Entropy of holomorphic and rational maps: a survey

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Abstract

We give a brief survey on the entropy of holomorphic self maps f of compact Kähler manifolds, and rational dominating self maps f of smooth projective varieties. We emphasize the connection between the entropy and the spectral radii of the induced action of f on the homology of the compact manifold. The main conjecture for the rational maps states that modulo birational isomorphism all various notions of entropy and the spectral radii are equal.

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1 Introduction

The subject of the dynamics of a map $f : X \to X$ has been studied by hundreds, or perhaps thousands, of mathematicians, physicists and other scientists in the last 150 years. One way to classify the *complexity* of the map f is to assign to it a number $h(f) \in [0, \infty]$, which called the *entropy* of f. The entropy of f should be an invariant with respect to certain *automorphisms* of X. The complexity of the dynamics of f should be reflected by $h(f)$, i.e. the larger $h(f)$ the more complex is its dynamics.

The subject of this short survey paper is mostly concerned with the entropy of a holomorphic $f : X \to X$, where X is a compact Kähler manifold, and the entropy of a rational map of $f : Y \dashrightarrow Y$, where Y is a smooth projective variety. In the holomorphic case the author [12, 13, 14] showed that entropy of f is equal to the logarithm of the spectral radius of the finite dimensional f_* on the total homology group $\mathrm{H}_*(X)$ over $\mathbb{R}$.

Most of the paper is devoted to the rational map $f : Y \dashrightarrow Y$ which can be assumed dominating. In this case we have some partial results and inequalities. We recall three possible definition of the entropy $h_B(f), h(f), h_F(f)$ which are related as follows: $h_B(f) \le h(f) = h_F(f)$. The analog of the *dynamical* homological spectral radius are given by $\rho_{\mathrm{dyn}}(f_*)$, $e^{\mathrm{lov}(f)}$ and $e^{H(f)}$, where the three quantities can be viewed as the volume growth. It is known that $h_F(f) \le \mathrm{lov}(f) \le H(f)$. $H(f)$ is a birational invariant. I.e. let $\hat{Y}$ be a smooth projective variety such that there exists $\iota : Y \dashrightarrow \hat{Y}$ which is a *birational* map. Then $f : Y \dashrightarrow Y$ can be lifted to dominating

$\hat{f} := \iota f \iota^{-1} : \hat{Y} \dashrightarrow \hat{Y}$, and $H(f) = H(\hat{f})$. However $h_F(f)$ does not have to be equal to $h_F(\hat{f})$. The main conjecture of this paper are the equalities

$$h_B(\hat{f}) = h(\hat{f}) = h_F(\hat{f}) = \mathrm{lov}(\hat{f}) = H(\hat{f}) = \rho_{\mathrm{dyn}}(\hat{f}_*), \tag{1.1}$$

for some $\hat{f}$ birationally equivalent to f. For polynomial automorphisms of $\mathbb{C}^2$, which are birational maps of $\mathbb{P}^2$, the results of the papers [16, 36, 8] prove the above conjecture for $\hat{f} = f$. Some other examples where this conjecture holds are given in [21, 22].

The pioneering inequality of Gromov $h_F(f) \le \mathrm{lov}(f)$ [20] uses basic results in entropy theory, Riemannian geometry and complex manifolds. Author's results are using basic results in entropy theory, algebraic geometry and the results of Gromov, Yomdin [38] and Newhouse [31]. From the beginning of 90's the notion of *currents* were introduced in the study of the dynamics of holomorphic and rational maps in several complex variables. See the survey paper [35]. In fact the inequality $\mathrm{lov}(f) \le H(f)$ proved in [8, 9, 10] and [23], as well as most of the results in are derived [21, 22], are using the theory of currents.

The author believes that in dealing with the notion of the entropy solely, one can cleverly substitute the theory of currents with the right notions of algebraic geometry. All the section of this paper except the last one are not using currents. It seems to the author that to prove the conjecture (1.1) one needs to prove a correct analog of Yomdin's inequality [38].

We now survey briefly the contents of this paper. §2 deals with the entropy of $f : X \to X$, where first X is a compact metric space and f is continuous, and second X is compact Kähler and f is holomorphic. §3 is devoted to the study of three definitions of entropy of a continuous map $f : X \to X$, where X is an arbitrary subset of a compact metric space Y. In §4 we discuss rational dominating maps $f : Y \dashrightarrow Y$, where Y is a smooth projective variety. §5 discusses various notions and results on the entropy of rational dominating maps. In §6 we discuss briefly the recent results, in particular the inequality $\mathrm{lov}(f) \le H(f)$ which uses currents.

It is impossible to mention all the relevant existing literature, and I apologize to the authors whose papers were not mentioned. It is my pleasure to thank S. Cantat, V. Guedj, J. Propp, N. Sibony and C.-M. Viallet for pointing out related papers.

2 Entropy of continuous and holomorphic maps

The first rigorous definition of the entropy was introduced by Kolmogorov [27]. It assumes that X is a probability space $(X, \mathcal{B}, \mu)$, where f preserves the probability measure μ. It is denoted by $h_\mu(f)$, and is usually referred under the following names: *metric* entropy, *Kolmogorov-Sinai* entropy, or *measure entropy*. $h_\mu(f)$ is an invariant under measure preserving invertible automorphism $A : X \to X$, i.e. $h_\mu(f) = h_\mu(A \circ f \circ A^{-1})$.

Assume that X is a compact metric space and $f : X \to X$ a continuous map. Then Adler, Konheim and McAndrew defined the *topological* entropy $h(f)$ [1]. $h(f)$ has a *maximal characterization* in terms of measure entropies f. Let $\mathcal{B}$ be the Borel sigma algebra generated by open set in X. Denote by $\Pi(X)$ the compact space of probability measures on $(X, \mathcal{B})$. Let $\Pi(f) \subseteq \Pi(X)$ be the compact set of all f-invariant probability measures. (Krylov-Bogolyubov theorem implies that $\Pi(f) \neq \emptyset$.) Then the *variational principle* due to Goodwyn, Dinaburg and Goodman [18, 7, 17] states $h(f) = \max_{\mu\in\Pi(f)} h_\mu(f)$. Hence $h(f)$ depends only to the topology induced by the metric on X. In particular, $h(f)$ is invariant under any homeomorphism $A : X \to X$.

The next step is to consider the case where X is a compact smooth manifold and $f : X \to X$ is a differentiable map, i.e. $f \in \mathrm{C}^r(X)$, where r is usually at least 1. The most remarkable subclasses of f are strongly hyperbolic maps, and in particular axiom A diffeomorphisms [34]. The dynamics of an Axiom A diffeomorphism on the nonwandering set can be coded as a *subshift of a finite type* (SOFT), hence its entropy is given by the exponential growth of the periodic points of f, i.e. $h(f) = \limsup_{k\to\infty} \frac{\log \mathrm{Fix}\, f^k}{k}$, where Fix f^k the number of periodic points of f of period k.

It is well known in topology that Fix f^k can be estimated below by the Lefschetz number of f^k. Let $\mathrm{H}_*(X)$ denote the total homology group of X over $\mathbb{R}$, i.e. $\mathrm{H}_*(X) = \oplus_{i=0}^{\dim_{\mathbb{R}} X} \mathrm{H}_i(X)$, the direct sum of the homology groups of X of all dimensions with coefficients in $\mathbb{R}$. Then f induces the linear operator $f_* : \mathrm{H}(X) \to \mathrm{H}(X)$, where $f_{*,i} : \mathrm{H}_i(X) \to \mathrm{H}_i(X), i = 0, \ldots, \dim_{\mathbb{R}} X$. The *Lefschetz number* of f^k is defined as $\Lambda(f^k) := \sum_{i=0}^{\dim_{\mathbb{R}} X} (-1)^i \mathrm{Trace} f^k_{*,i}$. Intuitively, $\Lambda(f^k)$ is the algebraic sum of k-periodic points of f, counted with their multiplicities.

Denote by $\rho(f_*)$ and $\rho(f_{*,i})$ the spectral radius of f_* and $f_{*,i}$ respectively. Recall that $\rho(f_{*,i}) = \limsup_{k\to\infty} |\mathrm{Trace} f^k_{*,i}|^{\frac{1}{k}}$ and $\rho(f_*) = \max_{i=0,\ldots,\dim_{\mathbb{R}} X} \rho(f_{*,i})$. Hence

$$\limsup_{k\to\infty} \frac{\log |\Lambda(f^k)|}{k} \leq \log \rho(f_*).$$

The arguments in [34] yield that for any f in the subset H of an Axiom A diffeomorphism, (H is defined in [34]), one has the inequality $|\mathrm{Trace} f^k_{*,i}| \leq \mathrm{Fix}\, f^k$ for each $= 1, \ldots, \dim_{\mathbb{R}} X$. ($H$ is C^0 dense in $\mathrm{Diff}^r(X)$ [34, Thm 3.1].) Hence for any $f \in H$ one has the inequality [34, Prop 3.3]

$$h(f) \geq \log \rho(f_*). \tag{2.1}$$

It was conjectured in [34] that the above inequality holds for any differentiable f.

Let $\deg f$ be the *topological degree* of $f : X \to X$. Then $|\deg f| = \rho(f_{*,\dim_{\mathbb{R}} X})$. Hence $\rho(f_*) \geq |\deg f|$. It was shown by Misiurewicz and Przytycki [30] that if $f \in \mathrm{C}^1(X)$ then $h(f) \geq |\deg f|$. However this inequality may fail if $f \in \mathrm{C}^0(X)$. The entropy conjecture (2.1) for a smooth f, i.e. $f \in \mathrm{C}^\infty(X)$, was proved by Yomdin [38]. Conversely, Newhouse [31] showed that for $f \in \mathrm{C}^{1+\varepsilon}(X)$ the *volume growth* of smooth submanifolds of f is an upper bound for $h(f)$. See also a related upper bound in [32].

This paper is devoted to study the entropy of f where X is a complex Kähler manifold and f is either holomorphic map, or X is a projective variety and f is a rational map dominating map. We first discuss the case where f is holomorphic.

Let $\mathbb{P}$ be the complex projective space. Then $f : \mathbb{P} \to \mathbb{P}$ is holomorphic if and only if $f|\mathbb{C}$ is a rational map. Hence $\deg f$ is the cardinality of the set $f^{-1}(z)$ for all but a finite number of $z \in \mathbb{C}$. So $\deg f = \rho(f_*)$ in this case. Lyubich [28] showed that $h(f) = \log \deg f$. Gromov in preprint dated 1977, which appeared as [20], showed that if $f : \mathbb{P}^d \to \mathbb{P}^d$ is holomorphic then $h(f) = \log \deg f$. It is well known in this case $\rho(f_*) = \deg f$.

In [12] the author showed that if X is a complex projective variety and $f : X \to X$ is holomorphic, then $h(f) = \log \rho(f_*)$. Note that one can view f_* a linear operator on $H_*(X, \mathbb{Z})$, i.e. the total homology group with integer coefficients. Hence f_* can be represented by matrix with integer coefficients. In particular, $\rho(f_*)$ is an algebraic integer, i.e. the entropy is the logarithm of an algebraic integer. (This fact was observed in [3] for certain rational maps.) In [14] the author extended this result to a compact Kähler manifold.

Examples of the dynamics of biholomorphic maps $f : X \to X$, where X is a compact $K3$ surface which is Kähler but not necessary a projective variety, are given in [6, 29]. See also [9] for higher dimensional examples. In summary, the entropy of a holomorphic self map f of a compact Kähler manifold is determined by the spectral radius of the induced action of f on the total homology of X.

3 Definitions of entropy

In this paper Y will be always a compact matrix space with the metric $\mathrm{dist}(\cdot, \cdot) : Y \times Y \to \mathbb{R}_+$. Let $X \subseteq Y$ be a nonempty set, and assume that $f : X \to X$ is a continuous map with respect the topology induced by the metric dist on X. For $x, y \in X$ and $n \in \mathbb{N}$ let

$$\mathrm{dist}_n(x, y) = \max_{k=0,\ldots,n-1} \mathrm{dist}(f^k(x), f^k(y)).$$

So $\mathrm{dist}_1(x, y) = \mathrm{dist}(x, y)$ and the sequence $\mathrm{dist}_n(x, y), n \in \mathbb{N}$ is nondecreasing. Hence for each $n \in \mathbb{N}$ dist_n is a distance on X. Furthermore, each metric dist_n induces the same topology X as the metric dist. For $\varepsilon > 0$ a set $S \subseteq X$ is called (n, ε) separated if $\mathrm{dist}_n(x, y) \geq \varepsilon$ for any $x, y \in S, x \neq y$. For any set $K \subseteq X$ denote by $N(n, \varepsilon, K) \in \mathbb{N} \cup \{\infty\}$ the maximal cardinality of (n, ε) separated set $S \subseteq K$. Clearly, $N(n_1, \varepsilon, K) \geq N(n_2, \varepsilon, K)$ if $n_1 \geq n_2$, $N(n, \varepsilon_1, K) \geq N(n, \varepsilon_2, K)$ if $0 < \varepsilon_1 \leq \varepsilon_2$, and $N(n, \varepsilon, K_1) \geq N(n, \varepsilon, K_2)$ if $K_1 \supseteq K_2$.

We now discuss a few possible definitions of the entropy of f. Let $K \subseteq X$. Then

$$h(f, K) := \lim_{\varepsilon \searrow 0} \limsup_{n \to \infty} \frac{\log N(n, \varepsilon, K)}{n}. \tag{3.1}$$

We call $h(f, K)$ the *topological entropy* of $f|K$. (Note that $h(f, K) = \infty$ if $N(n, \varepsilon, K) = \infty$ for some $n \in \mathbb{N}$ and $\varepsilon > 0$.) Equivalently, $h(f, K)$ can be viewed as the *exponential growth* of the maximal number of (n, ε) separated sets (in n).

Clearly $h(f, K_1) \geq h(f, K_2)$ if $X \supseteq K_1 \supseteq K_2$. Then $h(f) := h(f, X)$ is the *topological entropy* of f.

Bowen's definition of the entropy of f, denoted here as $h_B(f)$, is given as follows [37, §7.2]. Let $K \subseteq X$ be a compact set. Then K is a compact set with respect to dist_n. Hence $N(n, \varepsilon, K) \in \mathbb{N}$. Then $h_B(f, X)$ is the supremum of $h(f, K)$ for all compact subsets K of X. I.e.

$$h_B(f, X) = \sup_{K \Subset X} h(f, K).$$

When no ambiguity arises we let $h_B(f) := h_B(f, X)$. Clearly, if X is compact then $h_B(f) = h(f)$. (It is known that for a compact X $h(f) \in [0, \infty]$, i.e. [37].)

Since $N(n, \varepsilon, K) \le N(n, \varepsilon, X)$ for any $K \subseteq X$ it follows that $h(f) \ge h_B(f)$. The following example, pointed out to me by Jim Propp, shows that it is possible that $h(f) > h_B(f)$. Let $Y := \{z \in \mathbb{C},\ |z| \le 1\}, X := \{z \in \mathbb{C},\ |z| < 1\}$ be the closed and the open unit disk respectively in the complex plane. Let $2 \le p \in \mathbb{N}$ and assume that $f(z) := z^p$. It is well known that $h(f, Y) = \log p$. It is straightforward to show that $h(f, X) = h(f, Y)$. Let $K \subset X$ be a compact set. Let $D(0, r)$ be the closed disk or radius $r < 1$, centered at 0, such that $K \subseteq D(0, r)$. Since $f(D(0, r)) \subseteq D(0, r)$ it follows that $h_B(f, X) \le h(f, D(0, r)) = 0$.

Our last definition of the entropy of h, denoted by $h_F(f, X)$, or simply $h_F(f)$ is based on the notion of the orbit space. Let $\mathcal{Y} := Y^{\mathbb{N}}$ be the space of the sequences $\mathbf{y} = (y_i)_{i \in \mathbb{N}}$, where each $y_i \in Y$. We introduce a metric on $\mathcal{Y}$:

$$\mathrm{d}(\{x_i\}, \{y_i\}) := \sum_{i=1}^{\infty} \frac{\mathrm{dist}(x_i, y_i)}{2^{i-1}}, \quad \{x_i\}_{i \in \mathbb{N}}, \{y_i\}_{i \in \mathbb{N}} \in \mathcal{Y}.$$

Then $\mathcal{Y}$ is a compact metric space, whose diameter is twice the diameter of Y. The *shift* transformation $\sigma : \mathcal{Y} \to \mathcal{Y}$ is given by $\sigma(\{y_i\}_{i \in \mathbb{N}}) = \{y_{i+1}\}_{i \in \mathbb{N}}$. Then $\mathrm{d}(\sigma(\mathbf{x}), \sigma(\mathbf{y})) \le 2\mathrm{d}(\mathbf{x}, \mathbf{y})$, i.e. σ is a Lipschitz map. Given $x \in X$ then the f-*orbit* of x, or simply the orbit of x, is the point $\mathrm{orb}\, x := \{f^{i-1}(x)\}_{i \in \mathbb{N}} \in \mathcal{Y}$. Denote by $\mathrm{orb}\, X \subseteq \mathcal{Y}$, the *orbit space*, the set of all f-orbits. Note that $\sigma(\mathrm{orb}\, x) = \mathrm{orb}\, f(x)$. Hence $\sigma(\mathrm{orb}\, X) \subseteq \mathrm{orb}\, X$. $\sigma|\mathrm{orb}\, X$, the restriction of σ to the orbit space, is "equivalent" to the map $f : X \to X$. I.e. let $\omega : X \to \mathrm{orb}\, X$ be given by $\omega(x) := \mathrm{orb}\, x$. Clearly ω is a homeomorphism. Then the following diagram is commutative:

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ \omega \downarrow & & \omega \downarrow \\ \mathcal{X} & \xrightarrow{\sigma} & \mathcal{X} \end{array}$$

Let $\mathcal{X}$ be the closure of $\mathrm{orb}\, X$ with respect to the metric d defined above. Since $\mathcal{Y}$ is compact, $\mathcal{X}$ is compact. Clearly $\sigma(\mathcal{X}) \subseteq \mathcal{X}$. Following [12, §4] we define $h_F(f, X)$ to be equal to the topological entropy of $\sigma|\mathcal{X}$:

$$h_F(f, X) := h(\sigma|\mathcal{X}) = h(\sigma, \mathcal{X}).$$

When no ambiguity arises we let $h_F(f) := h_F(f, X)$. Since the closure of $\mathrm{orb}\, X$ is $\mathcal{X}$, it is not difficult to show that $h_F(f) = h(\sigma, \mathrm{orb}\, X)$.

Observe first that if X is a compact subset of Y then $h_F(f)$ is the topological entropy $h(f)$ of f. Indeed, since f is continuous and X is compact $\mathcal{X} = \mathrm{orb}\, X$. Since ω is a homeomorphism, the variational principle implies that $h(f) = h_F(f)$.

We observe next that $h(f) \le h_F(f)$. Let

$$\mathrm{d}_n(\mathbf{x}, \mathbf{y}) := \max_{k=0,\ldots,n-1} \mathrm{d}(\sigma^k(\mathbf{x}), \sigma^k(\mathbf{y})).$$

Then $\mathrm{dist}_n(x, y) \le \mathrm{d}_n(\mathrm{orb}\, x, \mathrm{orb}\, y)$. Hence $N(n, \varepsilon, X) \le N(n, \varepsilon, \mathcal{X})$. Hence $h(f) \le h_F(f)$. The arguments of the proof [22, Lemma 1.1] show that $h(f) = h_F(f)$. (In [22] $h^{\mathrm{Bow}}_{\mathrm{top}}(f)$ is our $h(f)$, and $h^{\mathrm{Gr}}_{\mathrm{top}}(f)$ is the topological entropy with respect to the metric $\mathrm{d}'(\{x_i\}, \{y_i\}) := \sup_{i \in \mathbb{N}} \frac{\mathrm{dist}(x_i, y_i)}{2^i}$. Since d and d′ induce the Tychonoff topology on $Y^{\mathbb{N}}$ it follows that $h^{\mathrm{Gr}}_{\mathrm{top}}(f) = h_F(f)$.)

Our discussion of various topological entropies for $f : X \to X$ is very close to the discussion in [25]. The notion of the entropy $h_F(f)$ can be naturally extended to the definition of the entropy of a semigroup acting on X [15]. See [5] for other definition of the entropy of a free semigroup and [11] for an analog of Misiurewicz-Przytycki theorem [30].

4 Rational maps

In this section we use notions and results from algebraic geometry most of which can be found in [19]. Let $\mathbf{z} = (z_0, z_1, \ldots, z_n)$, sometimes denotes as $(z_0 : z_1 : \ldots : z_n)$, be the homogeneous coordinates the n-dimensional complex projective space $\mathbb{P}^n$. Recall that a map $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ is called a rational map if there exists $n+1$ nonzero coprime homogeneous polynomials $f_0(\mathbf{z}), \ldots, f_n(\mathbf{z})$ of degree $d \in \mathbb{N}$ such that $\mathbf{z} \mapsto f_h(\mathbf{z}) := (f_0(\mathbf{z}), \ldots, f_n(\mathbf{z}))$. Equivalently f lifts to a homogeneous map $f_h : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$. The set of singular points of f, denoted by $\mathrm{Sing}\, f \subset \mathbb{P}^n$, sometimes called the *indeterminacy locus* of f, is given by the system $f_0(\mathbf{z}) = \ldots = f_n(\mathbf{z}) = 0$. $\mathrm{Sing}\, f$ is closed subvariety of $\mathbb{P}^n$ of codimension 2 at least. f is holomorphic if and only if $\mathrm{Sing}\, f = \emptyset$, i.e. the above system of polynomial equations has only the solution $\mathbf{z} = \mathbf{0}$.

Let Y be an irreducible algebraic variety. It is well known that Y can be embedded as an irreducible subvariety of $\mathbb{P}^n$. For simplicity of notation we will assume that Y is an irreducible variety of $\mathbb{P}^n$. So Y can be viewed as a homogeneous irreducible variety $Y_h \subset \mathbb{C}^{n+1}$, given as the zero set of homogeneous polynomials $p_1(\mathbf{z}) = \ldots = p_m(\mathbf{z}) = 0$. $y \in Y$ is called *smooth* if Y is a complex compact manifold in the neighborhood of y. A nonsmooth $y \in Y$ is called a *singular* point. The set of singular points of Y, denoted by $\mathrm{Sing}\, Y$, is a strict subvariety of Y. Y is called *smooth* if $\mathrm{Sing}\, Y = \emptyset$. Otherwise Y is called *singular*.

Let $f : Y \dashrightarrow Y$ be a rational map. Then one can extend f to $\underline{f} : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ such that $\mathrm{Sing}\, \underline{f} \cap Y$ is a strict subvariety of Y and $\underline{f}|(Y \backslash \mathrm{Sing}\, \underline{f}) = f|(Y \backslash \mathrm{Sing}\, \underline{f})$. $\underline{f}$ is not unique, but the f can be viewed as $\underline{f}|Y$. $\mathrm{Sing}\, f \subset Y$ is the set of the points where f is not holomorphic. $\mathrm{Sing}\, f$ is strict projective variety of X, $(\mathrm{Sing}\, f \subseteq \mathrm{Sing}\, \underline{f} \cap Y)$, and each irreducible component of $\mathrm{Sing}\, f$ is at least of codimension 2. The assumption $f : Y \dashrightarrow Y$ means that $\mathbf{w} := f_h(\mathbf{z}) \in Y_h$ for each $\mathbf{z} \in Y_h$. It is known that $Y_1 := \mathrm{Cl}\, f_h(Y_h)$, the closure of $f_h(Y_h)$, is a homogeneous irreducible subvariety of Y. Furthermore either $Y_1 = Y (= Y_0)$, in this case f is called a *dominating* map, or $\dim Y_1 < \dim Y_0$. In the second case the dynamics of $f_0 := f$ is reduced to the dynamics of the rational map $f_1 : Y_1 \dashrightarrow Y_1$. Continuing in the same manner we deduce that there exists a finite number of strictly descending irreducible subvarieties $Y_0 := Y \supsetneqq \ldots \supsetneqq Y_k$ such that $f_k : Y_k \dashrightarrow Y_k$ is a rational dominating map. (Note that Y_k may be a singular variety.) Thus one needs only to study the dynamics of a rational dominating map $f : Y \dashrightarrow Y$, where Y may be a singular variety.

The next notion is the resolution of singularities of Y and f. An irreducible projective variety Z birationally equivalent to Y if the exists a birational map $\iota : Z \dashrightarrow Y$. Z is called a *blow up* of Y if there exists a birational map $\pi : Z \to Y$ such π is holomorphic. Y is called a blow down of Z. Hironaka's result claims that any irreducible singular variety Y has a smooth blow up Z. Let $f : Y \dashrightarrow Y$ be a rational dominating map. Let Y be a birationally equivalent to Z. Then f lifts to a rationally dominating map $\hat{f} : Z \dashrightarrow Z$. Hence to study the dynamics of f one can assume that $f : Y \dashrightarrow Y$ is rational dominating map and Y is smooth. Hironaka's theorem implies that there exists a smooth blow up Z of Y such that f lifts to a holomorphic map $\tilde{f} : Z \to Y$. Then one has the induced dual linear maps on the homologies and the cohomologies of Y and Z:

$$\tilde{f}_* : \mathrm{H}_*(Z) \to \mathrm{H}_*(Y), \quad \tilde{f}^* : \mathrm{H}^*(Y) \to \mathrm{H}^*(Z).$$

We will view the homologies $\mathrm{H}_*(Y), \mathrm{H}_*(Z)$ as homologies with coefficients in $\mathbb{R}$, and hence the cohomologies $\mathrm{H}^*(Y), \mathrm{H}^*(Z)$, which are dual to $\mathrm{H}_*(Y), \mathrm{H}_*(Z)$, as de Rham cohomologies of differential forms. (It is possible to consider these homologies and cohomologies with coefficients in $\mathbb{Z}$ [12].) Recall that the Poincaré duality isomorphism $\eta_Y : \mathrm{H}_*(Y) \to \mathrm{H}^*(Y)$, which maps a k-cycle to closed $\dim Y - k$ form. ($\eta_Y^* = \eta_Y$.) Then one defines $f^* : \mathrm{H}^*(Y) \to \mathrm{H}^*(Y)$ and its dual $f_* : \mathrm{H}_*(Y) \to \mathrm{H}_*(Y)$ as

$$f^* := \eta_Y \pi_* \eta_Z^{-1} \tilde{f}^*, \quad f_* := \tilde{f}_* \eta_Z^{-1} \pi^* \eta_Y.$$

It can be shown that f_*, f^* do not depend on the resolution of f, i.e. on Z. Let $\rho(f_*) = \rho(f^*)$ be the spectral radii of f_*, f^* respectively. (As noted above f_*, f^* can be represented by matrix with integer entries. Hence $\rho(f_*)$ is an algebraic integer.) Then the *dynamical* spectral radius of f_* is defined as

$$\rho_{\mathrm{dyn}}(f_*) = \limsup_{m\to\infty} (\rho((f^m)_*))^{\frac{1}{m}}. \tag{4.1}$$

(Note that $\rho_{\mathrm{dyn}}(f_*)$ is a limit of algebraic integers, so it may not be an algebraic integer.)

Assume that $f : Y \to Y$ is holomorphic. Then f_*, f_* are the standard linear maps on homology and cohomology of Y. So $(f^m)_* = (f_*)^m, (f^m)^* = (f^*)^m$ and $\rho_{\mathrm{dyn}}(f_*) = \rho(f_*)$. It was shown by the author that $\log \rho(f_*) = h(f)$ [12]. This equality followed from the observation that $h(f)$ is the *volume growth* induced by f. View Y as a submanifold of $\mathbb{P}^n$, is endowed the induced Fubini-Study Riemannian metric and with the induced Kähler $(1, 1)$ closed form κ. Let $V \subseteq Y$ be any irreducible variety of complex dimension $\dim V \geq 1$. Then the volume of V is given by the Wirtinger formula $\mathrm{vol}(V) = \frac{1}{(\dim V)!} \int_V \kappa^{\dim V} (= \kappa^{\dim V}(V))$. Let $L_k \subset \mathbb{P}^n$ be a linear space of codimension k. ($L_0 := \mathbb{P}^n$.) Assume that L_k is in general position. Then $L_k \cap V$ is a variety of dimension $\dim V - k$. For $k < \dim V$ the variety $L_k \cap V$ is irreducible. For $k = \dim V$ the variety $L_k \cap V$ consists of a fixed number of points, independent of a generic L_k, which is called the degree of V, and denoted by $\deg V$. It is well known that $\deg V = \mathrm{vol}(V)$. The homology class of $L_k \cap V$, denoted by $[L_k \cap V]$, is independent of L_k. Since $\mathrm{vol}(L_k \cap V)$ can be expressed in terms of the cup product $\langle [L_k \cap V], [\kappa^{\dim V - k}] \rangle$, or equivalently as $\deg L_k \cap V$, this volume is an *integer*, which is independent of the choice of a generic L_k. Thus the j-th volume growth, of the subvariety $L_{\dim Y - j} \cap Y$ of dimension j, induced by f is given by

$$\begin{aligned} \beta_j &:= \limsup_{m\to\infty} \frac{\log \langle (f^m)_* [L_{\dim Y - j} \cap Y], [\kappa^j] \rangle}{m}, \quad j = 1, \ldots, \dim Y, \\ H(f) &:= \max_{j=1,\ldots,\dim Y} \beta_j. \end{aligned} \tag{4.2}$$

(See [12, (2)] and [13, (2.8)].) From the well known equality $\rho(f_*) = \lim_{m\to\infty} ||f_*^m||^{\frac{1}{m}}$, for any norm $|| \cdot ||$ on $\mathrm{H}_*(Y)$, it follows that $H(f) \leq \rho(f_*)$. Newhouse's result [31] claims that $h(f) \leq H(f)$. Combining this inequality with Yomdin's inequality [38] $h(f) \geq \log \rho(f_*)$ we deduced in [12]:

$$H(f) = \log \rho(f) = h(f), \tag{4.3}$$

which is a logarithm of an algebraic integer.

Let $\mathrm{K} \subset \mathrm{H}_*(Y)$ be the cone generated by the homology classes $[V]$ corresponding to all irreducible projective varieties $V \subseteq Y$. Note that $f_*(\mathrm{K}) \subseteq \mathrm{K}$. Let $\mathrm{H}_{*,a}(Y) := \mathrm{K} - \mathrm{K} \subset \mathrm{H}_*(Y)$ be the subspace generated by the homology classes of projective varieties in Y. Then $f_* : \mathrm{H}_{*,a}(Y) \to \mathrm{H}_{*,a}(Y)$ and denote $f_{*,a} := f_*|\mathrm{H}_{*,a}(Y)$. Using the theory of nonnegative operators on finite dimensional cones K, e.g. [4], it follows that $H(f) = \log \rho(f_{*,a})$.

Assume again that $f : Y \dashrightarrow Y$ is rational dominant. Then $f_*(K) \subseteq K$ so $f_* : \mathrm{H}_{*,a}(Y) \to \mathrm{H}_{*,a}(Y)$ and denote $f_{*,a} := f_*|\mathrm{H}_{*,a} * (Y)$. Hence we can define $H(f)$, the volume growth induced by f, as in (4.2) [12, 13]. Similar quantities were considered in [33, 3]. It is plausible to assume that $H(f) = \log \rho_{\mathrm{dyn}}(f_*)$ and we conjecture a more general set of equalities in the next section.

It was shown in [12] that the results on of Friedland-Milnor [16] imply the inequalities

$$(f^m)_{*,a} \leq (f_{*,a})^m \text{ for all } m \in \mathbb{N}, \tag{4.4}$$

for certain polynomial biholomorphisms of $\mathbb{C}^2$, (which are birational maps of $\mathbb{P}^2$.)

It was claimed in [12, pp. 367] that if (4.4) holds then the sequence $(\rho((f^m)_{*,a}))^{\frac{1}{m}}$, $m \in \mathbb{N}$ converges. (This is probably wrong. One can show that under the assumption (4.4) for all rational dominant maps $f : Y \dashrightarrow Y$ one has $\rho((f^q)_{*,a})^p \geq \rho((f^{pq})_{*,a})$ for any $p, q \in \mathbb{N}$.) It was also claimed in [12, Lemma 3] that (4.4) holds in general. Unfortunately this result is false, and a counterexample is given in [23, Remark 1.4]. Note that if $f : Y \to Y$ holomorphic then equality in (4.4) holds. Hence all the results of [12] hold for holomorphic maps.

5 Entropy of rational maps

Let $f : Y \dashrightarrow Y$ be a rational dominating map. (We will assume that f is not holomorphic unless stated otherwise.) In order to define the entropy of f we need to find the largest subset $X \subseteq Y \backslash \mathrm{Sing}\, f$ such that $f : X \to X$. Let X_k is the collection of all $x \in Y$ such that $f^j(x) \in Y \backslash \mathrm{Sing}\, f$ for $j = 0, 1, \ldots, k$. Then X_k is open and $Z_k := Y \backslash Y_k$ is a strict subvariety of Y. Clearly $X_k \supseteq X_{k+1}, Y_k \subseteq Y_{k+1}$ for $k \in \mathbb{N}$. Then $X := \cap_{k=1}^{\infty} X_k$ is G_δ set. Let κ the the closed $(1,1)$-Kähler form on Y. Then $\kappa^{\dim Y}$ is a canonical volume form on Y. Hence $\kappa^{\dim Y}(X) = \kappa^{\dim Y}(Y)$, i.e. X has the full volume.

Since Y is a compact Riemannian manifold, Y is a compact metric space. Thus we can define the three entropies $h(f), h_B(f), h_F(f)$ in §3. So

$$h_B(f) \leq h(f) = h_F(f).$$

Assume that $f : \mathbb{C}^n \to \mathbb{C}^n$ is a polynomial dominating map. Then f lifts to a rational dominating map $f : \mathbb{P}^n \dashrightarrow P^n$, which may be holomorphic. Hence $X \supseteq \mathbb{C}^n$. Assume that f is a proper map of $\mathbb{C}^n$. Recall that one point compactification of $\mathbb{C}^n$, denoted by $\mathbb{C}^n \cup \{\infty\}$, is homeomorphic to the $2n$ sphere S^{2n}. Then f lifts to a continuous map $f_s : \mathrm{S}^{2n} \to \mathrm{S}^{2n}$. Thus we can define the entropy $h(f_s)$. It is not hard to show that $h(f_s) \le h_F(f)$.

Let orb $X \subset Y^{\mathbb{N}}$ be the orbit space of f, and let $\mathcal{X}$ be its closure. $\mathcal{X}$ is closely related to the graph construction discussed in [20, 12, 13, 14, 15] as well as in other papers. Denote by $\Gamma(f) \subset Y^2$ the closure of the set $\{(x, f(x)),\ x \in Y\backslash \mathrm{Sing}\, X\}$ in Y^2. Then $\Gamma(f)$ is an irreducible variety of dimension $\dim Y$ in Y^2. Note that the projection of Γ on the first or second factor of Y in Y^2 is Y. Without a loss of generality we may assume that $\Gamma(f)$ is smooth.

Otherwise let $\pi : Z \to Y$ be a blow up of Y such that $f : Y \dashrightarrow Y$ lifts to a holomorphic map $\tilde{f} : Z \to Y$. Let $\Gamma_1(f) := \{(z, \tilde{f}(z)) :\ z \in Z\} \subset Z \times Y$. Then $\Gamma_1(f)$ is smooth variety of dimension $\dim Y$. Note that $\hat{\pi} : Z^2 \to Z \times Y$ given by $(z, w) \mapsto (z, \pi(w))$ is a blow up of $Z \times Y$. Lift $\tilde{f}$ to $\hat{f} : Z \dashrightarrow Z$. Then $\Gamma(\hat{f}) \subset Z^2$ is a blow up $\Gamma_1(f)$, hence $\Gamma(\hat{f})$ is smooth.

Let $\Gamma \subset Y^2$ be a closed irreducible smooth variety of dimension $\dim Y$ such that the projection of Γ on the first or second component is Y. Define

$$
\begin{aligned}
&Y^k(\Gamma) := \{(x_1, \ldots, x_k) \in Y^k,\ (x_i, x_{i+1}) \in \Gamma \text{ for } i = 1, \ldots, k-1\}, k = 2, \ldots, \\
&Y^{\mathbb{N}}(\Gamma) := \{(x_1, \ldots, x_k, \ldots) \in Y^{\mathbb{N}},\ (x_i, x_{i+1}) \in \Gamma \text{ for } i \in \mathbb{N}\}.
\end{aligned}
$$

Note that $Y^k(\Gamma)$ is an irreducible variety of dimension $\dim Y$ in Y^k for $k = 2, \ldots$. Note that $Y^{\mathbb{N}}(\Gamma)$ is a σ invariant compact subset of $Y^{\mathbb{N}}$, i.e. $\sigma(Y^{\mathbb{N}}(\Gamma)) \subseteq Y^{\mathbb{N}}(\Gamma)$. Let $h(\Gamma) = h(\sigma|Y^{\mathbb{N}}(\Gamma))$. Y, viewed as a submanifold of $\mathbb{P}^n$, is endowed the induced Fubini-Study Riemannian metric and with the Kähler $(1,1)$ form κ. Then Y^k has the corresponding induced product Riemannian metric, and Y^k is Kähler, with the $(1,1)$ form κ_k . Let $\mathrm{vol}(Y^k(\Gamma)) = \kappa_k^{\dim Y}(Y^k(\Gamma))$ be volume of the variety $Y^k(\Gamma)$. Then the volume growth of Γ is given by

$$
\mathrm{lov}(\Gamma) := \limsup_{k \to \infty} \frac{\log \mathrm{vol}(Y^k(\Gamma))}{k}. \tag{5.1}
$$

The fundamental inequality due to Gromov [20]

$$
h(\Gamma) \le \mathrm{lov}(\Gamma). \tag{5.2}
$$

Since the paper of Gromov was not available to the general public until the appearance of [20], the author reproduced Gromov's proof of (5.2) in [13, 14]. Using the above inequality Gromov showed that $h(f) \le \log \deg f$ for any holomorphic $f : \mathbb{P}^n \to \mathbb{P}^n$.

Let $f : Y \dashrightarrow Y$ be rational dominating. Then $\mathcal{X} = Y^{\mathbb{N}}(\Gamma(f))$. Hence

$$
h_F(f) = h(\Gamma(f)). \tag{5.3}
$$

If $\Gamma(f)$ is smooth then Gromov's inequality yields that

$$
h_F(f) \le \mathrm{lov}(f) := \mathrm{lov}(\Gamma(f)). \tag{5.4}
$$

Conjecture 5.1 *Let Y be a smooth projective variety and $f : Y \dashrightarrow Y$ be a rational dominating map. Then there exists a smooth projective variety $\hat{Y}$ and a birational map $\iota : Y \dashrightarrow \hat{Y}$, such that the lifting $\hat{f} : \hat{Y} \dashrightarrow \hat{Y}$ satisfies (1.1).*

We now review briefly certain notions, results and conjectures in [14, S3]. Let $\Gamma \subset Y^2$ be as above, and denote by $\pi_i(\Gamma) \to Y$ the projection of Γ on the i-th component of Y in $Y \times Y$ for $i = 1, 2$. Since $\dim \Gamma = \dim Y$ and $\pi_1(\Gamma) = \pi_2(\Gamma) = Y$, then $\deg \pi_i$ is finite and $\pi_i^{-1}(y)$ consists of exactly $\deg \pi_i$ distinct points for a generic $y \in Y$ for $i = 1, 2$. One can define a linear map $\Gamma_* : \mathrm{H}_*(Y) \to \mathrm{H}_*(Y)$ given by $\Gamma_* : \pi_1^* \eta_\Gamma^{-1} \pi_2^* \eta_Y$. (This is an analogous definition of f_*, where $f : Y \dashrightarrow Y$ is dominating.) One can show that $\Gamma_*(\mathrm{H}_{*,a}(Y)) \subseteq \mathrm{H}_{*,a}(Y)$. Let $\Gamma_{*,a} := \Gamma_* | \mathrm{H}_{*,a}(Y)$.

$\Gamma \subset Y^2$ is called a *proper* if each π_i is finite to one. Assume that Γ is proper. Then

$$\log \rho(\Gamma_{*,a}) \geq \mathrm{lov}(\Gamma). \tag{5.5}$$

It is conjectured that for a proper Γ

$$\log \rho(\Gamma_{*,a}) = \mathrm{lov}(\Gamma) = h(\Gamma). \tag{5.6}$$

Note that if $f : Y \to Y$ is dominating and holomorphic then $\Gamma(f)$ is proper, $\Gamma_{*,a} = f_* | \mathrm{H}_{*,a}(Y)$ and the above conjecture holds.

We close this section with observations and remarks which are not in [14]. Assume that $f : Y \dashrightarrow Y$ be a rational dominating and $Z := \Gamma(f) \subset Y^2$ smooth. Then $\pi_1 : \Gamma(f) \to Y$ is a blow up of Y, and $\pi_2 : \Gamma \to Y$ can be identified with $\tilde{f} : Z \to Y$. It is straightforward to show that $f_* = \Gamma(f)_*$.

It seems to the author that the arguments given in [14, Proof Thm 3.5] imply that (5.5) holds for any smooth variety $\Gamma \subset Y^2$ of dimension $\dim Y$ such that $\pi_1(Y) = \pi_2(Y) = Y$. Suppose that this result is true. Let $f : Y \dashrightarrow Y$ be rational and dominating. Assume that $\Gamma(f) \subset Y^2$ is smooth. Then (5.5) would imply that $\mathrm{lov}(f) \leq \log \rho(f_*)$. Applying the same inequality to $(f^k)_*$ and combining it with (5.4) one would able to deduce:

$$h_F(f) \leq \mathrm{lov}(f) \leq \log \rho_{\mathrm{dyn}}(f_*). \tag{5.7}$$

6 Currents

Many recent advances in complex dynamics in several complex variables were achieved using the notion of a *current*. See for example the survey article [35]. Recall that on an m-dimensional manifold M a current of degree $m - p \geq 0$ is a linear functional on all smooth p-differential forms $\mathcal{D}^p(M)$ with a compact support, where p is a nonnegative integer.

Let $f : Y \dashrightarrow Y$ be a meromorphic dominating self map of a compact Kähler manifold of complex dimension $\dim Y$, with the $(1,1)$ Kähler form κ. Let $f^*\kappa$ be a pullback of κ. Then $f^*\kappa$ is a current on $Y \backslash \mathrm{Sing}\, f$. Define the *p-dynamic degree* of f by

$$\lambda_p(f) := \limsup_{k \to \infty} \Big(\int_{Y \backslash \mathrm{Sing}\, f^k} (f^k)^* \kappa^p \wedge \kappa^{\dim Y - p} \Big)^{\frac{1}{k}}, \quad p = 1, \ldots, \dim Y.$$

It is shown in [8] that the dynamical degrees are invariant with respect to a bimeromorphic map $\iota : Y \dashrightarrow Z$, where Z is a compact Kähler manifold. (See also [23] for the case where Y, Z are projective varieties.) Moreover

$$\mathrm{lov}(f) \leq \max_{p=1,\ldots,\dim Y} \log \lambda_p(f). \tag{6.1}$$

Assume that Y is a projective variety. It can be shown that the dynamic degree $\lambda_p(f)$ is equal to $e^{\beta_{\dim Y - p}}$ for $p = 1, \ldots, \dim Y$, which are defined in (4.2), where $\beta_0 := \beta_{\dim Y}$. Hence

$$H(f) = \max_{p=1,\ldots,\dim Y} \log \lambda_p(f), \tag{6.2}$$

where $H(f)$ is defined in (4.2). Thus $H(f)$ can be viewed as the *algebraic entropy* of f [3]. [24, Lemma 4.3] computes $H(f)$ for a large class of automorphisms of $\mathbb{C}^k$, and see also [10, 23]. Combine (5.4) with (6.1) and (6.2) to deduce the inequality $h_F(f) \le H(f)$, which was conjectured in [13, Conjecture 2.9].

Consider the following example $f : \mathbb{C}^2 \to \mathbb{C}^2, (z, w) \mapsto (z^2, w + 1)$ [22, Example 1.4]. Since f is proper we have $f_s : S^4 \to S^4$. Clearly S^4 is the domain of attraction of the fixed point $f_s(\infty) = \infty$. Hence $h(f_s) = 0$. Lift f to $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$. Then f has a singular point $\mathbf{a} := (0, 1, 0)$ and any other point at the line at infinity $(1, w, 0)$ is mapped to a fixed point $\mathbf{b} := (1, 0, 0)$. So $X = \mathbb{P}^2 \backslash \{a\}$, and $\Gamma(f) = \{(\mathbf{z}, f(\mathbf{z})) : \mathbf{z} \in \mathbb{P}^2 \backslash \{a\}\} \cup \{(\mathbf{a}, (z : w : 0)) : (z : w) \in \mathbb{P}\}$, which is equal to the blow up of $\mathbb{P}^2$ at $\mathbf{a}$. On $(\mathbb{P}^2)^{\mathbb{N}}(\Gamma(f))$ σ has two fixed points: $(\mathbf{a}, \mathbf{a}, \ldots), (\mathbf{b}, \mathbf{b}, \ldots)$. The set $\mathcal{X}_0 := ((\mathbf{x}, f(\mathbf{x}), \ldots,) : \mathbf{x} \in A_0 := \{(z, w, 1), |z| \le 1\}\}$ is in the domain of the attraction of $(\mathbf{a}, \mathbf{a}, \ldots)$. The set $(\mathbb{P}^2)^{\mathbb{N}}(\Gamma(f)) \backslash (\mathcal{X}_0 \cup \{(\mathbf{a}, \mathbf{a}, \ldots)\}$ is in the domain of the attraction of $(\mathbf{b}, \mathbf{b}, \ldots)$. Hence $h(f) = 0$. Observe that $\hat{f} : (\mathbb{P} \times \mathbb{P}) \to (\mathbb{P} \times \mathbb{P})$, given as $((z : s), (w : t)) \mapsto ((z^2 : s^2), (w + t : t))$, is the lift of f to $(\mathbb{P} \times \mathbb{P})$. $\hat{f}$ is holomorphic and $h(\hat{f}) = H(\hat{f}) = \log 2$. Since $\mathbb{P} \times \mathbb{P}$ is birational to $\mathbb{P}^2$ it follows that $H(f) = \log 2 > h_F(f) = 0$. In particular $h_F(f)$ is not a birational invariant [22]. Note that Conjecture 5.1 is valid for this example. Additional examples in [21, 22] support the Conjecture 5.1.

Assume now that $f : \mathbb{C}^2 \to \mathbb{C}^2$ is a polynomial automorphism, hence f is proper. It is shown in [16] that $h(f_s) = h(f, K)$ for some compact subset of $\mathbb{C}^2$. Furthermore the results of [16] and [36] imply that $h(f, K) = H(f)$. One easily deduce that $H(f) = \rho_{\mathrm{dyn}}(f_*)$. Clearly $h_B(f) \ge h(f, K)$. Then the inequalities $h_F(f) \le \mathrm{lov}(f) \le H(f)$ yield Conjecture 5.1. See [2, 9, 26] for additional results on entropy of certain rational maps.

The inequality (6.1) and its suggested variant (5.7) can be viewed as Newhouse type upper bounds [31] which shows that the volume growth bounds from above the entropy of a rational dominating map. In order to prove Conjecture 5.1 one needs to prove a suitabe Yomdin type lower bound for the entropy of f.

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