Problem 1. (30pts) Let $G$ be a regular 6-gon: hexagon. Assume that $D_6$ is the dihedral group of rigid motions that acts on the vertices of the hexagon.

a. (15pts.) Find the cycle index polynomial of $D_6$.

b. (5pts.) Find the number of inequivalent coloring of the hexagon in three colors, (red, blue and green), under the action of $D_6$.

c. (10pts.) Assume that the 6 vertices of the hexagon are colored in 2-colors of red, blue and green. What is the number of inequivalent colors under the action of $D_6$?

Problem 2. (30pts) Let $X = \{x_1, \ldots, x_v\}$ be a set of $v$ distinct elements. Balanced Incomplete Block Design on $X$ are $b$ strict subset of $X$, $B_1, \ldots, B_b$ having each $k$ elements, where $2 \leq k < v$. Each pair $\{x_i, x_j\}$, where $1 \leq i < j \leq k$, appears exactly in $\lambda \geq 1$ blocks.

a. (10pts) Show that each element $x_i$ appears in $r = \frac{\lambda(v-1)}{k-1}$ blocks.

b. (5pts) Show that $b_k = vr$.

c. (5pts) Does there exist a BIBD with parameters $b = 10, v = 8, r = 5, k = 4$?

d. (10 pts) Show that $B = \{0, 1, 3\}$ is a difference set in $\mathbb{Z}_7$. Construct the corresponding BIBD at identify $v, k, \lambda, b, r$.

Problem 3. (25pts) Recall that $n$-th Catalan number $C_n, n = 0, 1, \ldots$ is given by the formula $C_n = \frac{1}{n+1} \binom{2n}{n}$.

a. (10pts) Show that $\frac{1}{\sqrt{1 - 4x}}$ is generating function for $C_0, C_1, \ldots$.

b. (10pts) A person comes with one dollar bill in his wallet to casino and plays the following game. Each time he bets one dollar. If he wins he gets two dollars back, (his dollar that he bet and an extra dollar for the win.) Otherwise the he loses a dollar. He must leave the casino when he has 0 dollars, otherwise he leaves when he wants to. The person played 10 times and he left with 1 dollar. How many possible ways this can happen? (What is the number all possible sequences of wins and losses this can happen?) Explain briefly.

c. (5pts) What is the probability of the event that happened in (b). Assume that the person has probability $\frac{1}{2}$ to win each time. (Even if you did not find the exact answer in (b), name the answer $a$ and find the probability.)

Problem 4. (25pts)

a. (15pts) Find the general solution of the recurrence equation $h_n = 6h_{n-1} - 9h_{n-2}$.

b. (5pts) Find a particular solution satisfying $h_0 = 1, h_1 = 0$.

c. (5pts) Find a solution to $h_n = 6h_{n-1} - 9h_{n-2} + 1$.

Problem 5. (25 pts.)

a. (5pts) Given a sequence $h_0, h_1, \ldots$ what is the definition of the exponential generating function?

b. (15pts) Determine the number $h_n$ of $n$-digit numbers consisting 2, 4, 6, 8 such that 2 and 8 occur an even number of times. (You may use the exponential generating function.)

c. (5pts) The function $\frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)} = \sum_{n=0}^{\infty} h_n x^n$ is a generating function of certain partitions of $n$. Which kinds of partitions?

Problem 6. (25 pts.)
a. (5pts) How many nonnegative integer solution are to the equation \(x_1 + \ldots + x_r = s\), where \(s\) is a given nonnegative integer. Justify briefly your formula.

b. (10pts) Find the number of nonnegative integer solutions to the equation \(x_1 + x_2 + x_3 + x_4 = 20\), where \(x_1 \geq 4, x_2 \leq 5, x_3 \leq 7\).

c. (10pts) There are 20 new graduate students in Math. Department that need to take exactly one of the five offered special topics graduate courses \(\{1, 2, 3, 4, 5\}\). How many are there choices so that there would be at least one student enrolled in each of the topics courses. (Do not attempt to find the exact number, a solution with several terms is fine.)

**Problem 7.** (40 pts.)
a. (5pts) Arrange the following 5 number in an increasing order \(\binom{50}{10}, \binom{50}{25}, \binom{50}{30}, \binom{50}{35}, \binom{50}{45}\).

b. (10pts.) Is it true that \(\binom{n}{k} = \binom{n-1}{n-k} + \binom{n-1}{n-k-1}\)? If yes, give a short proof. If no give a counterexample.

c. (10pts) What is the coefficient of \(x^3y^4z^2w\) in the expansion of \((x - y + 2z - 3w)^{10}\).

d. (5pts) There are 5 people at the party, such that among any 3 persons at least some two persons know each other and at least some two persons do not know each other. (In other words among any 3 not all are strangers and not all know each other.) An extra person arrives. Is it always true that there are now 3 people that know each other? If not what exactly is true?

e. (10pts) A child watches TV a whole number of hours, at least one hour, each day for seven weeks. He does not watch more than 11 hours in any week. Prove that there exists some period of consecutive days that the child watches exactly 20 hours of TV. (Hint: Consider the sequence \(a_i\), which is the number of hours that child watched TV from day 1 to the end of day \(i\).)