Eigenvalue inequalities, log-convexity and scaling: old results and new applications, a tribute to Sam Karlin

Shmuel Friedland
Univ. Illinois at Chicago

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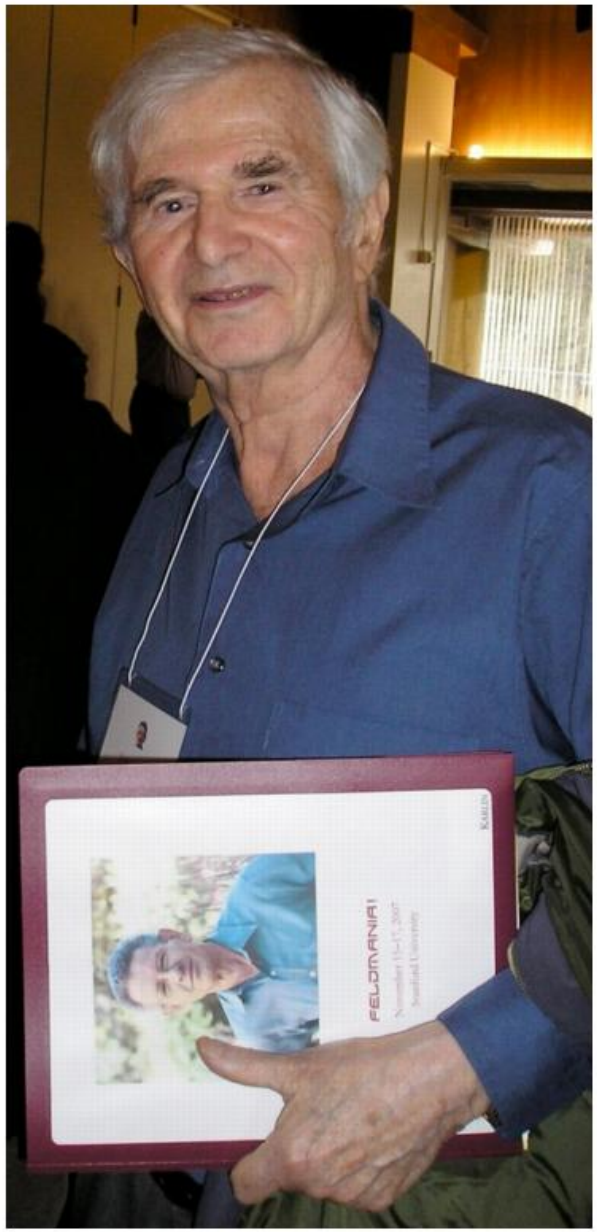

Figure: Karlin

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He died Dec. 18, 2007 at Stanford Hospital after a massive heart

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$\mathbf{z}_j = C\mathbf{z}_{j-1}, j = 1, \ldots,$ i.e. $\mathbf{z}_j = C^j \mathbf{z}_0, j = 1, \ldots,$

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No species extinct if $\rho(DA) > 1$.

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$\mathbf{d} = (d_1, \ldots, d_n) > \mathbf{0}, D = D(\mathbf{d}) := \operatorname{diag}(d_1, \ldots, d_n)$

$\rho(D(\mathbf{d})A) \geq \rho(A) \prod_{i=1}^n d_i^{x_i(A)y_i(A)}$

If A has positive diagonal then equality holds iff $D(\mathbf{d}) = aI_n$.

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(weighted arithmetic-geometric inequality)

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$\log \rho(DA) = \sum_{i=1}^n \mathbf{x}_i(A)\mathbf{y}_i(A)(\log d_i + \frac{(A\mathbf{x}(DA))_i}{x_i(DA)}) \geq$
$\log \rho(A) + \sum_{i=1}^n \mathbf{x}_i(A)\mathbf{y}_i(A) \log d_i$

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$A^{-1} = rI - B,\ 0 \le B,\ \rho(B) < r.$

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Weaker than Friedland 81 for inverse of M-matrix

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Example 2: $A = \begin{bmatrix} 0 & * \\ * & 0 \end{bmatrix}$ always rescalable to doubly stochastic with many more solutions than in THM 3.

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Contrary to claim in FK75 I do not know how to prove THM3 for FI matrices
Reason: why $f(\mathbf{z}) := \sum_{i=1}^n w_i \log \frac{(A\mathbf{z})_i}{z_i}$ blows to ∞ on $\partial\Pi_n$, or attains minimum in the interior of Π_n?

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Proof:
$\sum_{i=1}^n w_i \log \frac{d_i y_i}{(AD(\mathbf{d})\mathbf{y})_i} = \sum_{i=1}^n w_i \log \frac{y_i}{(D(\mathbf{c})AD(\mathbf{d})\mathbf{y})_i} + \sum_{i=1}^n w_i \log(c_i d_i)$

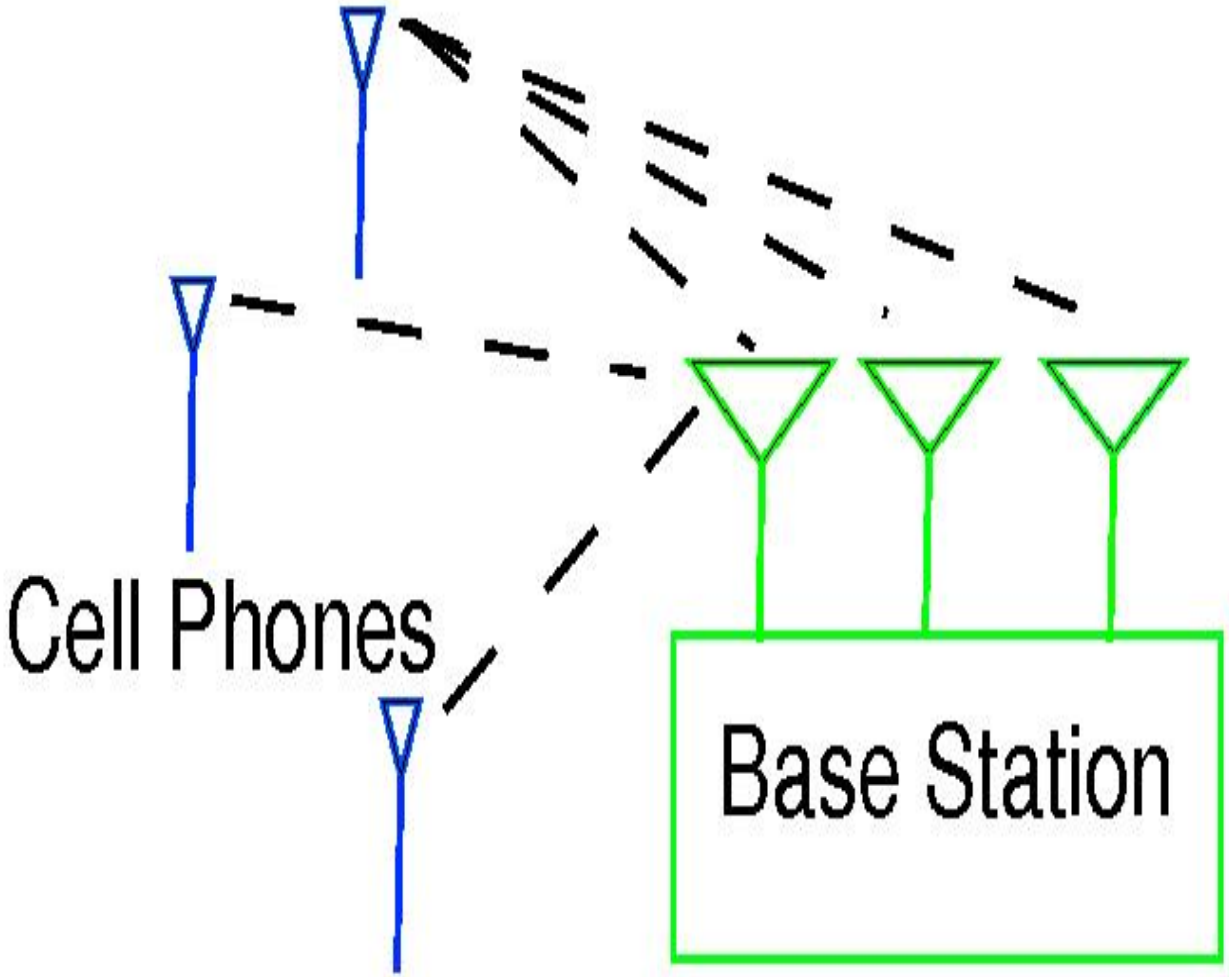

Figure: Cell phones communication

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n wireless users. Each transmits with power $p_i \in [0, \bar{p}_i]$, which can be regulated

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If $\sum_{j \neq i} w_j > w_i > 0$ for $i = 1, \ldots, n$
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Extensions to operators Friedland-Porta 04

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