ be two sequences of positive integers which converge to ∞. We claim that

$$\limsup \frac{\log w((n_i, m_i), \Gamma)}{n_i m_i} \leq h.$$

Pick a positive $\delta > 0$. Pick a positive integer m so that $\frac{\log \rho(1,m)}{m} < h + \delta$. Let $K >> 1$ so that

$$\forall n > K \max_{1 \leq k \leq m} (\frac{\log w((n,k), \Gamma)}{n} - \log \rho(1,k)) < \delta.$$

Assume that $m_i, n_i > K$. Set $m_i = p_i m + q_i, 1 \leq q_i \leq m$. Consider a word of length n_i corresponding to SFT induced by $\Gamma(1, m_i)$. This word splits (from bottom to top) as p_i words induced by $\Gamma(1,m)$ and a word induced by $\Gamma(1, q_i)$ of length n_i respectively. Hence, $w((n_i, m_i), \Gamma) \leq w((n_i, m), \Gamma)^{p_i} w((n_i, q_i), \Gamma)$. That is

$$\frac{\log w((n_i, m_i), \Gamma)}{n_i m_i} \leq \frac{\log w((n_i, m), \Gamma)}{n_i m} + \frac{\log w((n_i, q_i), \Gamma)}{n_i m_i} \leq$$
$$\frac{\log \rho(1,m)}{m} + \frac{\delta}{m} + \frac{\max_{1 \leq k \leq m} \rho(1,k) + \delta}{m_i}, m_i, n_i > K.$$

Thus, $\limsup_{m_i, n_i \to \infty} \frac{\log w((n_i, m_i), \Gamma)}{m_i n_i} < h + 2\delta$. These arguments prove the theorem for $\rho(1,m)$. Similar arguments verify the theorem for $\rho(2,m)$.

We now consider the periodic solutions. Assume first that $per(\Gamma^N) \neq \emptyset$ for some $N = (N_1, N_2), N_1 > 1, N_2 > 1$. It then follows that

$$per(\Gamma^M) \neq \emptyset, M = (N_1 + i(N_1 - 1), N_2 + j(N_2 - 1)), i, j = 0, ...,. \quad (2.4)$$

We then claim that $\rho p(1, N_2) \geq 1, \rho p(2, N_1) \geq 1$. Consider first the matrix $Ap(1, N_2)$. If $\rho p(1, N_2) = 0$, i.e. $Ap(1, N_2)$ is nilpotent, we could not have arbitrary long words in the SFT induced by $p(\Gamma(1, N_2))$. This contradicts (2.4) for $j = 0$. Similarly, $\rho p(2, N_1) \geq 1$. Assume now that $\rho p(1, N_2) \geq 1$ for some $N_2 > 1$. Then the SFT induced by $p(\Gamma(1, N_2))$ has at least one periodic word of length $N_1 > 1$, i.e. $per((p(\Gamma(1, N_2))^{N_1}) \neq \emptyset$. As every periodic word of length N_1 in the SFT corresponding to $p(\Gamma(1, N_2))$ is an element of $per(\Gamma^{(N_1, N_2)})$ we deduce in particular $per(\Gamma^{(N_1, N_2)}) \neq \emptyset$. That is,

$$hp(\Gamma) = -\infty \iff \rho p(1,m) = \rho p(2,m) = 0, m = 2, ...,.$$

Assume now that $hp(\Gamma) \geq 0$. We now prove the theorem for $\rho p(1,m)$. Consider the SFT induced by $p(\Gamma(1,m))$. Then $wp((l,m), \Gamma)$ is the number of periodic words of length l of this SFT. As $\rho p(1,m) \geq 1$ we know that for any $\delta > 0$ there exists $l = l(\delta)$ so that

$\frac{\log wp((l,m),\Gamma)}{l} \geq \log \rho p(1,m) - \delta$. Assume that $\{m_i\}_1^\infty$ is a strictly increasing sequence of positive integers so that

$$\limsup_{m\to\infty} \frac{\log \rho p(1,m)}{m} = \lim_{i\to\infty} \frac{\log \rho p(1,m_i)}{m_i}.$$

Let $\{l_i\}_1^\infty$ be a stricly increasing sequence so that $\frac{\log wp((l_i,m_i),\Gamma)}{l_i} \geq \log \rho p(1,m_i) - 1, i = 2, ...,$. We then deduce $\limsup_{m\to\infty} \frac{\log \rho p(1,m)}{m} \leq hp(\Gamma)$. The analogous result for $\rho p(2,m)$ is proved similarly.

Let $d > 2$. Assume that $(1, ..., 1) < M \in \mathbf{Z}^{d-1}$. Partition the box $B(M)$ to p nontrivial boxes of dimensions $M^i \in \mathbf{Z}_+^{d-1}, i = 1, ..., p$. We denote this fact by $M = \cup_1^p M^i$. We then have the following generalization of (2.3).

$$\log \rho(k,M) \leq \sum_1^p \log \rho(k,M^i), k = 1, ..., d. \qquad (2.3)'$$

Similarly, all assertions of the theorem for $d > 2$ are derived in an analogous way. $\diamond$

(2.5) Theorem. *Let $d \geq 2$ and assume that $\Gamma_i \subset < n > \times < n >, i = 1, ..., d$. Consider the $\mathbf{Z}^d$-SFT given by $\Gamma = (\Gamma_1, ..., \Gamma_d)$. Then*

$$h_{com}(\Gamma) = h(\Gamma).$$

To prove the theorem we need the following result.

(2.6) Lemma. *Let the assumptions of Theorem 2.5 hold. Assume furthermore that $\Gamma^\infty \neq \emptyset$. Let $M, N_1, N_2 \in \mathbf{Z}^d$ and assume that $(1, ..., 1) \leq M \leq N_1 \leq N_2$. Then*

$$\pi_{B(2M)+N_1-M}(\Gamma^{2N_1}) \supset \pi_{B(2M)+N_2-M}(\Gamma^{2N_2}).$$

Assume that $f \in \Gamma^{2M}$. Then

$$\exists g \in \Gamma^\infty \pi_{B(2M)} g = f \iff \forall N f \in \pi_{B(2M)+N-M}(\Gamma^{2N}).$$

Proof. The first claim of the lemma is trivial. Assume that $g \in \Gamma^\infty$. Let $f = \pi_{B(2M)} g$. Clearly, $\forall N f \in \pi_{B(2M)+N-M}(\Gamma^{2N})$. The reverse implication is proved by using Köning's argument as in the proof of Theorem 1.3. $\diamond$

Proof of Theorem 2.5 By Theorem 1.3 $h_{com}(\Gamma) = -\infty \iff h(\Gamma) = -\infty$. Thus, it is enough to consider the case $h_{com}(\Gamma) \geq 0$. As $w(N,\Gamma) \geq w(N,\Gamma^\infty)$ Theorem 1.5 implies that $h_{com}(\Gamma) \geq h(\Gamma)$. Thus $h_{com}(\Gamma) = 0 \Rightarrow h(\Gamma) = 0$. Hence, it is left to prove the theorem in the case $h_{com}(\Gamma) > 0$. For simplicity of the exposition we consider the case $d = 2$.

Fix $k \geq 1$ and let $m \geq k$. Consider the graph $\Gamma(1,2m)$. It represents a SFT induced by an infinite horizontal strip of width $2m$ in the vertical direction. Erase from the above infinite horizontal strip $m-k$ first and last infinite rows. We then obtain a $S(2m)(2k)$ a SFT induced by the graph $\Gamma(1,2m)$. Furthermore, $S(2m)(2k)$ is a subshift of $S(2k)$ induced by $\Gamma(1,2m)$. Clearly, we have the inclusions

$$S(2k) \supset S(2(k+1))(2k) \supset \cdots \supset S(2m)(2k) \supset \cdots.$$

Fix a box of dimension $(2l,2k)$ and let $w_{2m}(2l,2k)$ be the projection of $S(2m)(2k)$ on this box. Clearly

$$w((2l,2k),\Gamma) > w_{2(k+1)}(2l,2k) > \cdots > w_{2m(k)}(2l,2k) = w_{2(m(k)+1)}(2l,2k) = \dots .$$

Köning's argument yield that

$$w((2l,2k),\Gamma^\infty) = w_{2m(k)}(2l,2k).$$

We claim that

$$w((2l,2k),\Gamma^\infty)^{p-2m(k)} \geq \frac{\rho(1,p2k)^{2l}}{\rho(1,2k)^{2l2m(k)}}, p >> 1. \tag{2.7}$$

To prove this inequality consider the infinite horizontal strip of width $p2k$ where $p > 2m(k)$. In this strip pick up a box of dimension $(rl,p2k)$ where $r >> 1$. It then follows that

$$w((rl,p2k),\Gamma) \approx K_1(rl)^{s_1}\rho(1,p2k)^{rl}$$

for some fixed integer s_1. We view the above strip as composed of p infinite strips of width $2k$. For $m(k)$ the most upper strips and for $m(k)$the most lower strips the number of words in the box $(rl,2k)$ does not exceed

$$w((rl,2k),\Gamma) \approx K_2(rl)^{s_2}\rho(1,2k)^{rl}.$$

We now consider all the other infinite horizontal strip of width $2k$. Observe that they are all SFT contained in $S(2m(k))(2k)$. Denote by $C(l,2k)$ all distinct projections of Γ^∞ on a box $B(l,2k)$. Denote by $\Delta(l,2k) \subset C(l,2k) \times C(l,2k)$ the following graph induced by all distinct projections of Γ^∞ on the box $B((2l,2k))$. That is $(x,y) \in \Delta(l,2k)$ iff (x,y) is obtained by the projection on $B(2l,2k)$ of some possible configuration in Γ^∞. Let $w(t,\Delta(l,2k))$ be the number of words of length t in the SFT induced by $\Delta(l,2k)$. Set $\tilde\rho(l,2k) = \rho(A(\Delta(l,2k))$ It then follows that for $r >> 1$

$$w(r,\Delta(l,2k)) \approx K_3 r^{s_3}\tilde\rho(l,2k)^r.$$

We next claim that

$$w((2l,2k),\Gamma^\infty) \geq \tilde\rho(l,2k)^2. \tag{2.8}$$

Indeed, we trivially have that $w(2r,\Delta(l,2k)) \le w((2l,2k),\Gamma^\infty)^r$. Use the asymptotic value of $w((2r,2k),\Delta(l,2k))$ for $r >> 1$ to deduce (2.8). From the definitions of $m(k)$ and $\tilde\rho(l,2k)$ it follows that for $p > 2m(k)$

$$w((2rl,p2k),\Gamma) \le w((2rl,2k),\Gamma)^{2m(k)} w(2r,\Delta(l,2k))^{p-2m(k)}.$$

Use the asymptotic equalites for the above words and the inequality (2.8) to deduce (2.7). Take the $2lp-th$ root of (2.7) and let $p \to \infty$. Use Theorem 2.1 to deduce that

$$\liminf_{l\to\infty} \frac{\log w((2l,2k),\Gamma^\infty)}{2l} \ge 2kh_{com}(\Gamma).$$

Hence,

$$h(\Gamma) = \limsup_{k,l\to\infty} \frac{\log w((l,k),\Gamma^\infty)}{kl} \ge \liminf_{k\to\infty} \frac{1}{2k} \liminf_{l\to\infty} \frac{\log w((2l,2k),\Gamma^\infty)}{2l} \ge h_{com}(\Gamma).$$

Thus, $h(\Gamma) = h_{com}(\Gamma)$ and the proof of the theorem is completed. $\diamond$

§3. A symmetricity condition

(3.1) Theorem. *Let* $\Gamma_i \subset < n > \times < n >, i = 1, ..., d$, *and consider* $\mathbf{Z}^d$*-SFT given by* $\Gamma = (\Gamma_1, ..., \Gamma_d)$. *Assume that* $\Gamma_1, ..., \Gamma_{d-1}$ *are symmetric. Then* $hp(\Gamma) = h(\Gamma)$.

Proof. We prove the theorem by the induction on d. Assume first that $d = 2$. From Theorem 1.3 we deduce that $\rho(2,2) = 0 \Rightarrow h(\Gamma) = -\infty$. Assume that $\rho(2,2) \ge 1$. We now show that $hp(\Gamma) \ge 0$. Observe first that $per(\Gamma(2,2)^l) \ne \emptyset$ for some $l > 1$. In particular, $per(\Gamma_2^l)) \ne \emptyset$, i.e. $\rho(A(\Gamma_2)) = \rho(2,1) \ge 1$. Consider $p(\Gamma(1,l))$. The above assumption means that $p(\Gamma(1,l))$ has at least one edge. As Γ_1 is symmetric we deduce that $p(\Gamma(1,l))$ is also a symmetric matrix. Hence, $\rho p(1,l) \ge 1$. Theorem 2.1 implies that $hp(\Gamma) \ge 0$. Thus $hp(\Gamma) = h(\Gamma) = -\infty \iff \rho(2,2) = 0$.

In what follows we assume that $\rho(2,2) \ge 1$. We now prove that $hp(\Gamma) = h(\Gamma)$. Clearly, $hp(\Gamma) \le h(\Gamma)$. As we showed that $hp(\Gamma) \ge 0$ it is enough to consider the case $h(\Gamma) > 0$. Note that Theorems 2.1 and Theorem 2.5 yield that $\rho(2,m) > 1, m = 2, ...,$. Fix $m \ge 1$. Let $wp(l)$ be the number of periodic words in the SFT induced by $\Gamma(2,2m+1)$ of length l. Set

$$Lp_2(l) = card(per(\Gamma_2^m)), B = (b_{ij})_1^{Lp_2(l)} = A(p(\Gamma(1,l))),\ B^{2m} = (b_{ij}^{(2m)})_1^{Lp_2(l)}.$$

It then follows that $wp(l) = \sum_{i=j=1}^{Lp_2(l)} b_{ij}^{(2m)}$. Recall that B is a nonnegative symmetric matrix. Hence, its spectral norm is equal to its spectral radius $\rho p(1,l)$. As $wp(l) = eB^{2m}e^T, e = (1,...,1)$ we deduce that $wp(l) \le \rho p(1,l)^{2m} Lp_2(l)$. Observe next that

$trace(B^{2m}) = wp(2m+1, l)$. As B^{2m} is a symmetric matrix with nonnegative eigenvalues it follows that $trace(B^{2m}) \geq \rho p(1, l)^{2m}$. Combine the above inequalities to deduce

$$wp(l) \leq wp(2m+1, l) Lp_2(l) \leq wp(2m+1, l) n^{l-1}.$$

Fix $\delta, 0 < \delta$. Choose a strictly increasing sequence $\{l_m\}_1^\infty$ so that $\frac{\log wp(l_m)}{l_m} > \log \rho(2, 2m+1) - \delta$. Use Theorem 2.1 and the above inequalities to deduce

$$h(\Gamma) = \lim_{m\to\infty} \frac{\log \rho(2, 2m+1)}{2m+1} \leq \liminf_{m\to\infty} \frac{\log wp(l_m)}{(2m+1)l_m} \leq \liminf_{m\to\infty} \frac{\log wp(2m+1, l_m)}{(2m+1)l_m} \leq hp(\Gamma).$$

This proves the equality $hp(\Gamma) = h(\Gamma)$ for $d = 2$.

Assume that the result holds for $d \geq l \geq 2$ and let $d = l+1$. Choose $\delta > 0$ and $M = (M_1, ..., M_l) >> (1, ..., 1)$ so that $\frac{\log \rho(l+1, M)}{M_1 \cdots M_l} < h(\Gamma) + \delta$. (We are assuming the nontrivial case $\rho(l+1, M) \geq 1 \iff h(\Gamma) \geq 0$.) Furthermore, we assume that $M_1, ..., M_l$ are odd numbers. Choose $N_{l+1} >> 1$ so that $w(M_1, ..., M_l, N_{l+1})$- the total number of words in $(\Gamma(l+1, M))^{N_{l+1}}$ is not greater then $(1+\delta)^{N_{l+1}}$ times $wp_{l+1}(M_1, ..., M_l, N_{l+1}) = card(per(\Gamma(l+1, M)^{N_{l+1}}))$. Let $p_{l+1}(\Gamma(1, (M_2, ..., M_l, N_{l+1}))) \subset \Gamma(1, (M_2, ..., M_l, N_{l+1}))$ be the subgraph generated by all the words of length $(M_2, ..., M_l, N_{l+1})$ in the SFT induced by $(\Gamma_2, ..., \Gamma_{l+1})$ which are periodic with respect to the last coordinate. Note that this graph is symmetric. Moreover,

$$(p_{l+1}(\Gamma(1, (M_2, ..., M_l, N_{l+1}))))^{M_1} = per(\Gamma(l+1, M)^{N_{l+1}}) \; ne \emptyset.$$

The arguments of the proof for $d = 2$ show that $h(\Gamma)$ - the density of words of length $(N_1, ..., N_{l+1})$ is equal to the density of the words periodic in the last and the first coordinates. Let $p_{1,l+1}(\Gamma(2, (M_1, M_3, ..., M_l, N_{l+1}))) \subset \Gamma(2, (M_1, M_3, ..., M_l, N_{l+1}))$ be the subgraph generated by all the words of length $(M_1, M_3, ..., M_l, N_{l+1})$ in SFT induced by $(\Gamma_1, \Gamma_3, ..., \Gamma_{l+1})$ which are periodic in the first and the last coordinate. As Γ_2 is symmetric it follows that $p_{1,l+1}(\Gamma(2, (M_1, M_3, ..., M_l, N_{l+1})))$ is also symmetric. Use the previous arguments to deduce that $h(\Gamma)$ is the density of words periodic in $1, 2, l+1$ cordinates. Continue in this manner to deduce that $h(\Gamma) = hp(\Gamma)$. $\diamond$

Our results yield a new proof that the periodic entropy $hp(\Gamma)$ computed by Lieb [**Lie**] is equal to the standard entropy $h(\Gamma)$. See [**B-K-W**] for a specific proof of the above equality for the ice rule model in zero field.

Under the assumptions of Theorem 3.1 it is possible to give lower estimates for $h(\Gamma)$. To do that we need to introduce the following notation. Let $U \subset< d >$ be a set of cardinality p. We then agree that $U = \{i_1, ..., i_p\}, 1 \leq i_1 < \cdots < i_p \leq d$. For $N = (N_1, ..., N_d)$ set $N^U = (N_{i_1}, ..., N_{i_p})$. In particular, $N^{\overline{\{k\}}} = (N_1, ..., N_{k-1}, N_{k+1}, ..., N_d), k = 1, ..., d$. Assume the assumptions of Theorem 3.1. For any nontrivial set $U \subset< d >$ we consider the SFT on $\mathbf{Z}^{card(U)})$ induced on $\Gamma^U = (\Gamma_{i_1}, ..., \Gamma_{i_p})$. Suppose that $k \in U, V = U \backslash \{k\}, card(V) \geq 1$. Then $\Gamma(k, N^V)$ is graph induced by the SFT corresponding to Γ^U. Let $\rho(k, N^V)$ be the spectral radius of this graph. Given three pairwise disjoint sets $V, \{k\}, W \subset< d >$ we consider the following contraction of $\rho(k, N^{V\cup W})$ on V indices

$$\rho_V(k, N^W)) = \lim_{N_i\to\infty, i\in V} \rho(k, N^{V\cup W})^{\frac{1}{\prod_{i\in V} N_i}}.$$

Let $U = \{k\} \cup V$. Observe that $\log \rho_V(k, N^V) = h(\Gamma^U)$.

(3.2) Theorem. *Let* $\Gamma_i \subset <n> \times <n>, i = 1, ..., d$, *and consider the* $\mathbf{Z}^d$ *-SFT given by* $\Gamma = (\Gamma_1, ..., \Gamma_d)$. *Assume that* Γ_k *is symmetric. Then*

$$\rho(i, N^{\overline{\{i\}}}) \leq \rho_{\{i\}}(k, N^{\overline{\{i,k\}}})^{N_k - 1} \rho(i, N^{\overline{\{i,k\}}}),$$
$$N = (N_1, ..., N_d) \geq (1, ..., 1), i = 1, ..., k-1, k+1, ..., d.$$

Proof. Fix $N_j \geq 1, j = 1, ..., i-1, i+1, ..., d$. For a small positive $\delta > 0$ choose $N_i >> 1$ so that

$$(1-h)^{N_i} \rho(i, N^{\overline{\{i\}}})^{N_i} \leq w(N) = card((\Gamma(i, N^{\overline{\{i\}}}))^{N_i}),$$
$$\omega(N^{\overline{\{k\}}}) = card((\Gamma(i, N^{\overline{\{i,k\}}}))^{N_i}) \leq (1+\delta)^{N_i} \rho(i, N^{\overline{\{i,k\}}})^{N_i}.$$

Let $C = A(\Gamma(k, N^{\overline{\{k\}}}))$. Then C is $\omega(N^{\overline{\{k\}}}) \times \omega(N^{\overline{\{k\}}})$ symmetric matrix with the spectral norm equal to $\rho(k, N^{\overline{\{k\}}})$. Set $e = (1, ..., 1)$. The maximal characterization of the maximal eigenvalue of $C^{N_k - 1}$ yields

$$w(N, \Gamma) = eC^{N_k - 1} e^T \leq \rho(k, N^{\overline{\{k\}}})^{N_k - 1} \omega(N^{\overline{\{k\}}}).$$

Taking the $N_i - th$ root in the above inequality and letting $N_i \to \infty$ we deduce the theorem. $\diamond$.

Combine Theorems 3.1-3.2 for $d = 2, k = 1$ with Theorems 2.1 and 2.5 to obtain.

(3.3) Corollary. *Let* $\Gamma_1, \Gamma_2 \subset <n> \times <n>$. *Assume that* Γ_1 *is symmetric and consider the* $\mathbf{Z}^2$*-SFT induced by* $\Gamma = (\Gamma_1, \Gamma_2)$. *Then*

$$\frac{\log \rho(2, k)}{k-1} - \frac{\log \rho(2, 1)}{k-1} \leq hp(\Gamma) = h(\Gamma) \leq \frac{\log \rho(2, k)}{k}, \ k = 2, ..., .$$

The above Corollary under stronger assumptions is due to [**M-P2**]. Note that Corollary 3.3 enables one to calculate effectively the entropy $h(\Gamma)$ up to an arbitrary precision.

We now apply Theorem 3.2 for $d = 3$ assuming that Γ_2 is symmetric. Let $N_1 = p \geq 1, N_2 = q \geq 2, k = 2, i = 3$ to deduce

$$\frac{\log \rho(3, (p, q))}{p(q-1)} - \frac{\log \rho(3, p)}{p(q-1)} \leq \frac{\log \rho_{\{3\}}(2, p)}{p}.$$

Let $p \to \infty$. We then get the inequalites

$$\frac{\log \rho_{\{1\}}(3, q)}{q-1} - \frac{h(\Gamma^{\{1,3\}})}{q-1} \leq h(\Gamma). \qquad (3.4)$$

This yields a lower bound for $h(\Gamma)$ which converges to $h(\Gamma)$ as $q \to \infty$. To obtain computable lower bounds for $h(\Gamma)$ in terms of various $\rho(k, M)$ we assume that Γ_3 is symmetric. First observe that Theorem 2.1 gives an upper bound on $h(\Gamma^{\{1,3\}})$. Use Theorem 3.2 with $k = 3, i = 1, M_2 = q, M_3 = r$ to deduce

$$\frac{\log \rho(1,(q,r))}{r-1} - \frac{\log \rho(1,q)}{r-1} \le \log \rho_{\{1\}}(3,q).$$

Use the above inequalites in (3.4) to obtain a lower bound for $h(\Gamma)$ which in principle can be arbitrary close to $h(\Gamma)$. (Choose all the numbers entering in this inequality to be big enough.)

§4. Observations

Let $\Gamma \subset< n > \times < n >$ be a directed graph on n vertices. For any nontrivial set $V \subset< n >$ set $\Gamma(V) = \Gamma \cap V \times V$. Γ is called a strongly connected graph if any two vertices $i, j \in< n >$ are connected by a path in a graph. This is equivalent to the statement that $A(\Gamma)$ is an irreducible matrix. If Γ is not strongly connected then $< n >$ is decomposed to a disjoint union

$$\begin{aligned} &< n >= \cup_0^p U_i, U_i \cap U_j = \emptyset, 0 \le i < j \le p, card(U_i) \ge 1, i = 1, ..., p, \\ &A(\Gamma(U_0))^n = 0, (A(\Gamma(U_i)) + I)^n > 0, i = 1, ..., p. \end{aligned} \tag{4.1}$$

Here I stands for the identity matrix and $B > 0$ denote a real valued matrix whose all entries are positive. The set U_0 is called a transient set. That is, if we consider any path with edges in our graph Γ each transient vertex will appear at most once. Equivalently, any closed path will not contain any transient vertex, while for each vertex in $\cup_1^p U_i$ there exists a closed path which contains this vertex. The set $\cup_1^p U_i$ is the set of nontransient vertices. Moreover, each graph $\Gamma(U_i)$ is a strongly connected for $i = 1, ..., p$. Furthermore, $U_1, ..., U_p$ are maximal sets with this property. That is, for $1 \le i < j \le p$ either there is no path of Γ connecting U_i to U_j or U_j to U_i (or both). The reduced graph $red(\Gamma)$ is defined as follows. The states (vertices) of the reduced graph are the transient vertices U_0 and the new states $[U_1], ..., [U_p]$. Let $red(n) = card(U_0) + p$. Then $red(\Gamma) \subset< red(n) > \times < red(n) >$ does not have self loops, i.e. $(i, i) \notin red(\Gamma), i \in< red(n) >$. Furthermore $(i, j) \in red(\Gamma)$ iff there is at least one edge in Γ which goes from one vertex represented by the state i to one vertex represented by the state j. It then follows that $A(red(\Gamma))$ is a nilpotent matrix. Let $x = (x_j)_1^m \in \Gamma^m, m >> 1$. The generic picture dictated by the reduced graph is as follows. First we may have a couple of transient vertices $x_1, ..., x_{t_1} \in U_0, (x_i, x_{i+1}) \in red(\Gamma), i = 1, ..., t_1 - 1$. (It may happen that we do not have transient vertices, i.e. $t_1 = 0$.) Then we have a sequence of an arbitrary length k_1 $x_{t_1+1}, ..., x_{t_1+k_1} \in U_{j_1}, (x_{t_1}, [U_{j_1}]) \in red(\Gamma)$.. Then we may have another few transient states $x_{t_1+k_1+1}, ..., x_{t_1+k_1+t_2} \in U_0, ([U_{j_1}], x_{t_1+k_1+1}), (x_i, x_{i+1}) \in red(\Gamma), i = t_1 + k_1 + 1, ..., t_1 + k_1 + t_2 - 1$, $(t_2 \ge 0)$. This sequence may be followed by another arbitrary long sequence $x_{t_1+k_1+t_2+1}, ..., x_{t_1+k_1+t_2+k_2} \in U_{j_2}, (x_{t_1+k_1+t_2}, [U_{j_2}]) \in red(\Gamma)$. If $t_2 = 0$ we then have the condition $([U_{j_1}], [U_{j_2}]) \in red(\Gamma)$. This process may continue until we reach the final state of the reduced graph. In particular, the arbitrary long sequences belong to pairwise distinct components $U_{j_1}, ..., U_{j_l}$ whose order depends on the structure of the reduced graph. In particular, $1 \le l \le n$.

These properties can be deduced straightforward from the Frobenius normal form of a nonnegative matrix, e.g. [**Gan**]. Consult for example with [**F-S**]. In particular, $\rho(A(\Gamma)) = \max_{1 \le i \le p} \rho(A(\Gamma^{(i)}))$. A graph $\Gamma \subset <n> \times <n>$ is called nontransient if it does not have a transient set, i.e. $U_0 = \emptyset$. For a general graph we let $\Gamma' = \Gamma(\cup_1^p U_i)$ to be the nontransient part of Γ. As $h(\Gamma) = \log \rho(A(\Gamma))$ we deduce that $h(\Gamma) = \max_{1 \le i \le p} h(\Gamma(U_i)) = h(\Gamma')$. Finally observe that the periodic orbits under the shift correspond to closed paths in the graph Γ. Hence, any periodic orbit has vertices only in one $per((\Gamma(U_i))^N)$. We now show that some these results can be generalised to SFT in higher dimension.

(4.2) Lemma. *Let $\Gamma_i \subset <n> \times <n>, i = 1, ..., m$. Then one of the following mutually exclusive conditions hold:*
(i) For any nontrivial subset $V \subset <n>$ there exists $k \in <m>$ so that $\Gamma_k(V)$ has a nontrivial transient set of vertices in V.
(ii) There exist a maximal (nontrivial) subset $V \subset <n>$, so that $\Gamma_k(V)$ is a nontransient graph on V for $k = 1, ..., m$.

Proof. Let $U_{0,i} \subset <n>$ be the set of transient vertices of the graph $\Gamma_i, i = 1, ..., m$. If $U_{0,i} = \emptyset, i = 1, ..., m$, then we have the condition (ii) with $V = <n>$. Let $V_1 = <n> \backslash (\cup_1^m U_{0,i})$. If $V_1 = \emptyset$ then the condition (i) holds. Assume that $<n> \ne V_1 \ne \emptyset$. Repeat the above process for $\Gamma_i(V_1), 1 =, ..., m$ to deduce either (i) or (ii). $\diamond$

(4.3) Theorem. *$\Gamma_i \subset <n> \times <n>, i = 1, ..., d$, and consider $\mathbf{Z}^d$ -SFT given by $\Gamma = (\Gamma_1, ..., \Gamma_d)$. Assume first that condition (i) of Lemma 4.2 holds. Then $h(\Gamma) = -\infty$. Assume now that V is the maximal (nontrivial) subset of $<d>$ so that $\Gamma_k(V)$ is nontransient for $k = 1, ..., d$. Set $\Gamma(V) = (\Gamma_1(V), ..., \Gamma_d(V))$. Then $h(\Gamma) = h(\Gamma(V))$.*

Proof. Clearly, the theorem trivially holds if $h(\Gamma) = -\infty$. Assume that $h(\Gamma) \ge 0$. That is for each $N = (N_1, ..., N_d), N_i \ge 1, i = 1, ..., d$, $\rho(k, \overline{N^{\{k\}}}) \ge 1, k = 1, ..., d$. As in the proof of Lemma 4.2 consider the transient set $U_{0,k}$ for the graph Γ_k for $k = 1, ..., d$. If all $U_{0,k} = \emptyset$ then $V = <n>$ and the theorem is trivial in this case. Supppose that $U_{0,k} \ne \emptyset$. Fix $\overline{N^{\{k\}}}$. As $\rho(k, \overline{N^{\{k\}}}) \ge 1$ we know that $h(\Gamma(k, \overline{N^{\{k\}}})$ is given by the density of the periodic words $per(\Gamma(k, \overline{N^{\{k\}}})^{N_k})$. Observe next that every periodic word in $per(\Gamma(k, \overline{N^{\{k\}}})^{N_k})$ is induced by a word $f = (f_{(j_1, ... j_d)})_{j_1 = \cdots = j_d = 1}^{N_1, ..., N_d}$ such that

$$(f_{(j_1, ..., j_d)})_{j_k=1}^{N_k} \in per((\Gamma_k)^{N_k}), j_l = 1, ..., N_l, l = 1, ..., k-1, k+1, ..., d.$$

Hence, the coordinates of each vector $(f_{(j_1, ..., j_d)})_{j_k=1}^{N_k}$ belong to some set $U_{k,i}$ appearing in the decomposition (4.1) of the nontransient set for Γ_k. Note that the value of i may depend on $(j_1, ..., j_{k-1}, j_{k+1}, ..., j_d)$. In particular, all the coordinates of f are in the set

$V_1 =< n > \backslash U_{0,k}$. Let $\Gamma(V_1) = (\Gamma_1(V_1), ..., \Gamma_d(V_1))$. Theorems 2.1 and 2.5 yield that $h(\Gamma) = h(\Gamma(V_1))$. Repeat this process as in the proof of Lemma 4.2. If we obtain the condition (i) of Lemma 4.2 we deduce that $h(\Gamma) = -\infty$ which contradicts our assumption that $h(\Gamma) \geq 0$. Hence, the second condition of Lemma 4.2 holds. By the above arguments $h(\Gamma) = h(\Gamma(V))$ and the proof of the theorem is concluded. $\diamond$

Let $\Gamma_1, \Gamma_2 \subset < n >$. Set $X = (\Gamma_2)^\infty$. Then X is a closed compact space in the Tychonoff topology. (More precisely, X is a Cantor set.) Set $\Delta = \Delta(\Gamma_1, \Gamma_2) \subset X \times X$ be the following closed graph

$$\Delta = \{(x, y) : x = (x_i)_{i \in \mathbf{Z}}, (y_i)_{i \in \mathbf{Z}} \in X, (x_i, y_i) \in \Gamma_1, i \in \mathbf{Z}\}.$$

Define Δ^m, Δ^∞ as in the introduction. Note that

$$\Delta^m = \emptyset \iff \rho(2, m) = 0, m = 2, ...,$$
$$\Delta^\infty = \emptyset \iff \Gamma^\infty = \emptyset, \Gamma = (\Gamma_1, \Gamma_2).$$

Observe that if Γ_1 is symmetric then Δ is also symmetric.

In [**Fri1-2**] we studied the entropy $h(\Delta)$ of the shift σ restricted to Δ^∞. Here $\sigma((x_i)_{i \in \mathbf{Z}}) = (x_{i+1})_{i \in \mathbf{Z}}$. It is not difficult to show that if $h(\Gamma) > 0$ then $h(\Delta) = \infty$. Thus, $h(\Gamma)$ can be considered as the renormalization of the entropy $h(\Delta)$. More precisely if $N(k, \epsilon)$ is the number of $k - \epsilon$ separated sets then one can show that up to a multiplicative constant that the right renormalization is:

$$h(\Gamma) = \lim_{\epsilon \to 0} \limsup_{k \to \infty} \frac{\log N(k, \epsilon)}{k \log \frac{1}{\epsilon}}.$$

Moreover, the dynamics of $\mathbf{Z}^2$ shift restricted to Γ^∞ is related to the dynamics of the standard shift restricted to Δ^∞. It would be interesting to explore in more details this relation. Similar ideas apply to higher dimensional $\mathbf{Z}^d$ -SFT.

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