MCS 590 - HW 2

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2.2

(1) H is time independent, so Schrödinger's equation solves to

$$|\psi(t)\rangle = e^{-iHt/\hbar}|\psi(0)\rangle. \tag{1}$$

H can be diagonalized as $H = U\Lambda U^\dagger$, where

$$\Lambda = -\frac{\hbar}{2}\omega\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, U = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ i & -i\end{pmatrix},$$

so

$$e^{-iHt/\hbar} = Ue^{-i\Lambda t/\hbar}U^\dagger = U\begin{pmatrix}e^{it\omega/2} & 0\\ 0 & e^{-it\omega/2}\end{pmatrix}U^\dagger = \begin{pmatrix}\cos\frac{t\omega}{2} & \sin\frac{t\omega}{2}\\ -\sin\frac{t\omega}{2} & \cos\frac{t\omega}{2}\end{pmatrix}.$$

Therefore, (**??**) yields

$$|\psi(t)\rangle = \begin{pmatrix}\cos\frac{t\omega}{2} & \sin\frac{t\omega}{2}\\ -\sin\frac{t\omega}{2} & \cos\frac{t\omega}{2}\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix} = \begin{pmatrix}\sin\frac{t\omega}{2}\\ \cos\frac{t\omega}{2}\end{pmatrix}.$$

(2) The observable σ_z has eigenvalues $\lambda_1 = 1, \lambda_2 = -1$ with eigenvectors $|\lambda_1\rangle = (1,0)^\top, |\lambda_2\rangle = (0,1)^\top$. Thus,

$$|\psi(t)\rangle = \sin\frac{t\omega}{2}|\lambda_1\rangle + \cos\frac{t\omega}{2}|\lambda_2\rangle,$$

so the probability of observing $+1$ at time t is $\sin^2(t\omega/2)$.

(3) The observalbe σ_x has eigenvalues $\lambda_1 = 1, \lambda_2 = -1$ with eigenvectors $|\lambda_1\rangle = \frac{1}{\sqrt{2}}(1,1)^\top, |\lambda_2\rangle = \frac{1}{\sqrt{2}}(1,-1)^\top$. Thus, if we have

$$|\psi(t)\rangle = c_1|\lambda_1\rangle + c_2|\lambda\rangle, \tag{2}$$

then $|c_1|^2$ is the probability of observing +1 at time t. (**??**) is simply a system of 2 equations in c_1, c_2, which can be solved trivially to yield $c_1 = \left(\sin\frac{t\omega}{2} + \cos\frac{t\omega}{2}\right)/\sqrt{2}$. Thus, the probability of observing +1 at time t is

$$|c_1|^2 = \frac{1}{2}\left(\sin\frac{t\omega}{2} + \cos\frac{t\omega}{2}\right)^2 = \frac{1}{2}\left(1 + 2\sin\frac{t\omega}{2}\cos\frac{t\omega}{2}\right) = \frac{1}{2}(1+\sin(tw)).$$

2.4

First assume ρ is pure. Then by theorem 2.1, $\rho^2 = \rho$, so

$$\operatorname{tr}\rho^2 = \operatorname{tr}\rho = 1.$$

Now assume $\operatorname{tr}\rho^2 = 1$. Let $\lambda_1 \geq \ldots \geq \lambda_n \geq 0$ be the eigenvalues of ρ. Note that they are all nonnegative real numbers since ρ is Hermitian and positive semidefinite. Then $\lambda_1^2 \geq \ldots \geq \lambda_n^2 \geq 0$ are the eigenvalues of ρ^2. Thus, we have

$$\operatorname{tr}\rho = \lambda_1 + \ldots + \lambda_n = 1 \tag{3}$$

and

$$\operatorname{tr}\rho^2 = \lambda_1^2 + \ldots + \lambda_n^2 = 1. \tag{4}$$

Subtracting (**??**) from (**??**), we get

$$\sum_{i=1}^{n} \lambda_i(1-\lambda_i) = 0. \tag{5}$$

Observe that since $\operatorname{tr}\rho = 1$, we have $0 \leq \lambda_i \leq 1$ for all i. Thus, each term in (**??**) is nonnegative. Since they sum to 0, we conclude $\lambda_i \in \{0, 1\}$ for all i. But by (**??**), exactly one of the λ_i (namely, λ_1) must be 1 and the rest 0; therefore, the eigendecomposition of ρ is

$$\rho = \sum_{i=1}^{n} \lambda_i |\lambda_i\rangle\langle\lambda_i| = \lambda_1|\lambda_1\rangle\langle\lambda_1|,$$

so ρ is pure.

2.5

Clearly ρ is hermitian and $\operatorname{tr}\rho = 1$. Its eigenvalues are $\lambda_1 = \lambda_2 = \lambda_3 = (1-p)/4, \lambda_4 = (1+3p)/4$, which are all ≥ 0 for any $p \in [0,1]$, so ρ is positive semidefinite. Therefore, ρ is a density matrix.

Now assume $p > 1/3$. Consider ρ as a 2×2 block matrix of 2×2 blocks ρ_{ij}. Then

$$\rho^{\mathrm{pt}} = \begin{pmatrix} \rho_{11}^{\top} & \rho_{12}^{\top} \\ \rho_{21}^{\top} & \rho_{22}^{\top} \end{pmatrix} = \begin{pmatrix} \frac{1+p}{4} & 0 & 0 & 0 \\ 0 & \frac{1-p}{4} & \frac{p}{2} & 0 \\ 0 & \frac{p}{2} & \frac{1-p}{4} & 0 \\ 0 & 0 & 0 & \frac{1+p}{4} \end{pmatrix}.$$

The eigenvalues of ρ^{pt} are $\mu_1 = \mu_2 = \mu_3 = (1+p)/4, \mu_4 = (1-3p)/4$. Since $p > 1/3$, $\mu_4 < 0$. Thus, we have

$$N(\rho) = \frac{\sum_{i=1}^{4} |\mu_i| - 1}{2} = \frac{\mu_1 + \mu_2 + \mu_3 - \mu_4 - 1}{2} = \frac{3p-1}{4} > \frac{3 \cdot \frac{1}{3} - 1}{4} = 0.$$

2.7

The density matrix is

$$\rho = |\psi'\rangle\langle\psi'| = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}.$$

Each ρ_{ij} is a 2×2 block. The partial trace over $\mathcal{H}_1$ is thus

$$\mathrm{tr}_1(\rho) = \begin{pmatrix} \mathrm{tr}(\rho_{11}) & \mathrm{tr}(\rho_{12}) \\ \mathrm{tr}(\rho_{21}) & \mathrm{tr}(\rho_{22}) \end{pmatrix} = \frac{1}{2} I_2.$$

2.8

We write

$$\rho_1 = \frac{1}{4}|\psi_1\rangle\langle\psi_1| + \frac{3}{4}|\psi_2\rangle\langle\psi_2|.$$

We will use $|\psi_1\rangle, |\psi_2\rangle$ as the basis for the new Hilbert space as well. According to (2.53) we get

$$|\Psi\rangle = \frac{1}{2}|\psi_1\rangle \otimes |\psi_1\rangle + \frac{\sqrt{3}}{2}|\psi_2\rangle \otimes |\psi_2\rangle$$

2.9

Unitary transformations map orthonormal vectors to orthonormal vectors. Thus, $U|\phi_k\rangle$ are orthonormal, so $|\Psi'\rangle$ is a purification of ρ_1.

2.10

Observation 1. *Let U be unitary and A be Hermitian and positive semidefinite. Then $B = UAU^\dagger$ is Hermitian and positive semidefinite, and $\sqrt{B} = U\sqrt{A}U^\dagger$.*

Proof. Trivially B is Hermitian. A and B are similar, so they have the same eigenvalues. Thus, B is positive semidefinite since A is.

Now observe that

$$(U\sqrt{A}U^\dagger)^2 = U\sqrt{A}U^\dagger U\sqrt{A}U^\dagger = U\sqrt{A}\sqrt{A}U^\dagger = UAU^\dagger = B.$$

$\sqrt{B}$ is the *unique* matrix whose square is B, so this proves $U\sqrt{A}U^\dagger = \sqrt{B}$. □

By observation **??**, we have

$$\sqrt{U\rho_1U^\dagger}U\rho_2U^\dagger\sqrt{U\rho_1U^\dagger} = U\sqrt{\rho_1}U^\dagger U\rho_2U^\dagger U\sqrt{\rho_1}U^\dagger = U\sqrt{\rho_1}\rho_2\sqrt{\rho_1}U^\dagger.$$

Since $\sqrt{\rho_1}\rho_2\sqrt{\rho_1}$ is Hermitian and positive semidefinite, again we may apply observation **??** to get

$$\sqrt{\sqrt{U\rho_1U^\dagger}U\rho_2U^\dagger\sqrt{U\rho_1U^\dagger}} = \sqrt{U\sqrt{\rho_1}\rho_2\sqrt{\rho_1}U^\dagger} = U\sqrt{\sqrt{\rho_1}\rho_2\sqrt{\rho_1}}U^\dagger.$$

Finally, $U\sqrt{\sqrt{\rho_1}\rho_2\sqrt{\rho_1}}U^\dagger$ is similar to $\sqrt{\sqrt{\rho_1}\rho_2\sqrt{\rho_1}}$, so they have the same trace.

2.11

ρ_1 is a diagonal matrix, so $\sqrt{\rho_1} = \text{diag}\left(\frac{1}{\sqrt{2}}, 0, 0, \frac{1}{\sqrt{2}}\right)$ and

$$\sqrt{\rho_1}\rho_2\sqrt{\rho_1} = \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} := B.$$

To find $\sqrt{B}$, we first diagonalize it using methods from HW 1:

$$B = U\Lambda U^\dagger,$$

where

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

and $\Lambda = \frac{1}{2}\operatorname{diag}(1,0,0,0)$. Since Λ is diagonal, we have $\sqrt{\Lambda} = \frac{1}{\sqrt{2}}\operatorname{diag}(1,0,0,0) = \sqrt{2}\Lambda$ and hence

$$\sqrt{B} = U\sqrt{\Lambda}U^\dagger = U(\sqrt{2}\Lambda)U^\dagger = \sqrt{2}U\Lambda U^\dagger = \sqrt{2}B.$$

Therefore,

$$F(\rho_1, \rho_2) = \operatorname{tr}(\sqrt{B}) = \frac{\sqrt{2}}{4}(1+1) = \frac{\sqrt{2}}{2}.$$