MCS 421 -Combinatorics, **Final Test**,
May 6, 2013, Instructor: S. Friedland
Name and e-mail address:

No books or notes. Show all your work. Write solutions in the exam booklet without copying the problems. You can use a result (x) of any part of the problem, to show other part of any problem. **Unjustified** answer yields no credit.

Problem 1. (40 pts.)
a. (5pts) Arrange the following 5 number in an increasing order $\binom{50}{10}, \binom{50}{25}, \binom{50}{26}, \binom{50}{30}, \binom{50}{45}$.
b. (10pts.) Is it true that $\binom{n}{k} = \binom{n-1}{n-k} + \binom{n-1}{n-k-1}$? If yes, give a short proof. If no give a counterexample.
c. (10pts) What is the coefficient of $x^3y^4z^2w$ in the expansion of $(x - y + 2z - 3w)^{10}$.
d. (5pts) There are 5 people at the party, such that among any 3 persons at least some two persons know each other and at least some two persons do not know each other. (In other words among any 3 not all are strangers and not all know each other.) An extra person arrives. Is it always true that there are now 3 people that know each other? If not what exactly is true?
e. (10pts) A child watches TV a whole number of hours, at least one hour, each day for seven weeks. He does not watch more then 11 hours in any week. Prove that there exists some period of consecutive days that the child watches exactly 20 hours of TV. (**Hint**: Consider the sequence a_i which is the number of hours that child watched TV from day 1 to the end of day i.)

Problem 2. (25 pts.)
a. (5pts) How many nonnegative integer solution are to the equation $x_1 + \ldots + x_r = s$, where s is a given nonnegative integer. Justify briefly your formula.
b. (10pts) Find the number of nonnegative integer solutions to the equation $x_1 + x_2 + x_3 + x_4 = 20$, where $x_1 \geq 4, x_2 \leq 5, x_3 \leq 7$.
c. (10pts) There are 20 new graduate students in Math. Department that need to take exactly one of the five offered special topics graduate courses $\{1, 2, 3, 4, 5\}$. How many are there choices so that there would be at least one student enrolled in each of the topics courses. (**Hint**: Use inclusion-exclusion and do not attempt to find the exact number, a solution with several terms is fine.)

Problem 3. (20pts) Recall that $n - th$ Catalan number $C_n, n = 0, 1, \ldots$ is given by the formula $C_n = \frac{1}{n+1}\binom{2n}{n}$.
a. (10pts) Show that $\frac{1-\sqrt{1-4x}}{2x}$ is generating function for $C_0, C_1, \ldots$.
b. (10pts) A big city lawyer works 10 blocks north and 10 blocks east of her place of residence. Every day she walks 20 blocks to work. How many routes are possible if she never crosses (but may touch) the diagonal line from home to office?

Problem 4. (30pts) Recall that a *balanced block design* consists of v elements, b blocks, each block has k elements, each variety in r blocks, λ pairs of each variety.

a. (10pts) Define a Steiner triple system and explain briefly why the relations $r = \frac{\lambda(v-1)}{2}, b = \frac{\lambda v(v-1)}{6}$ hold.
b. (20) Suppose you want to construct a Steiner triple system from $v \geq 4$ elements using a difference set in $\mathbb{Z}_v$. Find all possible values of v and for each possible value of v one corresponding Steiner triple arising this way.

Problem 5 (25 points)

a. (5 points) What is a Latin square on n elements $\mathbb{Z}_n := \{0, 1, \ldots, n-1\}$?
b. (5 points) Give an example of Latin square for any integer $n \geq 1$.
b. (5 points) What is a pair of orthogonal Latin squares?
c. (10 points) Show that if $n \geq 3$ is odd then there exists a pair of orthogonal Latin squares.

Problem 6. (30pts) Let G be a regular 6-gon: hexagon. Assume that D_6 is the dihedral group of rigid motions that acts on the vertices of the hexagon.
a. (15pts.) Find the cycle index polynomial of D_6.
b. (5pts.) Find the number of inequivalent coloring of the hexagon in three colors, (red, blue and green), under the action of D_6
c. (10pts.) Assume that the 6 vertices of the hexagon are colored in 2-colors of red, blue and green. What is the number of inequivalent colors under the action of D_6?