Hausdorff dimension, strong hyperbolicity and complex dynamics

S. Friedland and G. Ochs

University of Illinois at Chicago and University of Bremen

September 17, 1997

§0. Introduction

Let X be a compact metric space and assume that $f : X \to X$ is a continuous map. Denote by Ω the nonwandering set of f. An interesting and a nontrivial invariant of f is $HD(\Omega)$-the Hausdorff dimension of Ω. It is usually a highly nontrivial problem to find $HD(\Omega)$. The seminal work of Bowen [**Bow2**] gives $HD(\Omega)$ as the solution to $P(t\phi) = 0$ for some special expanding maps. Here $P(g)$ denotes the topological pressure. See also [**Rue2**] and the recent works [**Bar**] and [**Fri2**]. Denote by $\mathcal{E}$ the set of all f-invariant ergodic probability Borel measures on M. Let $HD(\mu), \mu \in \mathcal{E}$ be the Hausdorff dimension of μ

$$HD(\mu) = \inf_{Y, \mu(Y)=1} HD(Y).$$

It is known that $HD(\mu)$ is easy to compute in many general cases. See for example [**Man**], [**You**], [**L-Y**] and [**Fri**, 1-2]. In the above references $HD(\mu)$ is given in terms of entropy of f (along a foliation) and the Lyapunov exponents. As the support of μ lies in Ω it follows that $HD(\Omega) \geq HD(\mu)$. Hence

$$HD(\Omega) \geq \sup_{\mu \in \mathcal{E}} HD(\mu), . \tag{0.1}$$

In fact in the examples studied in [**Bow2**] and [**Rue2**] one has the equality in (0.1). In these cases $HD(\Omega) = HD(\mu^*)$ and μ^* is a unique Gibbs measure given by thermodynamics formalism which is equivalent (absolutely continuous) to the Hausdorff measure on Ω. See also [**Fri2**]. In general a strict inequality holds in (0.1). To motivate our results consider the following example.

Let M be a compact surface equipped with a Riemannian metric. Assume that $f : M \to M$ is smooth diffeomorphism, i.e. $f \in \text{Diff}^{1+\alpha}(M), \alpha > 0$. For $\mu \in \mathcal{E}$ let $h(\mu), \quad \lambda_1(\mu) \geq \lambda_2(\mu)$, be the measure (metric) entropy and the corresponding Lyapunov exponents of f. Assume that $h(\mu) > 0$. Then Young's theorem [**You**] claims

$$HD(\mu) = \frac{h(\mu)}{\lambda_1(\mu)} + \frac{h(\mu)}{-\lambda_2(\mu)}. \tag{0.2}$$

Assume that f is an Axiom A diffeomorphism. Then Ω is a finite union of basic hyperbolic sets. Assume for simplicity that $f|\Omega$ is topologically transitive, i.e. Ω consists of one basic set. The result of McCluskey-Manning [**M-M**] is equivalent to.

$$HD(\Omega) = \sup_{\mu \in \mathcal{E}} \frac{h(\mu)}{\lambda_1(\mu)} + \sup_{\mu \in \mathcal{E}} \frac{h(\mu)}{-\lambda_2(\mu)}). \tag{0.3}$$

The corresponding suprema are achieved for the Gibbs measures μ^u, μ^s. Usually $\mu^u \neq \mu^s$, i.e. (0.1) is not sharp.

Our paper is divided roughly to three parts. In the first part (§1-§2) we give sufficient conditions on an Axiom A surface diffeomorphism for which $\mu^u = \mu^s := \mu^*$. We show that these conditions are satisfied by certain area-preserving Hénon maps.

The second part (§3-§4) introduces the notion of a Strong Axiom A diffeomorphisms of manifolds M with $n = \dim M > 2$. An Axiom A diffeomorphism $f : M \to M$ is strongly hyperbolic if the tangent space $T(x)$ splits to

$$T(x) = \sum_{i=1}^{n} \oplus E_i(x), \quad x \in \Omega,$$

where each $E_i(x)$ depends continously on x. Furthermore, possible rates of growth of $Df(u), u \in E_i(x)$ are located in a closed interval $I_i, i = 1, ...n$, and $I_i \cap I_j = \emptyset$ for $i \neq j$. In particular, for any $\mu \in \mathcal{E}$ f has n distinct Lyapunov exponents.

$$\lambda_1(\mu) > \cdots \lambda_n(\mu).$$

Let r^u, r^s be the dimensions of the unstable manifold and stable manifold respectively. The results of Ledrappier and Young [**L-Y**, I-II] and Barreira, Pesin and Schmeling [**B-P-S**] yield:

$$HD(\mu) = \sum_{i=1}^{r^u} \frac{h_i^+(\mu)}{\lambda_i(\mu)} - \sum_{i=1}^{r^s} \frac{h_i^-(\mu)}{\lambda_{r^u+i}}.$$

Here $h_i^+(\mu), h_j^-(\mu)$ is the entropy of f along the unstable and stable manifolds corresponding to the $i-th, j-th$ expanding and contracting direction.

We show that the notion of strong hyperbolicity is structurally stable. Hence a small neighborhood of a Strong Axiom A diffeomorphism $f : M \to M$ consists of Strong Axiom A diffeomorphisms. A simple way to find such f is as follows. Let $f_i : M_i \to M_i, i = 1, ..., k$, be k Axiom A surface diffeomorphisms. Assume furthermore that the rates of expansions and contractions of any pair f_i, f_j lie in nonintersecting closed intervals. Then $f_1 \times \cdots f_k : M_1 \times \cdots M_k \to M_1 \times \cdots M_k$ is a Strong Axiom A diffeomorphism.

The last part of this paper (§5-§6) applies the above ideas to the study of the dynamics of some proper polynomial maps $f : \mathbf{C}^2 \to \mathbf{C}^2$ which extend to holomorphic self-maps of $\mathbf{CP}^2$. More precisely let $J(f)$ be the closure of all repelling periodic points of f. In one complex variables $J(f)$ is exactly the Julia set of f. Using the known structural stability results for hyperbolic sets of endomorphisms (in particular for repellers) we show that $J(f)$ has many properties like the standard Julia set for small neighborhoods of certain f which basically have the structure of $f_1 \times f_2$. We prove the Ω-orbit stability theorem for the above classes of polynomial maps in $\mathbf{C}^2$.

§1. Equilibrium measures for surface diffeomorphisms

Let N be a compact smooth manifold of dimension n. Assume that $g : N \to N$ is a C^1 diffeomorphism. For $\mu \in \mathcal{E}$ we denote by $h(\mu)$ the μ-entropy of g and

$$\lambda_1(\mu) \geq \cdots \geq \lambda_n(\mu)$$

denote the n Lyapunov exponents of g. A map g satisfies Axiom A if $\Omega(g)$ is a hyperbolic set, i.e. for each $x \in \Omega(g)$ the tangent bundle T_xN splits as a direct sum of the contracting and expanding bundles- $E^s(x) \oplus E^u(x)$ and this decomposition is continuous in $x \in \Omega(g)$. Furthermore, the set of periodic points of g is dense in Ω. It is known that

$$\Omega(g) = \cup_{i=1}^k \Lambda_i, \tag{1.1}$$

where each Λ_i is a closed g-invariant set. Moreover, $g|\Lambda_i$ is topologically transitive and $g : \Lambda_i \to \Lambda_i$ is homeomorphic to a subshift of finite type (SFT) with respect some Markov partition. Λ_i is called a basic set. Let $\mathcal{E}_i$ be the set of all g-invariant ergodic measures supported on Λ_i. Then $\mathcal{E} = \cup_{i=1}^k \mathcal{E}_i$ is the set of all g-invariant ergodic measures.

Let M be a compact real surface and $f \in \mathrm{Diff}^1(M)$. Assume that $\mu \in \mathcal{E}$. Denote by $h(\mu)$ the measure (metric) entropy of f. Let $\lambda_1(\mu) \geq \lambda_2(\mu)$ be the corresponding Lyapunov exponents. Assume that $h(\mu) > 0$. Then Margulis-Ruelle inequality gives

$$\lambda_1(\mu) \geq h(\mu) > 0 > -h(\mu) \geq \lambda_2(\mu).$$

The fundamental result of L. Young [**You**] claims that (0.2) holds.

Suppose furthermore that f is an Axiom A diffeomorphism. Then $\mu \in \mathcal{E}_i, \mathrm{supp}(\mu) \subset \Lambda_i$ for some $1 \leq i \leq k$ Denote by $\Lambda_i^+(\mu), \Lambda_i^-(\mu)$ the future and the past μ- generic points. ($x \in M$ is called the future (past) generic point if for any continuous function $\phi : M \to \mathbf{R}$ the average of ϕ on the f-forward (backward) orbit of x converges to $\int \phi d\mu$.) It was proved by Manning [**Man**] that $\frac{h(\mu)}{\lambda_1(\mu)}, \frac{h(\mu)}{-\lambda_2(\mu)}$ are the Hausdorff dimension of $W^u(x) \cap \Lambda_i^+(\mu), W^s(x) \cap \Lambda_i^-(\mu)$ for any $x \in \Lambda_i$. ($W^u(x), W^s(x)$ denote the unstable and the stable manifold through $x \in \Omega$.) any unstable manifold and the stable manifolds with respect to μ. See [**Man**]. The results of McCluskey-Manning [**M-M**] implies the following theorem.

Theorem 1.2. *Let M be a compact surface. Assume that $f: M \to M$ is a C^1 Axiom A diffeomorphism where $\Omega = \Omega(f)$ has decomposition* (1.1) *to the basic sets. Then*

$$\begin{aligned} &\delta_i^u = \sup_{\mu \in \mathcal{E}_i} \frac{h(\mu)}{\lambda_1(\mu)} = \frac{h(\mu_i^u)}{\lambda_1(\mu_i^u)}, \\ &\delta_i^s = \sup_{\mu \in \mathcal{E}_i} \frac{h(\mu)}{-\lambda_2(\mu)}) = \frac{h(\mu_i^s)}{\lambda_2(\mu_i^s)}, \\ &HD(W^u(x) \cap \Lambda_i) = \delta_i^u, \quad HD(W^s(x) \cap \Lambda_i) = \delta_i^s, \quad x \in \Lambda_i, \\ &HD(\Lambda_i) = \delta_i^u + \delta_i^s, \quad i = 1, ..., k, \\ &HD(\Omega) = \max_{1 \le i \le k} \delta_i^u + \delta_i^s. \end{aligned} \tag{1.3}$$

Assume furthermore that $f \in \mathrm{Diff}^{1+\alpha}(M), \alpha > 0$. Then the measures μ_i^u, μ_i^s are unique Gibbs measure for $i = 1, ..., k$.

We now discuss the conditions which yield the equalities

$$\mu_i^u = \mu_i^s, i = 1, ..., k.$$

A simple sufficient condition is

$$\lambda_1(\mu) + \lambda_2(\mu) = 0, \quad \mu \in \mathcal{E}. \tag{1.4}$$

In that case

$$\begin{aligned} &HD(\mu) = \frac{2h(\mu)}{\lambda_1(\mu)}, \\ &\sup_{\mu \in \mathcal{E}} \frac{h(\mu)}{\lambda_1(\mu)} + \sup_{\mu \in \mathcal{E}} \frac{h(\mu)}{-\lambda_2(\mu)} = \sup_{\mu \in \mathcal{E}} HD(\mu). \end{aligned} \tag{1.5}$$

Lemma 1.6. *Let $f: M \to M$ be a C^1 Axiom A diffeomorphism of a compact Riemannian manifold M of dimension n. Then the following are equivalent.*
(a) $\sum_{i=1}^n \lambda_i(\mu) = 0, \quad \mu \in \mathcal{E}$.
(b) $|det(D(f^m(x)))| = 1, \quad f^m(x) = x$ for any periodic point x of f.

Proof.
(a) $\Rightarrow$ (b) Let $f^m(x) = x$. Then there exists a unique $\mu \in \mathcal{E}$ which is uniformly distributed on $x, f(x), ..., f^{m-1}(x)$. The moduli of the eigenvalues of $D(f^m(x))$ (which are independent of a basis in $T(x)$) are $e^{m\lambda_1(\mu)}, ..., e^{m\lambda_n(\mu)}$. Hence $|det(D(f^m(x))| = e^{m \sum_1^n \lambda_i(\mu)} = 1$.
(b) $\Rightarrow$ (a) Set

$$\phi_1(x) = \log |Df(x)|_{W^u(x)}|, \quad \phi_2(x) = -\log |Df(x)|_{W^s(x)}|, \quad x \in \Omega.$$

Observe

$$|det(Df(x))| = e^{\phi_1(x) - \phi_2(x)}. \tag{1.7}$$

Let $\mu \in \mathcal{E}$. Assume that

$$\lambda_1(\mu) \ge \cdots \lambda_{r^+(\mu)} > 0 > \lambda_{r^+(\mu)+1} \ge \cdots \lambda_n(\mu).$$

It follows that

$$\sum_{i=1}^{r^+(\mu)} \lambda_i(\mu) = \int_\Omega \phi_1 d\mu, \quad -\sum_{r^+(\mu)+1}^{n} \lambda_i(\mu) = \int_\Omega \phi_2 d\mu. \tag{1.8}$$

Assume that $f^m(x) = x$. Let μ be the ergodic measure equally distributed on the periodic orbit $x, f(x), ..., f^{m-1}(x)$. The assumption (b) and (1.7)-(1.8) yields that

$$\int_\Omega \phi_1 d\mu = \int_\Omega \phi_2 d\mu. \tag{1.9}$$

Recall that Ω has the decomposition (1.1) to the basic sets and $f : \Lambda_i \to \Lambda_i$ is homeomorphic to a subshift of finite type which is topologically transitive. Hence any $\mu \in \mathcal{E}_i$ is a weak limit of convex combinations of ergodic measures supported on periodic points. As $\phi_1(x), \phi_2(x)$ are continuous it follows that (1.9) holds for any $\mu \in \mathcal{E}$. Use (1.8) to deduce (a).

◇

Theorem 1.10. *Let $f : M \to M$ be a C^1 Axiom A diffeomorphism of a compact surface M. Suppose that f satisfies the condition (b) of Lemma 1.6. Then any extremal measure μ_i^u (given by (1.3)) is also an extremal measure μ_i^s for $i = 1, ..., k$ and vice versa. In particular,*

$$\begin{aligned} &\delta_i^u = \delta_i^s, \\ &H(\Lambda_i) = \sup_{\mu \in \mathcal{E}_i, h(\mu) > 0} HD(\mu), \\ &H(\Omega) = \sup_{\mu \in \mathcal{E}, h(\mu) > 0} HD(\mu). \end{aligned}$$

Assume furthermore that $f \in \mathrm{Diff}^{1+\alpha}$. Then the unique Gibbs measure $\mu_i : \mu_i^u = \mu_i^s$ is equivalent to the Hausdorff measure on Λ_i for $i = 1, ..., k$.

Proof. In view of Theorem 1.2, Lemma 1.6 and (1.5) we need to discuss only the case where $f \in \mathrm{Diff}^{1+\alpha}$. Young's formula (0.2) yields that $HD(\mu_i)$-the μ_i-Hausdorff dimension of Λ_i is equal to $HD(\Lambda_i)$. Moreover it is proved in [**You**] that μ_i a.e.

$$\lim_{\epsilon \to 0^+} \frac{\log \mu_i(B(x,\epsilon))}{\log \epsilon} = \frac{h(\mu_i)}{\lambda_1(\mu_i)} + \frac{h(\mu_i)}{-\lambda_2(\mu_i)}.$$

Here

$$B(x, \epsilon) = \{y : \quad y \in M, \quad \mathrm{dist}(x, y) \le \epsilon\}.$$

Hence the $HD(\Lambda_i)$-Hausdorff measure of Λ_i exist and is absolutely continuous with respect to μ_i. See for example [**Fal**].

◇

Another condition for $\mu_i^u = \mu_i^s$ for the maximal i which satisfies the equality $HD(\Omega) = \delta_i^u + \delta_i^s$ can be deduced from Pesin's formula [**Pes**]. Assume that $f \in \mathrm{Diff}^{1+\alpha}$. Suppose furthermore that f preserves a probability measure ν which is absolutely continuous with respect to the area measure dv given by some Riemannian metric on M. Then Pesin's formula claims

$$h(\nu) = \int \lambda_1(x) d\nu = \int -\lambda_2(x) d\nu.$$

Assume that $h(\nu) > 0$. Consider the ergodic decomposition of ν, e.g. [**Wal**]. In view of Margulis-Ruelle inequality for most of $\mu \in \mathcal{E}$ appearing in the ergodic decomposition of ν we have the equality

$$0 < h(\mu) = \lambda_1(\mu) = -\lambda_2(\mu) \Rightarrow HD(\mu) = 2 \Rightarrow H(\Omega(f)) = 2.$$

Assume in addition that f is an Axiom A diffeomorphism. Margulis-Ruelle inequality yields that $\delta_i^u, \delta_i^s \le 1$. Hence the above μ is extremal and is equal to unique μ_i^u and μ_i^s for some i. It follows that there exists $I \subset \{1, ..., k\}$ such that

$$\mu = \sum_{i\in I} \mu_i,$$
$$d\mu_i = \rho_i d\mu, \quad \rho_i(1-\rho_i) = 0, i \in I, \quad \sum_{i\in I} \rho_i = 1,$$
$$h(\mu_i) > 0, HD(\mu_i) = 2, i \in I.$$

Each $\mu_i, i \in I$ is the unique Gibbs measure which is equal to $\mu_i^u = \mu_i^s, i \in I$.

We now give a sufficient condition which implies the conditions of Lemma 1.6. Let M be a compact Riemannian manifold and $f : M \to M$ a continuous map. Let $M_0 \subset M$ be an open set. Then $f \in \mathrm{Diff}^r(M_0)$ if $f(M_0) = M_0$ and $f|M_0$ is C^r diffeomorphism of M_0. Assume that $f \in \mathrm{Diff}^1(M_0)$. We say that f is Axiom A diffeomorphism on M_0 if

$$\Omega(f) = \Omega_0 \cup \Omega_1, \quad \Omega_0 \subset M_0, \quad \Omega_1 \subset M\backslash M_0, \quad \Omega_0 \cap \Omega_1 = \emptyset,$$

f is hyperbolic on Ω_0 and Ω_0 is the closure of the periodic points of f (which must be in Ω_0). Assume that $f \in \mathrm{Diff}^1(M_0)$. Suppose furthermore that $M\backslash M_0$ is a finite set. Let $\mu \in \mathcal{E}$ and assume that $h(\mu) > 0$. It then follows that μ is supported on Ω_0.

Lemma 1.11. *Let $f : M \to M$ be a continuous map of a compact Riemannian manifold M. Suppose that $M_0 \subset M$ is an open set and f is C^1 Axiom A diffeomorphism of M_0. Assume furthermore that f preserves a σ-finite measure on M_0 of the form*

$$d\nu = w dv, \quad w(x) \in C(M_0), w > 0, \quad x \in M_0, \tag{1.12}$$

where dv is the volume measure induced by the Riemannian metric. Then any ergodic measure μ of f supported on Ω_0 satisfies the condition (a) of Lemma 1.10.

Proof. According to the proof of Lemma 1.6 it is enough to show that for any periodic point $x \in \Omega_0$ the condition (b) of Lemma 1.6 holds. Clearly f^m preserves μ. Use the form (1.12) of μ and the assumption that $f^m(x) = x \in \Omega_0$ to deduce that $|D(f^m)(x)| = 1$.

◇

Combine the above results to obtain the following theorem.

Theorem 1.13. *Let $f : M \to M$ be a continuous map of a compact surface M. Suppose that $M_0 \subset M$ is an open set and f is $C^{1+\alpha}, \alpha > 0$ Axiom A diffeomorphism of M_0. Assume furthermore that $M\backslash M_0$ is a finite set. Let $\Omega_0 = \cup_1^k \Lambda_i$ be the decomposition of Ω_0 to the basic sets. Then equalities (1.3) hold. Furthermore μ_i^u, μ_i^s are unique Gibbs measures on Λ_i for $i = 1, ..., k$. Assume furthermore that f preserves a measure on M_0 of the form (1.12). Then then $\mu_i^u = \mu_i^s := \mu_i$ and μ_i is equivalent to the Hausdorff measure on Λ_i for $i = 1, ..., k$.*

§2. Complex surfaces and Hénon maps

Let N be a compact complex manifold of complex dimension n and assume that $g : N \to N$ is a holomorphic map. Suppose that $\mu \in \mathcal{E}$. As the real dimension of N is $2n$, the complex structure of the tangent bundle TN implies the following conditions:

$$\lambda_1(\mu) = \lambda_2(\mu) \geq \cdots \geq \lambda_{2n-1}(\mu) = \lambda_{2n}(\mu). \tag{2.1}$$

Suppose furthermore that g is an Axiom A diffeomorphism. Then the stable and unstable manifolds $W^s(x), W^u(x), x \in \Omega$ are complex manifolds.

Theorem 2.2. *Let M be a compact complex surface. Assume that $f : M \to M$ is a holomorphic Axiom A diffeomorphism where $\Omega = \Omega(f)$ has decomposition* (1.1) *into the basic sets. Then*

$$\begin{aligned}
&\delta_i^u = \sup_{\mu\in\mathcal{E}_i, h(\mu)>0} \frac{h(\mu)}{\lambda_1(\mu)} = \frac{h(\mu_i^u)}{\lambda_1(\mu_i^u)},\\
&\delta_i^s = \sup_{\mu\in\mathcal{E}_i, h(\mu)>0} \frac{h(\mu)}{-\lambda_4(\mu)}) = \frac{h(\mu_i^s)}{-\lambda_4(\mu_i^s)},\\
&HD(W^u(x)\cap\Lambda_i) = \delta_i^u, \quad HD(W^s(x)\cap\Lambda_i) = \delta_i^s, \quad x\in\Lambda_i,\\
&HD(\Lambda_i) = \delta_i^u + \delta_i^s, \quad i = 1, ..., k,\\
&HD(\Omega) = \max_{1\le i\le k} \delta_i^u + \delta_i^s.
\end{aligned} \tag{2.3}$$

The measures μ_i^u, μ_i^s are unique Gibbs measure for $i = 1, ..., k$.

Suppose furthermore that f satisfies the condition (b) of Lemma 1.6. Then $\mu_i := \mu_i^u = \mu_i^s$ and μ_i is equivalent to the Hausdorff measure on Λ_i for $i = 1, ..., k$.

Proof. Let $\mu \in \mathcal{E}$. Then the equality (2.1) holds with $n = 2$. It is enough to consider the case where $h(\mu) > 0$. Hence $W^u(x), W^s(x), x \in \Omega$ are real two dimensional manifolds. The arguments of [**Man**] yield that $\frac{h(\mu)}{\lambda_1(\mu)}, \frac{h(\mu)}{-\lambda_4(\mu)}$ are the Hausdorff dimension of $W^u(x) \cap \Lambda_i^+(\mu), W^s(x) \cap \Lambda_i^-(\mu)$ for any $x \in \Lambda_i$. Use the arguments of [**M-M**] to deduce (2.3). (One should replace the intervals in the arguments of [**Man**] and [**M-M**] by corresponding disks. See Verjovsky and Wu [**V-W**].) As f is smooth we obtain that the measures μ_i^u, μ_i^s are unique for $i = 1, ..., k$. The last claim of the Theorem follows from Lemma 1.6 applied to this particular case.

◇

Theorem 2.2 is meaningful for special compact complex surfaces, as most of compact complex surfaces have a small group of automorphisms-$Aut(M)$ (complex diffeomorphisms). In most cases $Aut(M)$ is finite. Indeed, assume first that M is a real compact Riemann surface of genus g. Suppose furthermore that M is endowed with a complex structure, i.e. M is one dimensional compact complex manifold. If $g = 0$, i.e. M is the Riemann sphere then $Aut(M)$ is the group of Möbius transformations. If $g = 1$, i.e. M is a complex torus, then the group of translations ($z \to z + a$ for the standard representation of M as a parallelogram in $\mathbf{C}$) has a finite index in $Aut(M)$. Finally, if $g > 1$ then $Aut(M)$ is finite. (Schwarz's theorem, e.g. [**F-K**].) In all these cases the dynamics of an automorphism is trivial. Consider next the two dimensional complex projective plane $\mathbf{CP}^2$. Then $Aut(\mathbf{CP}^2)$ is the group of invertible affine maps of $\mathbf{C}^2 \subset \mathbf{CP}^2$. See for example [**G-H**]. Again the dynamics of any automorphism is trivial. The most interesting case is the complex two dimensional torus $\mathbf{T}^2$. Again $Aut(\mathbf{T}^2)$ can be classified completely by the corresponding complex affine transformations. In certain cases, e.g. when the lattice in $\mathbf{C}^2$ defining $\mathbf{T}^2$ coincides with the standard lattice in $\mathbf{R}^4$ (given by the standard basis), $Aut(\mathbf{T}^2)$ will have elements with nontrivial dynamics given by Anosov diffeomorphisms. However the dynamics of $f \in Aut(\mathbf{T}^2)$ can be determined straightforward using the simple form of f.

We obtain interesting results when we relax the conditions of Theorem 2.2 by considering birational automorphisms of M. We now discuss this situation for the polynomial automorphisms of $\mathbf{C}^2$-$Aut(\mathbf{C}^2)$. The systematical study of the dynamics of $f \in Aut(\mathbf{C}^2)$ was initiated by Friedland and Milnor in [**F-M**] and continued in particular by Bedford and Smillie, e.g. [**B-S, 1-3**].

Let $f \in Aut(\mathbf{C}^2)$. Consider one point compactification of $\mathbf{C}^2$ which is homeomorphic to the four dimensional sphere $S^4 = \mathbf{C}^2 \cup \infty$. Then f lifts to a homeomorphism map $\hat{f} : S^4 \to S^4, \quad \hat{f}(\infty) = \infty$. That is $\hat{f}$ is smooth at all points of S^4 except ∞. In the notation of the previous section $\hat{f}$ is a smooth (holomorphic) diffeomorphism of $M_0 = \mathbf{C}^2$. The simplest example of an automorphism of $\mathbf{C}^2$ is a (generalized) Hénon map

$$\begin{aligned}
&H(x, y) = (y, p(y) - dx), \quad x, y \in \mathbf{C}, \quad d \neq 0,\\
&p(y) = y^n + \sum_{i=2}^{n} a_i y^{n-i}, \quad n \geq 2.
\end{aligned} \tag{2.4}$$

Note that if $d, a_2, ..., a_n \in \mathbf{R}$ then $H : \mathbf{R}^2 \to \mathbf{R}^2$. Thus, the original Hénon map is the case $n = 2, d, a_2 \in \mathbf{R}$ [**Hén1-2**]. Note that

$$\det(DH) = d.$$

Hence H is area preserving (in absolute value) iff $|d| = 1$. Note that the area of $\mathbf{C}^2$ or $\mathbf{R}^2$ is not finite. It was shown in [**F-M**] that any $f \in Aut(\mathbf{C}^2)$ is either conjugate to an elementary automorphism (with rather trivial dynamics) or to a composition of Hénon maps $g = H_1 \cdots H_p$. This product is essentially unique up to a cyclic permutation of the factors. The nonwandering set of $\hat{f}$ is of the form $\Omega(f) \cup \infty$, where $\Omega(f)$ is a compact set in $\mathbf{C}^2$. It was shown that in [**F-M**, §5] that given n and d there are many real Hénon maps which are n-fold horseshoes on $\Omega(H)$. For complex valued Hénon maps one has the following family of horseshoe maps: Fix all the parameters of (2.4) except a_n. Then there exists $r(d, a_2, ..., a_{n-1}) > 0$ so that for $|a_n| \geq r(d, a_2, ..., a_{n-1})$ H is an n fold horseshoe. For these maps we deduce that $\Omega(H)$ satifies all the conditions of Axiom A diffeomorphism. Following [**B-S2**] we call $f \in Aut(\mathbf{C}^2)$ hyperbolic if f is conjugate to a product of Hénon maps and $\Omega(f)$ is hyperbolic. That is $T_x\mathbf{C}^2 = E^s(x) \oplus E^u(x), x \in \Omega(f)$ and this decomposition is continuous. In that case the periodic points are dense in $\Omega(f)$ [**B-S2**, Cor. 6.13]. Assume that $f \in Aut(\mathbf{C}^2)$ has real coefficients. f is called real hyperbolic if f is conjugate to a product of Hénon maps, $\emptyset \neq \Omega(f) \cap \mathbf{R}^2$ is equal to the closure of its real periodic points, and the decomposition $T_x\mathbf{R}^2 = E^s(x) \oplus E^u(x), x \in \Omega(f) \cup \mathbf{R}^2$ is continuous. We thus can apply the results and the arguments of Theorems 1.13 and 2.2 to the corresponding automorphisms of $\mathbf{C}^2$.

Theorem 2.5. *Let $f \in Aut)(\mathbf{C}^2)$. Assume that f is either real hyperbolic or complex hyperbolic. Let $\Omega = \cup_1^k \Lambda_i$ be the decomposition of the nonwandering set of f into the basic sets. Then* (1.3) *and* (2.3) *respectively hold. If $|\det(Df)| = 1$ then $\mu_i := \mu_i^s = \mu_i^u, i = 1, ..., k$ and each μ_i is equivalent to the Hausdorff measure on the corresponding basic set.*

§3. Strict and strong hyperbolicity

Let M be an d-dimensional compact Riemannian manifold. Assume that $X \subset M$ is an invariant set of $f \in \text{Diff}^r(M), r \geq 1$. Then X is called a hyperbolic set if there exists a continuous splitting of the tangent bundle of M restricted to X which is Df invariant:

$$T_X M = E^s \oplus E^u; \quad Df(E^s) = E^s; \quad Df(E^u) = E^u;$$

for which there are constants $c > 0$ and $\rho_{+,1} > 1 > \rho_{-,1} > 0$ such that

$$||Df^n|_{E^s}|| < c\rho_{-,1}^n, \quad n \geq 0,$$
$$||Df^n|_{E^u}|| < c\rho_{+,1}^n, \quad n \leq 0.$$

Assume that X is a closed hyperbolic set for f. Then for each $x \in X$ there exist stable and unstable manifolds

$$W^s(x) = \{y : \quad y \in M, \quad \limsup_{n\to\infty} \frac{1}{n} \log \text{dist}(f^n(y), f^n(x)) \leq \log \rho_{-,1}\},$$
$$W^u(x) = \{y : \quad y \in M, \quad \limsup_{n\to\infty} \frac{1}{n} \log \text{dist}(f^{-n}(y), f^{-n}(x)) \leq -\log \rho_{+,1}\}.$$

$W^s(x), W^u(x)$ are immersed submanifolds of M which are as smooth as f and

$$T_x W^s(x) = E^s(x), \quad T_x W^u(x) = E^u(x), \quad x \in X.$$

For $\epsilon > 0$ set

$$W_\epsilon^s(x) = W^s(x) \cap B(x, \epsilon), \quad W_\epsilon^u(x) = W^u(x) \cap B(x, \epsilon).$$

Then there exists $\epsilon > 0$ that

$$B(x,\epsilon) = W^s_\epsilon(x) \times W^u_\epsilon(x), \quad x \in X.$$

See for example [**Shu**].

Assume that $X \subset M$ is an invariant hyperbolic set. Then f is called strictly hyperbolic if the above continuous splitting of $T_X M$ has a further continuous splitting with the following conditions.

$$\begin{aligned}
&E^s = \sum_{j=1}^{r^-} \oplus E^s_j, \quad Df(E^s_j) = E^s_j, \quad j = 1, ..., r^-,\\
&c'(\rho'_{-,j})^n \le ||Df^n|_{E^s_j}|| \le c\rho^n_{-,j}, \quad n \ge 0, \quad j = 1, ..., r^-,\\
&E^u = \sum_{j=1}^{r^+} \oplus E^u_j, \quad Df(E^u_j) = E^u_j, \quad j = 1, ..., r^+,\\
&c'\rho^n_{+,j} \le ||Df^n|_{E^u_j}|| \le c(\rho'_{+,j})^n, \quad n \le 0, \quad j = 1, ..., r^+,\\
&\rho_{+,1} > \rho'_{+,1} > \rho_{+,2} > \rho'_{+,2} > \cdots > \rho_{+,r^+} > \rho'_{+,r^+} > 1,\\
&1 > \rho_{-,1} > \rho'_{-,1} > \rho_{-,2} > \rho'_{-,2} > \cdots > \rho_{-,r^-} > \rho'_{-,r^-} \ge 0,\\
&c \ge c' > 0.
\end{aligned} \tag{3.1}$$

Note that hyperbolicity of X is equivalent to (3.1) with $r^+ = r^- = 1$.

The finer decomposition of X implies the finer decomposition of the stable and unstable manifold.

Theorem 3.2. *Let M be a compact Riemannian manifold and assume that $f \in \mathrm{Diff}^r(M), r \ge 1$. Let $X \subset M$ be a closed f-invariant hyperbolic set. Suppose furthermore that* (3.1) *holds. Then there exist the following decompositions of stable and unstable manifolds.*

$$\begin{aligned}
&V^s_i \subset W^s, \quad T_x V^s_i = E^s_i(x), \quad f(V^s_i(x)) = V^s_i(f(x)), \quad i = 1, ..., r^-,\\
&W^s(x) = V^s_1(x) \times \cdots \times V^s_{r^-}(x),\\
&V^u_i \subset W^u, \quad T_x V^u_i = E^u_i(x), \quad f(V^u_i(x)) = V^u_i(f(x)), \quad i = 1, ..., r^+,\\
&W^u(x) = V^u_1(x) \times \cdots \times V^u_{r^+}(x).
\end{aligned}$$

Each V^s_i, V^u_j are C^1 at least.

Proof. We first prove the decomposition of the unstable manifold. Our proof is based on the arguments given in [**Shu**, Appendix IV]. Let $S(X, T_X M)$ be the space of all continuous sections on X. That is $h \in S(X, T_X M)$ if for each $x \in X$, $h(x) \in T_x M$ and $h(x)$ is continuous. Then $S(X, T_X M)$ is a vector space using the pointwise addition. We let $S(X, T_X M)$ be a Banach space by introducing the max norm

$$||h|| = \sup_{x \in X} ||h(x)||.$$

Here we assume that on $T_x M$ we have the Hilbert norm induced by the Riemannian metric on M. Let

$$\begin{aligned}
&S^+_i = \{h: \quad h \in S(X, T_X M), \quad h(x) \in E^u_i(x)\}, \quad i = 1, ..., r^+,\\
&Y^+_i = \sum_{l=1}^{i} \oplus S^+_l, \quad i = 1, ..., r^+,\\
&S^-_i = \{h: \quad h \in S(X, T_X M), \quad h(x) \in E^s_i(x)\}, \quad i = 1, ..., r^-,\\
&Y^-_i = \sum_{l=i}^{r^-} \oplus S^+_l, \quad i = 1, ..., r^-.
\end{aligned}$$

Clearly

$$\begin{aligned}
&Df(S_i^+) = S_i^+, \quad i = 1, ..., r^+,\\
&Df(S_i^-) = S_i^-, \quad i = 1, ..., r^-,\\
&S(X, T_X M) = Y_{r^+}^+ \oplus Y_{r^-}^-,\\
&Y_i^+ = Y_{i-1}^+ \oplus S_i^+, \quad i = 2, ..., r^+.
\end{aligned}$$

We now recall briefly the proof of the existence of the unstable manifold as in [**Shu**, Ch.6]. In view of the above assumptions Df is strictly expanding on $Y_{r^+}^+$ and strictly contracting on $Y_{r^-}^-$. Let

$$D(\epsilon) = \{h : \quad h \in S(X, T_X M), \quad ||h|| \leq \epsilon\}.$$

Then f induces the following Lipschitz map $\tilde{f} : D(\epsilon) \to S(X, T_X M)$. Since M is compact there exists $\delta > 0$ so that the exponential map $exp_x : T_x M \cap C(x, \delta) \to M$ is $1 - 1$. Here by $C(x, \delta)$ we denote the closed ball of radius δ in $T_x M$ centered at 0 in the given Riemannian metric on M. We thus can identify the closure of the appropriate neighborhood of $x \in M$ with $T_x M \cap C(x, \delta)$. Hence there exists $0 < \delta_1 \leq \delta$ so that f carries the closed neighborhood of $x \in M$ corresponding to $T_x M \cap C(x, \delta_1)$ to a subset of the closed neighborhood of $f(x)$ corresponding to $T_{f(x)} M \cap C(f(x), \epsilon)$. Let $h \in D(\delta_1)$. Hence $h(x) \in T_x M \cap C(x, \delta_1)$. We then let $\tilde{f}(h(x)) \in T_{f(x)} M$ to be the unique solution of

$$exp_{f(x)}(\tilde{f}(h(x)) = f(exp_x(h(x))).$$

It is straightforward to show that since $f \in \mathrm{Diff}^1(M)$ $\tilde{f}$ is C^1 on $D(\delta_1)$. In particular $\tilde{f}$ is Lipschitz. Let 0 be the zero section in $S(X, T_X M)$. It then follows the $\tilde{f}(0) = 0$. Moreover, Df viewed as a linear operator on $S(X, T_X M)$ is the Fréchet derivative of $\tilde{f}$ at 0. Thus we can apply Theorem 5.2 of [**Shu**] as in the proof of Theorem 6.2. (We skip some of the technical details and oversimplify the ideas of the proof given in [**Shu**].) Set

$$\begin{aligned}
&S_i^+(\epsilon) = S_i^+ \cap D(\epsilon), \quad Y_i^+(\epsilon) = Y_i^+ \cap D(\epsilon), \quad i = 1, ..., r^+,\\
&S_i^-(\epsilon) = S_i^- \cap D(\epsilon), \quad Y_i^-(\epsilon) = Y_i^- \cap D(\epsilon), \quad i = 1, ..., r^-.
\end{aligned}$$

Then there exists $0 < \epsilon < \delta_1$ and a Lipschitzian $g : Y_{r^+}^+(\epsilon) \to Y_{r^-}^-(\epsilon)$ which gives the local unstable manifolds $W_\epsilon^u(x), x \in X$ as follows. First $Lip(g) \leq 1$. Second g can be viewed using the exponential map as

$$\hat{g}_x : E^u(x) \cap C(x, \delta') \to E^s(x), \quad x \in X.$$

Here $\hat{g}_x$ varies continuously in x. Finally $\tilde{f}$ maps the graph $\hat{g}_x$ into the graph of $\hat{g}_{f(x)}$. The exponential map of the graph of $\hat{g}_x$ gives the local unstable manifold $W_\epsilon^u(x)$. The r smoothness of g and hence of $W_\epsilon^u(x)$ is obtained from the appropriate smoothness of f as in [**Shu**]. Similar arguments applied for f^{-1} give the local stable manifolds $W_\epsilon^s(x), x \in X$. Moreover

$$B(x, \epsilon) = W_\epsilon^u(x) \times W_\epsilon^s(x), \quad x \in X.$$

Assume that $r^+ > 1$. We now show how to obtain the decomposition

$$W_\epsilon^u(x) = W_{r^+-1,\epsilon}^u(x) \times V_{r^+,\epsilon}^u(x), \quad x \in X.$$

Since $\tilde{f}$ acts on the graphs of $\hat{g}_x$ it follows that we have the following restriction

$$\tilde{f} : Y_{r^+}^+ \to Y_{r^+}^+.$$

Set $f_1 = \tilde{f}|Y_{r^+}^+$. Again $Df|Y_{r^+}^+$ is the Fréchet derivative of f_1. Consider the decomposition $Y_{r^+}^+ = Y_{r^+-1}^+ \oplus S_{r^+}^+$. Note that Df expands on $Y_{r^+-1}^+$ at the rate ρ'_{+,r^+-1} at least while Df expands on $S_{r^+}^+$ at the rate ρ_{+,r^+} at most. Thus we can apply Theorem III.2 as in the proof of Theorem IV.1 (Center and Strong Stable Manifolds for Invariant Sets). To be precise if we consider f_1^{-1} we then obtain $W^u_{r^+-1,\epsilon}$ as the super stable manifold and $V^+_{r^+,\epsilon}$ as the center unstable manifold. Hence $W^u_{r^+-1,\epsilon}$ is C^r while $V^+_{r^+,\epsilon}$ is C^1 at least. Furthermore

$$W^u_\epsilon(x) = W^u_{r^+-1,\epsilon}(x) \times V^+_{r^+,\epsilon}(x), \quad x \in X.$$

Continue this procedure to obtain the complete decomposition of W^u_ϵ:

$$\begin{aligned} W^u_{i,\epsilon}(x) &= W^u_{i-1,\epsilon}(x) \times V^+_{i,\epsilon}(x), \quad i = r^+, ..., 2, \\ W^u_\epsilon(x) &= W^u_{r^+,\epsilon}(x), \quad V^+_{1,\epsilon}(x) = W^u_{1,\epsilon}(x). \end{aligned} \tag{3.3+}$$

Apply these arguments to f^{-1} to obtain the following decomposition of the stable manifold:

$$\begin{aligned} W^s_{i,\epsilon}(x) &= W^s_{i-1,\epsilon}(x) \times V^-_{r^- -i+1,\epsilon}(x), \quad i = r^-, ..., 2, \\ W^s_\epsilon(x) &= W^s_{r^-,\epsilon}(x), \quad V^-_{r^-,\epsilon}(x) = W^s_{1,\epsilon}(x). \end{aligned} \tag{3.3−}$$

Then each $W^u_{i,\epsilon}(x), W^s_{j,\epsilon}(x)$ is C^r while each $V^+_{i,\epsilon}(x), V^-_{j,\epsilon}(x)$ is at least C^1.

Finally to define globally $V^+_i(x), V^-_j(x)$ we let

$$\begin{aligned} V^+_i(x) &= \cup_{n\geq 0} f^n V^+_{i,\epsilon}(f^{-n}(x)), \quad i = 1, ..., r^+, \\ V^-_i(x) &= \cup_{n\geq 0} f^{-n} V^-_{i,\epsilon}(f^n(x)), \quad i = 1, ..., r^-. \end{aligned}$$

The proof of the theorem is completed.

$\diamond$

It is well known that the submanifolds $W^s_i(x), W^u_j(x)$ have the following geometric meanings:

$$\begin{aligned} W^s_i(x) &= \{y: \quad y \in M, \quad \limsup_{n\to\infty} \frac{1}{n} \log \mathrm{dist}(f^n(y), f^n(x)) \leq \log \rho_{-,r^- -i+1}\}, \quad i = 1, ..., r^-, \\ W^u_i(x) &= \{y: \quad y \in M, \quad \limsup_{n\to\infty} \frac{1}{n} \log \mathrm{dist}(f^{-n}(y), f^{-n}(x)) \leq -\log \rho'_{+,i}\}, \quad i = 1, ..., r^+. \end{aligned}$$

See for example the arguments of [**Rue1**, §6]. As pointed out in [**Shu**, p'80], there is no special meaning of the centered unstable manifolds; thus we do not see why $V^u_i(x), V^s_j(x)$ are unique (except those which concide with $W^u_1(x), W^s_1(x)$).

We now show that a closed strict hyperbolic set X of f satisfying the conditions (3.1) is structurally stable in the sense of [**Shu**, Th.8.3]:

Theorem 3.4. *Let M be a compact Riemannian manifold and assume that $f \in \mathrm{Diff}^r(M), r \geq 1$. Let X be a closed f-invariant hyperbolic set. Assume that* (3.1) *holds. There is a neighborhood U_f of f in $\mathrm{Diff}^r(M)$ and a continuous function $\Phi : U_f \to C^0(X, M)$ such that:*
(1) $\Phi(f)$ is the inclusion, inc_X, of X in M.
(2) $\Phi(g)(X)$ is a hyperbolic set for any $g \in U_f$. Moreover for each $g \in U_f$ there exists a continuous decomposition (3.1) *of $T_{\Phi(g)(X)}M$ with*

$$\begin{aligned} &\rho_{+,1}(g) > \rho'_{+,1}(g) > \rho_{+,2}(g) > \rho'_{+,2}(g) > \cdots > \rho_{+,r^+}(g) > \rho'_{+,r^+}(g) > 1, \\ &1 > \rho_{-,1}(g) > \rho'_{-,1}(g) > \rho_{-,2}(g) > \rho'_{-,2}(g) > \cdots > \rho_{-,r^-}(g) > \rho'_{-,r^-}(g) > 0, \\ &c(g) \geq c'(g) > 0. \end{aligned}$$

(3) $\Phi(g)$ is a homeomorphism of X onto $\Phi(g)(X)$ and topologically conjugates the restriction of f to X to the restricition of g to $\Phi(g)(X)$.
(4) There is a constant K such that $d_{C^0}(\Phi(g), \mathrm{inc}_X) < K d_{C^0}(g, f)$.

Proof. The conditions (1)-(4) except the fine decomposition (3.1) of part (2) of the theorem is proven in Theorem 8.3 of [**Shu**]. We now prove the fine decomposition (3.1) for any $g \in U_f$. To do that we have to analyze carefully the proof of (2) in [**Shu**]. We already know that $\Phi(g)(X)$ is a hyperbolic set of g. We thus can use the ideas of the proof of Proposition 7.6 in [**Shu**]. We extend the splitting (3.1) to a neighborhood $\mathcal{N} \supset X$. We shall assume that U_f is chosen small enough to satisfy the perturbation conditions needed. Then

$$A(x) = (A_{ij}(x))_1^m = Df|_x, \quad B(x) = (B_{ij}(x))_1^m = Dg|_x, \quad m = r^+ + r^-, \quad x \in \Phi(g)X$$

are $m \times m$ block matrices. Furthermore, since x is close to X and g is a perturbation of f we assume that the tangent bundles at $f(x)$ and $g(x)$ are identical. Hence we have the inequalites

$$\begin{aligned}
&\rho'_{+,i} - \delta \le ||A_{ii}(x)|| \le \rho_{+,i} + \delta, \quad i = 1, ..., r^+,\\
&\rho'_{-,i-r^+} - \delta \le ||A_{ii}(x)|| \le \rho_{-,i-r^+} + \delta, \quad i = r^+ + 1, ..., m\\
&||A_{ij}(x)|| \le \delta, \quad , i \ne j, \quad i, j = 1, ..., m,\\
&||A_{ij}(x) - B_{ij}(x)|| < \delta, \quad i, j = 1, ..., m,\\
&x \in \Phi(g)(X).
\end{aligned}$$

δ is assumed to be a positive and arbitrarily small. Set

$$\begin{aligned}
&L(x) = (L_{ij}(x))_1^m,\\
&L_{ii}(x) = B_{ii}(x), \quad i = 1, ..., m,\\
&L_{ij}(x) = 0, \quad i \ne j, \quad i, j = 1, ..., m,\\
&x \in \Phi(g)X.
\end{aligned}$$

The matrices $L(x)$ induce the linear operator on $S(\Phi(g)X, T_{\Phi(g)X}M)$:

$$L(h)(g(x)) = L(x)h(x), \quad h \in S(\Phi(g)X, T_{\Phi(g)X}M), \quad x \in \Phi(g)X.$$

The above inequalities mean that the spectrum of L is concentrated on $r^+ + r^-$ distinct annuli in the complex plane:

$$\begin{aligned}
&\rho'_{+,i} - 2\delta \le |z| \le \rho_{+,i} + 2\delta, \quad i = 1, ..., r^+,\\
&\rho'_{-,i} - 2\delta \le |z| \le \rho_{-,i} + 2\delta, \quad i = 1, ..., r^-.
\end{aligned}$$

Furthermore the spectrum of L has a nonvoid intersection with each annulus given above. It now follows that the spectrum of B is concentrated $r^+ + r^-$ distinct closed annuli

$$\begin{aligned}
&\rho'_{+,i}(g) \le |z| \le \rho_{+,i}(g), \quad i = 1, ..., r^+,\\
&\rho'_{-,i}(g) \le |z| \le \rho_{-,i}(g), \quad i = 1, ..., r^-.
\end{aligned}$$

Let

$$\Pi_1^+(L), ..., \Pi_{r^+}^+(L), \Pi_1^-(L), ..., \Pi_{r^+}^-(L), \quad \Pi_1^+(B), ..., \Pi_{r^+}^+(B), \Pi_1^-(B), ..., \Pi_{r^+}^-(B),$$

be the spectral projections corresponding to L and B respectively on the above annuli. Set

$$\begin{aligned}
&S_i^+(L) = \Pi_i^+(L) S(\Phi(g)X, T_{\Phi(g)X}M), \quad S_i^+(g) = \Pi_i^+(B) S(\Phi(g)X, T_{\Phi(g)X}M), \quad i = 1, ..., r^+,\\
&S_i^-(L) = \Pi_i^-(L) S(\Phi(g)X, T_{\Phi(g)X}M), \quad S_i^-(g) = \Pi_i^-(B) S(\Phi(g)X, T_{\Phi(g)X}M), \quad i = 1, ..., r^-,\\
&S(\Phi(g)X, T_{\Phi(g)X}M) = \sum_{i=1}^{r^+} \oplus S_i^+(L) \oplus \sum_{i=1}^{r^-} \oplus S_i^-(L) = \sum_{i=1}^{r^+} \oplus S_i^+(g) \oplus \sum_{i=1}^{r^-} \oplus S_i^-(g).
\end{aligned}$$

The projection of $S_i^+(g), S_j^-(g)$ on $T_yM, y = \Phi(g)(x), x \in X$ induces the subspaces $E_i^u(g)(y), E_j^s(g)(y)$. As the projection of $S_i^+(L), S_j^-(L)$ on $T_yM, y = \Phi(g)(x), x \in X$ have the dimensions of $E_i^u(f), E_j^s(f)$ it follows that the dimensions of $E_i^u(g)(y), E_j^s(g)((y)$ do not depend on g. In particular (3.1) holds. The continuity of the decompostion (3.1) follows from the fact that each $S_i^+(g), S_j^-(g)$ is a closed subspace of continuous sections.

$\diamond$

The following lemma is straightforward.

Lemma 3.5. *Let M_1, M_2 be two compact Riemannian manifolds. Assume that $f_i : M_i \to M_i$ are $C^r, r \geq 1$, diffeomorphisms for $i = 1, 2$. Suppose that each X_i is a strict hyperbolic set satisfying the assumptions* (3.1) *with the constants depending on f_i as in the condition* (2) *of Theorem 3.4. Furthermore, the integers $r^+ = r^+(f_i), r^- = r^-(f_i)$ are functions of f_i. Assume that the following conditions are satisfied:*

$$[\rho'_{+,i}(f_1), \rho_{+,i}(f_1)] \cap [\rho'_{+,j}(f_2), \rho_{+,j}(f_2)] = \emptyset, \quad i = 1, ..., r^+(f_1), \quad j = 1, ..., r^+(f_2),$$
$$[\rho'_{-,i}(f_1), \rho_{-,i}(f_1)] \cap [\rho'_{-,j}(f_2), \rho_{-,j}(f_2)] = \emptyset, \quad i = 1, ..., r^-(f_1), \quad j = 1, ..., r^-(f_2).$$

Then the set $X_1 \times X_2$ is a strict hyperbolic set of C^r diffeomorphism $f = f_1 \times f_2 : M_1 \times M_2 \to M_1 \times M_2$ with

$$r^+(f) = r^+(f_1) + r^+(f_2), \quad r^-(f) = r^-(f_1) + r^-(f_2),$$
$$\{\rho_{+,i}(f)\}_1^{r^+(f)} = \{\rho_{+,i}(f_1)\}_1^{r^+(f_1)} \cup \{\rho_{+,i}(f_2)\}_1^{r^+(f_2)}, \quad \{\rho'_{+,i}(f)\}_1^{r^+(f)} = \{\rho'_{+,i}(f_1)\}_1^{r^+(f_1)} \cup \{\rho'_{+,i}(f_2)\}_1^{r^+(f_2)},$$
$$\{\rho_{-,i}(f)\}_1^{r^-(f)} = \{\rho_{-,i}(f_1)\}_1^{r^-(f_1)} \cup \{\rho_{-,i}(f_2)\}_1^{r^-(f_2)}, \quad \{\rho'_{-,i}(f)\}_1^{r^-(f)} = \{\rho'_{-,i}(f_1)\}_1^{r^-(f_1)} \cup \{\rho'_{-,i}(f_2)\}_1^{r^-(f_2)}.$$

Lemma 3.5 enables us to obtain strict hyperbolic sets from smaller dimension strict hyperbolic sets or even just hyperbolic sets.

Definition 3.6. *Let $f : M \to M$ be a $C^r, r \geq 1$, diffeomorphism. Assume that $X \subset M$ is an f-invariant hyperbolic set. Then X is called strongly hyperbolic if (3.1) holds where each $E_i^s(x), E_j^u(x), x \in X$ is a one dimensional subspace of T_xM. That is,*

$$\dim M = r^+ + r^-.$$

If M is a complex manifold and f is a complex diffemorphism then f is called strongly hyperbolic if each $E_i^s(x), E_j^u(x), x \in X$ is a one dimensional complex subspace of T_xM.

Theorem 3.4 implies that strongly hyperbolic sets are structurally stable. Lemma 3.5 implies that $X_1 \times X_2$ is strongly hyperbolic for $f_1 \times f_2$ if each X_i is strongly hyperbolic for f_i and the assumptions of Lemma 3.5 hold.

Recall that $f : M \to M$ is a $C^r, r \geq 1$, Axiom A diffeomorphism if $\Omega(f)$ is hyperbolic and is the closure of its periodic points. Then $\Omega(f)$ decomposes to k mutually disjoint basic sets $\cup_{i=1}^k \Lambda_i$. Each Λ_i is a closed f-invariant hyperbolic set. $f|\Lambda_i$ is topologically transitive and has a Markov partition. The sets $\Lambda_1, ..., \Lambda_k$ have no-cycle property if there is no cycle on $r > 1$ elements of $1, ..., k$ satisfying the condition.

$$W^u(\Lambda_{i_j}) \cap W^s(\Lambda_{i_{j+1}}) \neq \emptyset, \quad 1 \leq i_j \neq i_{j+1} \leq k, \quad j = 1, ..., r, \quad i_{r+1} = i_1. \tag{3.7}$$

Assume that f is an Axiom A diffeomorphism with no cycle property. Then $\Omega(f)$ is structurally stable. See for example [**Shu**, Cor. 8.24]. We say that f is a Strong Axiom A diffeomorhism if f is an Axiom A diffeomorphism and each basic set satisfies the assumptions of Definition 3.6. Theorem 3.4 implies that Strong Axiom A diffeomorphism with no-cycle property are structurally stable.

A standard example of a Strong Axiom A diffeomorphisms is the following one. Let $M = T^n = S^1 \times \cdots \times S^1$ be an n-dimensional torus. Assume that $f : T^n \to T^n$ is represented by an $n \times n$ unimodular matrix. Then f is a Strong Axiom A diffeomorphism iff the absolute values on the n eigenvalues of A are pairwise distinct.

We now point out the following construction of a class of Strong Axiom A diffeomorphisms. Let M be a compact surface and $f : M \to M$ be an Axiom A diffeomorphism. Assume furthermore that

(3.8) *f does not have an isolated cycle.*

Since f is an Axiom A diffeomorphism it follows that (3.8) is equivalent to the assumption that each basic set of f is infinite and is a subshift of a finite type. In particular, for each $x \in \Omega(f)$ the stable and the unstable manifolds are nonempty. Hence $E^s(x), E^u(x)$ are one dimensional and f is a Strong Axiom A diffeomorphism. The above arguments yield the following theorem:

Theorem 3.9. *Let f_i be a C^1 Axiom A diffeomorphism of the compact surface M_i with no-cycle property which satisfy the condition* (3.8) *for $i = 1, ...p$. Assume that*

$$\begin{aligned}
&c'(\rho'_{-,i})^n \le ||Df_i^n|_{E^s(f_i)}|| \le c\rho_{-,i}^n, \quad n \ge 0, \quad 0 < \rho'_{-,i} < \rho_{-,i} < 1,\\
&c'\rho_{+,i}^n \le ||Df_i^n|_{E^u(f_i)}|| \le c(\rho'_{+,i})^n, \quad n \le 0, \quad 1 < \rho'_{+,i} < \rho_{+,i},\\
&i = 1, ..., p.
\end{aligned}$$

Suppose furthermore that

$$[\rho'_{+,i}, \rho_{+,i}] \cap [\rho'_{+,j}, \rho_{+,j}] = \emptyset, \quad [\rho'_{-,i}, \rho_{-,i}] \cap [\rho'_{-,j}, \rho_{-,j}] = \emptyset, \quad 1 \le i < j \le p.$$

Then there exists a neighborhood U_f of $f = f_1 \times \cdots \times f_p$ in $\mathrm{Diff}^1(M), M = M_1 \times \cdots \times M_p$ *such that each $g \in U_f$ is a Strong Axiom A diffeomorphism.*

§4. The Hausdorff dimension of measures

Let M be a smooth n-dimensional compact Riemannian manifold. Assume that $f \in \mathrm{Diff}^1(M)$. For $\mu \in \mathcal{E}$ let

$$\lambda_1(\mu) = \cdots = \lambda_{n_1}(\mu) > \lambda_{n_1+1}(\mu) = \cdots = \lambda_{n_2}(\mu) > \cdots > \lambda_{n_{r-1}+1}(\mu) = \cdots = \lambda_{n_r}(\mu), \quad n_r = n$$

be the Lyapunov exponents of λ. Set

$$\begin{aligned}
&\chi_i(\mu) = \lambda_{n_i}(\mu), \quad i = 1, ..., r = r(\mu), \quad \chi(\mu) = \{\chi_1(\mu), ..., \chi_r(\mu)\},\\
&\chi^+(\mu) = \{x : \quad x \in \chi(\mu), \quad x > 0\}, \quad |\chi^+(\mu)| = r^+(\mu),\\
&\chi^-(\mu) = \{x : \quad x \in \chi(\mu), \quad x < 0\}, \quad |\chi^-(\mu)| = r^-(\mu),\\
&\chi^0(\mu) = \chi(\mu) \cap \{0\}.
\end{aligned}$$

The set χ is called the spectrum of μ. We shall assume that $h(\mu) > 0$ unless otherwise stated. Then the Margulis-Ruelle inequality claims

$$h(\mu) \le \min\Big(\sum_{i=1}^{r^+(\mu)} n_i\chi_i(\mu), \sum_{i=r(\mu)-r^-(\mu)+1}^{r(\mu)} -n_i\chi_i(\mu)\Big).$$

Hence $r^+(\mu), r^-(\mu) > 0$. μ is called hyperbolic if $\chi^0 = \emptyset$. According to Oseledec [**Ose**] there exists a Borel f-invariant set $\Gamma \subset M, \mu(\Gamma) = 1$ with the following properties.

$$\begin{aligned}
&T(x) = \sum_1^r \oplus U_i(x), \quad \mathrm{dim} U_i(x) = n_i, \quad i = 1, ..., r(\mu),\\
&\lim_{m\to\infty} \frac{1}{m} \log ||D(f^m(x))(u)|| = \chi_i(\mu), \quad u \in U_i(x)\backslash\{0\}, \quad i = 1, ..., r(\mu),\\
&\lim_{m\to\infty} \frac{1}{m} \log ||D(f^{-m}(x))(u)|| = -\chi_i(\mu), \quad u \in U_i(x)\backslash\{0\}, \quad i = 1, ..., r(\mu),\\
&x \in \Gamma.
\end{aligned} \tag{4.1}$$

Furthermore at each $x \in \Gamma$ we have the following filtration of the stable and the unstable manifolds

$$\begin{aligned}
&W_i^s(x,\mu) = \{y: \quad y \in M, \quad \limsup_{m\to\infty} \frac{1}{m} \log \operatorname{dist}(f^m(y), f^m(x)) \le \chi_{r-i+1}(\mu)\}, \quad i = 1, ..., r^-(\mu),\\
&W_1^s(x,\mu) \subset \cdots \subset W_{r^-(\mu)}^s(x,\mu) = W^s(x,\mu),\\
&W_i^u(x,\mu) = \{y: \quad y \in M, \quad \limsup_{m\to\infty} \frac{1}{m} \log \operatorname{dist}(f^{-m}(y), f^{-m}(x)) \le -\chi_i(\mu)\}, \quad i = 1, ..., r^+(\mu),\\
&W_1^u(x,\mu) \subset \cdots \subset W_{r^+(\mu)}^u(x,\mu) = W^u(x,\mu).
\end{aligned} \tag{4.2}$$

Each $W_i^s(x,\mu), W_j^u(x,\mu)$ is an immersed $C^{1,\theta}$ submanifold of M passing through x such that

$$T_x W_i^s(x,\mu) = \sum_{l=r(\mu)-i+1}^{r(\mu)} \oplus U_l(x), \quad i = 1, ..., r^-(\mu),$$

$$T_x W_i^u(x,\mu) = \sum_{l=1}^{i} \oplus U_l(x), \quad i = 1, ..., r^+(\mu).$$

This result is basically due to [**Pe1**]. See also Ruelle [**Rue1**]. If μ is hyperbolic then the neighborhood of each $x \in \Gamma$ is diffeomorphic to $W^s(x,\mu) \times W^u(x,\mu)$. In what follows we show the existence of a finer (strict) decomposition of $W^s(x,\mu), W^u(x,\mu)$ as in Theorem 3.2. If $0 \in \chi(\mu)$ then at each $x \in \Gamma$ there exists locally a center manifold $W^c(x,\mu)$ immersed in M such that

$$T_x W^c(x,\mu) = U_{r^+(\mu)+1}(x), \quad x \in \Gamma. \tag{4.3}$$

Moreover each neighborhood of $x \in \Gamma$ is diffeomorphic to $W^s(x,\mu) \times W^c(x,\mu) \times W^u(x,\mu)$. The proof of these results are along the line of the proof of Theorem 3.2. This is possible if we follow the ideas and results in [**F-H-Y**]. The case $0 \in \chi(\mu)$ is handled in the same way as in the proof of the center manifold [**Shu**, Th. IV.1]. Since the proofs in [**F-H-Y**] assume that $f \in \mathrm{Diff}^{1,\theta}(M)\theta \in (0,1]$ we adopt this assumption.

Theorem 4.4. *Let M be a compact Riemannian manifold and assume that $f \in \mathrm{Diff}^{1,\theta}(M), \theta \in (0,1]$. Suppose that $\mu \in \mathcal{E}$ and $h(\mu) > 0$. Let $\chi(\mu) = \{\chi_1(\mu), ..., \chi_r(\mu)\}, \quad \chi_1(\mu) > ... > \chi_r(\mu)$ be the spectrum of μ. Assume that $\Gamma \subset M$ is an f-invariant Borel set with $\mu(\Gamma) = 1$ which satisfies the Oseledec decomposition* (4.1). *Then* (4.2) *holds. Furthermore for each $x \in \Gamma$ there exist C^1 stable and unstable manifolds $V_i^-(x,\mu), V_j^+(x,\mu)$ immersed in M satisfying the following conditions.*

$$\begin{aligned}
&V_{r^-(\mu)-i+1}^-(x,\mu) \subset W_i^s(x,\mu), \quad T_x V_i^-(x,\mu) = U_{r(\mu)-r^-(\mu)+i}(x), \quad f(V_i^-(x,\mu)) = V^-(f(x),\mu),\\
&i = 1, ..., r^-(\mu),\\
&W_i^s(x,\mu) = V_{r^-(\mu)}^-(x,\mu) \times \cdots \times V_{r^-(\mu)-i+1}^-(x,\mu), \quad i = 1, ..., r^-(\mu),\\
&V_i^+(x,\mu) \subset W_i^u(x,\mu), \quad T_x V_i^+(x) = U_i(x), \quad f(V^+(x,\mu)) = V^+(f(x),\mu), \quad i = 1, ..., r^+(\mu),\\
&W_i^u(x,\mu) = V_1^+(x,\mu) \times \cdots \times V_i^+(x,\mu), \quad i = 1, ..., r^+(\mu).
\end{aligned}$$

Assume that μ is hyperbolic then for each neighborhood $x \in \Gamma$ is diffeomorphic to $W^s(x,\mu) \times W^u(x,\mu)$. If $0 \in \chi(\mu)$ then through each $x \in \Gamma$ passes the center manifold $W^c(x,\mu)$ satisfying the condition (4.3). *Each neighborhood $x \in \Gamma$ is diffeomorphic to $W^s(x,\mu) \times W^c(x,\mu) \times W^u(x,\mu)$.*

We now recall the results of Ledrappier and Young [**L-Y** II]. Let M be a compact smooth manifold of dimension n and assume that $f \in \mathrm{Diff}^2(M)$. Let the assumptions of Theorem 4.4 hold. Then one can define $h_i^u(\mu)$-the local entropy of f along $W_i^u(x,\mu)$ following [**B-K**]. Denote by $\mu_{u,i}^x$ the conditional measure induced by μ on $W_i^u(x,\mu)$ in a small neighborhood of x. See [**Rok**] and the remarks in [**L-Y** I]. For $y \in W_i^u(x,\mu)$ let $d_{u,i}(x,y)$ be the distance between x and y induced by the Riemannian metric on $W_i^u(x,\mu)$. Set

$$B_{u,i}(x,m,\epsilon) = \{y \in W_i^u(x,\mu): \quad d_{u,i}(f^k(x), f^k(y)) \le \epsilon, \quad 0 \le k < m\}.$$

Note that $B_{u,i}(x,1,\epsilon)$ is the closed ball in $W_i^u(x,\mu)$ of radius ϵ centered in x with respect to the Riemannian metric on $W_i^u(x,\mu)$. It is shown in [**L-Y** II] that μ a.e. one has the limit

$$\lim_{m\to\infty} -\frac{1}{m}\log \mu_{u,i}^x B_{u,i}(x,m,\epsilon) = h_i^u(\mu), \quad i=1,...,r^+(\mu). \tag{4.5}$$

Furthermore, the $\mu_{u,i}^x$ Hausdorff dimension of $W_i^u(x,\mu)$ is equal to

$$\delta_i^u(\mu) = \sum_{j=1}^{i} \frac{h_j^u(\mu)-h_{j-1}^u(\mu)}{\chi_j(\mu)}, \quad i=1,...,r^+(\mu). \tag{4.6}$$

Here $h_0^u(\mu)=0$. More precisely μ a.e.

$$\lim_{\epsilon\to 0^+} \frac{\log \mu_{u,i}^x B_{u,i}(x,1,\epsilon)}{\log \epsilon} = \delta_i^u(\mu), \quad i=1,...,r^+(\mu). \tag{4.7}$$

Similarly one defines $h_j^s(\mu), \delta_j^s(\mu), j=1,...,r^-(\mu)$.

Theorem 4.8. *Let the assumptions of Theorem 4.4 hold. Assume furthermore that* $f\in \mathrm{Diff}^2(M)$. *Set*

$$\begin{aligned} h_i^+(\mu) &= h_i^u(\mu)-h_{i-1}^u(\mu), \quad i=1,...,r^+(\mu),\\ h_i^-(\mu) &= h_i^s(\mu)-h_{i-1}^s(\mu), i=1,...,r^-(\mu). \end{aligned} \tag{4.9}$$

Assume in addition that μ is hyperbolic. Then

$$HD(\mu) = \sum_{i=1}^{r^+(\mu)} \frac{h_i^+(\mu)}{\chi_i(\mu)} - \sum_{i=1}^{r^-(\mu)} \frac{h_i^-(\mu)}{\chi_{r^+(\mu)+i}(\mu)}. \tag{4.10}$$

Proof. As μ is hyperbolic Theorem F in [**L-Y** II] claims that μ a.e.

$$\limsup_{\epsilon\to 0^+} \frac{\log \mu B(x,\epsilon)}{\log \epsilon} \le \delta_{r^+(\mu)}^u(\mu) + \delta_{r^-(\mu)}^s(\mu).$$

The recent results of Barreira, Pesin and Schmeling [**B-P-S**] imply μ a.e. the inequality

$$\liminf_{\epsilon\to 0^+} \frac{\log \mu B(x,\epsilon)}{\log \epsilon} \ge \delta_{r^+(\mu)}^u(\mu) + \delta_{r^-(\mu)}^s(\mu).$$

Hence (4.10) holds. $\diamond$

We view $h_i^+(\mu), h_j^-(\mu)$ as f-entropies along $V_i^+(x,\mu), V_j^-(x,\mu)$ respectively. Let $f\in \mathrm{Diff}^1(M)$ be an Axiom A diffeomorphism. Then any $\mu\in\mathcal{E}$ is hyperbolic. Moreover, the μ stable and unstable manifolds which are defined for $x\in\Omega(f)$ are equal to the stable and the unstable manifolds of f.

Corollary 4.11. *Let M be a smooth compact manifold and assume that* $f\in\mathrm{Diff}^2(M)$ *satisfies the Axiom A. Suppose that* $\mu\in\mathcal{E}$. *Then* (4.10) *holds.*

Suppose that $f\in\mathrm{Diff}^1(M)$ is a Strict Axiom A diffeomorphism with the decomposition (3.1). It follows that that for any $\mu\in\mathcal{E}$ the partition of $T_xM, x\in\Gamma\subset\Omega(f)$ to the Oseledec spaces (4.1) is obtained by splitting the corresponding subspaces of (3.1). Assume finally that f is a Strong Axiom A diffeomorphism.

Let $\Omega(f) = \cup_{j=1}^k \Lambda_j$ be the decomposition to the basic sets. It then follows that (3.1) gives the Oseledec spaces. In that case

$$W_i^s(x,\mu) = W_i^s(x), \quad i = 1, ..., r_j^-,$$
$$W_i^u(x,\mu) = W_i^u(x), \quad i = 1, ..., r_j^+,$$
$$x \in \Lambda_j, \quad j = 1, ..., k.$$

From the decomposition in Theorem 3.2 we obtain the equalities

$$V_i^-(x,\mu) = V_i^-(x), \quad \mu \in \mathcal{E}_j, \quad i = 1, ..., r_j^-,$$
$$V_i^+(x,\mu) = V_i^+(x), \quad \mu \in \mathcal{E}_j, \quad i = 1, ..., r_j^+,$$
$$x \in \Lambda_j, \quad j = 1, ..., k.$$

Note that each $V_i^-(x), V_j^+(x)$ are one dimensional.

Define on $\Omega(f)$ the following functions

$$\phi_i^u(x) = \log |Df|_{E_i^u}|, \quad i = 1, ..., r_j^+,$$
$$\phi_i^s(x) = -\log |Df|_{E_i^s}|, \quad i = 1, ..., r^- j,$$
$$x \in \Lambda_j, \quad j = 1, ..., k.$$

Thus for any $\mu \in \mathcal{E}_j$ we have the equalities

$$\lambda_i(\mu) = \chi_i(\mu) = \int \phi_i^u d\mu, \quad i = 1, ..., r_j^+,$$
$$\lambda_{r^+ + i}(\mu) = \chi_{r^+ + i}(\mu) = -\int \phi_i^s d\mu, \quad i = 1, ..., r_j^-.$$

As in §1 we consider the supremum

$$\delta_{i,j}^+ = \sup_{\mu \in \mathcal{E}_j} \frac{h_i^+(\mu)}{\lambda_i(\mu)}, \quad i = 1, ..., r_j^+,$$
$$\delta_{i,j}^- = \sup_{\mu \in \mathcal{E}_j} \frac{h_i^-(\mu)}{-\lambda_{r_j^+ + i}(\mu)}, \quad i = 1, ..., r^- j.$$

We conjecture that the following equalities hold.

$$HD(\Lambda_j) = \sum_{i=1}^{r^+} \delta_{i,j}^u + \sum_{i=1}^{r^-} \delta_{i,j}^u, \quad j = 1, ..., k.$$

As a first step toward proving this conjecture one should consider it for the class of Strong Axiom A diffeomorphsims given by Theorem 3.9.

§5. Strict and strong orbit hyperbolicity for endomorphisms

Let M be a compact smooth Riemannian manifold. Denote by $\mathrm{End}^r(M), r \geq 0$ the set of C^r endomorphisms $f : M \to M$. An f-invariant set: $X \supset f(X)$ is called hyperbolic if there is a continuous decomposition $T_X M = E^u \oplus E^s$ satisfying the standard assumptions (3.1) with $r^- = r^+ = 1$. It is well known that contrary to the diffeomorphism case a closed f-invariant hyperbolic set X is not structurally stable. See for example [**M-P**] and [**Prz**]. It was pointed out in [**M-P**] that one can use results for the hyperbolic sets of diffeomorphisms when f is a cover map by considering the lifting of f to the universal cover. It is more convenient to consider the following construction. See for example [**Q-Z**] and the references therein and [**Och**].

Let $X \subset M$ be a closed f-invariant set. Denote by X^f the full orbit space with the following metric:

$$X^f = \{x = (x_i)_{-\infty}^{\infty} : \quad x_i \in X, \quad f(x_i) = x_{i+1}, \quad i \in \mathbf{Z}\};$$
$$\mathrm{dist}(x, y) = \sum_{i \in \mathbf{Z}} \frac{d(x_i, y_i)}{2^{|i|}}, \quad x, y \in X^f;$$
$$\pi : X^f \to X, \quad \pi(x) = x_0, x = (x_i)_{i \in \mathbf{Z}} \in X^f.$$

Note that X^f is a compact space. Define

$$W^s(X) = \{x_0 \in M : \quad \lim_{m \to \infty} d(f^m(x_0), X) = 0\},$$
$$W^u(X) = \{x_0 \in M : \quad x_0 = \pi(x), x = (x_i)_{i \in \mathbf{Z}} \in M^f, \lim_{m \to -\infty} d(x_m, X) = 0\}.$$

Assume in addition that Y is a closed set f-invariant set. Then $X^f \cup Y^f \subset (X \cup Y)^f$. In particular for $x \in X^f, y \in Y^f$ we can define $\mathrm{dist}(x, y)$ as above.

As usual let $\sigma : X^f \to X^f$ be the shift map. Then $f : X \to X$ is a factor of π, i.e. $f\pi = \pi\sigma$. Let $E = \pi_X^* M$ be the pull back of the tangent bundle of $T_X M$ by $\pi : X^f \to X$. Denote by

$$E_x = \pi_x^* T_X M \overset{\pi_*}{\underset{\pi_x^*}{\rightleftarrows}} T_{x_0} M$$

the natural isomorphism between fibres E_x and $T_{x_0} M$:

$$\xi = (x, v) \overset{\pi_*}{\underset{\pi_x^*}{\rightleftarrows}} v, \quad x \in X^f, \xi \in E_x, v \in T_{x_0} M.$$

A fibre preserving map on E with respect to σ is defined as

$$\pi_{\sigma(x)}^* \circ Df \circ \pi_* : E_x \to E_{\sigma(x)}, \quad x \in X^f.$$

By abusing the notation we denote by Df the above cocycle on E. We call X orbit hyperbolic if there exits a continuous decomposition the vector bundle $E = E^u \oplus E^s$ over X^f invariant under D which satisfies (3.1) with $r^+ = r^- = 1$. Note that (3.1) yields that $det(Df(x)) \neq 0$. Clearly, an f-invariant hyperbolic set X is orbit hyperbolic. As in §3 we define a strict (strong) hyperbolicity and a strict (strong) orbit hyperbolicity of f invariant set X for real or complex endomorphism f of M.

As in the case of diffeomorphisms one can show an f-invariant compact orbit hyperbolic set is structurally stable. See for example [**C-H-Y**], [**Liu**] and [**Rue3**]. Using the arguments of §3 for structural stability of strict (strong) hyperbolic sets we obtain.

Theorem 5.1 *Let M be a compact smooth manifold and $f \in \mathrm{End}^1(M)$. Assume that $X \subset M$ is a compact set, $f(X) = X$ and suppose that X is strictly (strongly)orbit hyperbolic set. Then there exists a neighborhood O of X and $\epsilon_0 > 0$ satisfying the following conditions. For any $0 < \epsilon < \epsilon_0$ there exists an f-neighborhood $U_{f,\epsilon} \subset \mathrm{End}^1(M)$ such that for any $g \in U$ there exists a unique compact set $Y \subset O, g(Y) = Y$ with the following properties. g is strictly (strongly) hyperbolic on Y^g such that the partition of $E(g)$ is conformal with the partition $E(f)$, i.e. $r^+(X) = r^+(Y), r^-(X) = r^-(Y)$. Moreover, there is a homeomorphism $\phi : X^f \to Y^g$ which commutes with the corresponding shifts on X^f, Y^g such that*

$$\mathrm{dist}(x, \phi(x)) < \epsilon, \quad x \in X^f, \phi(x) \in Y^g.$$

Recall that a compact invariant hyperbolic set X with respect to a C^1-endomorphism $f : M \to M$ is called an expander if $E = E^u$. Generalizing the results of [**M-P**] and [**Prz**] it was shown by Zhang [**Zha**] that a compact invariant expanding set is structurally stable. We thus deduce

Theorem 5.2 *Let M be a compact smooth manifold and $f \in \mathrm{End}^1(M)$. Assume that X is a compact set, $f(X) = X$ and suppose that X is strictly (strongly) expanding set. Then there exists a neighborhood O of X and $\epsilon_0 > 0$ satisfying the following conditions. For any $0 < \epsilon < \epsilon_0$ there exists an f-neighborhood $U_{f,\epsilon} \subset \mathrm{End}^1(M)$ such that for any $g \in U_{f,\epsilon}$ there exists a unique compact set $Y \subset O, g(Y) = Y$ with the following properties. g is strictly (strongly) expanding on Y such that the partition of $E(g)$ is conformal with the partition $E(f)$, i.e. $r^+(X) = r^+(Y)$. Moreover, there is a homeomorphism $\phi : X \to Y$ which commutes with $f|_X, g|Y$ such that $d(x, \phi(x)) < \epsilon, x \in X$.*

Definition 5.3. *Let M be a compact smooth manifold. $f \in \mathrm{End}^1(M)$ is called a Strict (Strong) Axiom A endomorphism if the following conditions hold.*
(a) $\Omega(f) = \cup_{i=1}^k \Lambda_i, \Lambda_i \cap \Lambda_j = \emptyset, 1 \le i < j \le k$, *each Λ_i is an f-invariant closed strict (strong) orbit hyperbolic set such that $f : \Lambda_i \to \Lambda_i$ is topologically transitive*;
(b) $\bar{P}(f) = \Omega(f)$.

Assume that $f \in \mathrm{End}^1(M)$ is an Axiom A endomorphism. Then f has no cycle property if the cycle condition (3.7) does not hold. Following [**C-H-Y**] and [**Liu**] we have the following orbit Ω-stability result.

Theorem 5.4 *Let M be a smooth compact manifold. Assume that $f \in \mathrm{End}^1(M)$ is a Strict (Strong) Axiom A endomorphsim with no cycle property. Then there exists $\epsilon_0 > 0$ so that for any $0 < \epsilon < \epsilon_0$ there exists an f-neighborhood $U_{f,\epsilon} \subset \mathrm{End}^1(M)$ such that any $g \in U_{f,\epsilon}$ is an Axiom A endomorphism. There is a homeomorphism $\phi : \Omega(f)^f \to \Omega(g)^g$ which commutes with the corresponding shifts on $\Omega(f)^f, \Omega(g)^g$ and*

$$\mathrm{dist}(x, \phi(x)) < \epsilon, \quad x \in \Omega(f)^f, \phi(x) \in \Omega(g)^g.$$

g is strictly (strongly) hyperbolic on each basic set $\Lambda_i(g)$ such that the partition of $E(g)$ is conformal with the partition $E(f)$ on $\Lambda_i(f)$.

§6. Dynamics of certain polynomial maps in $\mathbf{C}^2$

The dynamics of a rational map $f : \mathbf{CP} \to \mathbf{CP}$ is a well studied subject. The main notions here are the Julia set $J(f)$ and the Fatou domains. Recall that the Julia set is the closure of all repelling periodic points while the number of non-repelling cycles is at most $2deg(f) - 2$. (Here by $deg(f)$ we denote the degree of f.) Consult with [**Bea**] for a good reference on the dynamics of rational maps. Julia set $J(f)$ is called hyperbolic if there exists $m \ge 1$ so that

$$|Df^m(z)| \ge \rho > 1, z \in J(f).$$

A rational map f is called *hyperbolic* if $deg(f) \ge 2$ and $J(f)$ is hyperbolic. The following lemma is known (e.g. [**Bea**, Ch.7-8]).

Lemma 6.1. *Let $f : \mathbf{CP} \to \mathbf{CP}$ be a rational hyperbolic map. Then*
(a) All $\Lambda_1, ..., \Lambda_l$ ($l \le 2deg(f) - 2$) non-repelling cycles are attracting;
(b) $\mathbf{CP} \backslash \cup_{i=0}^l \Lambda_i$ is the domain of attraction of the cycles $\Lambda_1, ..., \Lambda_l$.

Let $d \ge 2$ be an integer and denote by $\mathcal{U}^d$ the space of all rational maps of degree d. Note that $\mathcal{U}^d = \mathbf{CP}^{2d+1} \backslash V$ where V is corresponds to the variety to rational maps of degree less than d.

Theorem 6.2. *Let $f : \mathbf{CP} \to \mathbf{CP}$ be a rational hyperbolic map. Then $\Omega(f) = \cup_{i=0}^l \Lambda_i(f)$ and $\Lambda_0(f), ..., \Lambda_l(f)$ satisfy the no cycle condition. There exists $\epsilon_0 > 0$ so that for any $0 < \epsilon < \epsilon_0$ there exists an f-neighborhood $U_{f,\epsilon} \subset \mathcal{U}^d$ such that any $g \in U_{f,\epsilon}$ is a rational hyperbolic map with $\Omega(g) = \cup_{i=0}^l \Lambda_i(g)$. Moreover, there is a homeomorphism $\phi : \Omega(f) \to \Omega(g)$ which commutes with $f|\Omega(f), g|\Omega(g)$ such that $d(x, \phi(x)) < \epsilon, x \in \Omega(f)$.*

Proof. The following equalities imply straightforward that $\Lambda_0(f),...,\Lambda_i(f)$ satisfy the no cycle condition.

$$W^s(\Lambda_0(f)) = \Lambda_0(f),$$
$$W^u(\Lambda_i(f)) = \Lambda_i(f), \quad i = 1,...,l.$$

We now show Ω-stability of f. Choose an f neigborhood $U \subset \mathcal{U}^d$ so that for any $g \in U$ g has l attracting cycles $\Lambda_1(g),...,\Lambda_l(g)$ which are perturbation of the attracting cycles $\Lambda_1(f),...,\Lambda_l(f)$. Furthermore, there exists a neighborhood O of $J(f)$ such that $\mathbf{CP}\backslash O$ is in the domain of attraction of $\Lambda_1(g),...,\Lambda_l(g)$. By choosing U small enough O can be chosen as small as needed. Theorem 5.2 implies the existence of a sufficiently small neighborhood $O \supset J(f)$ that contains every $J(g), g \in U_{f,\epsilon}$, which will repel any point $x \in O\backslash J(g)$ outside O. Combine the above facts with Theorem 5.2 to deduce the theorem. $\diamond$

We now consider a polynomial map $f : \mathbf{C}^2 \to \mathbf{C}^2$. The dynamics of a general polynomial map is terra incognita. Note that contrary to the one dimensional case there exist nonconstant polynomial maps of $\mathbf{C}^2$ which are not proper. Assume that f is proper. Then the study of the dynamics of f is divided into two cathegories. The first one is when f is a polynomial automorphism of $\mathbf{C}^2$. We discussed some of the dynamical properties of these maps in §2. See [**F-M**] and [**B-S**, 1-3] and the references there. We are not going to discuss this case here. The other case which was studied is when f lifts to a holomorphic map $\tilde{f} : \mathbf{CP}^2 \to \mathbf{CP}^2$. See for example [**F-S**], [**H-P**] and [**Hei**]. Recall that $\tilde{f}$ is holomorphic iff $f(z_1, z_2) = (f_1(z_1, z_2), f_2(z_1, z_2))$ satisfy the conditions

$$\begin{aligned}
&f_1(z_1, z_2) = g_1(z_1, z_2) + h_1(z_1, z_2), \quad g_1 = \prod_{i=1}^{d}(\alpha_i z_1 + \beta_i z_2), \quad d = deg(g_1) > deg(h_1),\\
&f_2(z_1, z_2) = g_2(z_1, z_2) + h_2(z_1, z_2), \quad g_2 = \prod_{i=1}^{d}(\gamma_i z_1 + \delta_i z_2), \quad d = deg(g_2) > deg(h_2),\\
&\frac{\alpha_i z_1 + \beta_i z_2}{\gamma_j z_1 + \delta_j z_2} \neq Constant, \quad i, j = 1,...,d.
\end{aligned} \tag{6.3}$$

This claim follows quite straightforward if one recalls that $\mathbf{C}^2$ have projective coordinates (z_0, z_1, z_2) so that $\mathbf{C}^2$ is given by the coordinates $(1, z_1, z_2)$. The line at infinity $\mathbf{CP}$ (the Riemann sphere) is given by the projective coordinates $(0, z_1, z_2)$. Then the restricition of $\tilde{f}$ to the line at infinity is given by the rational map:

$$q(z) = \frac{g_1(z, 1)}{g_2(z, 1)}, z \in \mathbf{C}. \tag{6.4}$$

We now discuss briefly a few possible definitions of the Julia set $J(f) \subset \mathbf{C}^2$ of a polynomial map f which satisfies conditions (6.3). The first natural definition follows the one dimensional case [**F-S**]. Let $J_1(\tilde{f}) \subset \mathbf{CP}^2$ be the closed set where the sequence $\tilde{f}^m, m = 1,...,$ is not normal. According to [**F-S**] $J_1(\tilde{f})$ is always connected. Consider the map $Q = (z_1^2, z_2^2)$. It is not hard to show in the homogeneous coordinates we have the following characterization of $J_1(\tilde{Q})$:

$$J_1(\tilde{Q}) = \{z : \quad z = (z_0, z_1, z_2) \in \mathbf{CP}^2, \quad z_p = |z_q| = 1 > |z_r|, \quad \{p, q, r\} = \{0, 1, 2\}\}.$$

In particular $J_1(\tilde{Q}) \cap \mathbf{C}^2 \supset J(z_1^2) \times J(z_2^2) = S^1 \times S^1$. Since $J_1(\tilde{f})$ is connected and as $J_1(\tilde{f})$ must always contain the one-dimensional Julia set of $q(z_1)$ it follows that $J_1(f) = J_1(\tilde{f}) \cap \mathbf{C}^2$ will be an unbounded set. We expect the Julia set of f to be bounded. Moreover we want:

$$J(f) = J(f_1) \times J(f_2), \quad f(z_1, z_2) = (f_1(z_1), f_2(z_2)), \quad deg(f_1) = deg(f_2) > 1. \tag{6.5}$$

Let $S^4 = \mathbf{C}^2 \cup \infty$ be a one point compactification of $\mathbf{C}^2$. Then f lifts to a continuous map $\hat{f} : S^4 \to S^4$ where $\hat{f}(\infty) = \infty$. In fact ∞ is a superattracting point of $\hat{f}$. Let $A(f, \infty) \subset \mathbf{C}^2$ be the domain of attraction of ∞. It follows that $\partial A(f, \infty)$ is a compact totally invariant set of f:

$$f(A(f, \infty)) = A(f, \infty) = f^{-1}(A(f, \infty)).$$

Moreover, in one dimensional case $\partial A(f,\infty) = J(f)$. It is easy to see that $\partial A(Q,\infty)$ is much bigger than $S^1 \times S^1 = J(Q_1) \times J(Q_2)$.

As in [**H-P**] one can construct certain invariant currents or measure, e.g. the equlibrium measure of $A(f,\infty)^c = \mathbf{C}^2 \backslash A(f,\infty)$, and declare that their support (which is contained in $\partial A(f,\infty)$) to be the Julia set of f. (This definition was suggested by J. Hubbard). It takes some work to show that (6.5) holds in this case.

Another approach was suggested in [**Hei**]. One defines the Julia set by the nonnormality of iterations $f^m, m = 1, ...,$ restricted to any possible one dimensional foliation of a neighborhood of $x \in \mathbf{C}^2$. It is shown in [**Hei**] that this definition satisfies the property (6.5). Yet Heinemann definition seems to be unconstructive. We are looking for a simple dynamic definition of the Julia set of f. Let $z \in \mathbf{C}^2$ be a periodic point of f of period m. Then $z, f(z), ..., f^{m-1}(z)$ is called a repelling cycle if the two eigenvalues of $Df^m(z)$ are outside the closed unit disk in $\mathbf{C}$.

Definition 6.6. *Let f be a polynomial map of $\mathbf{C}^2$ of the form* (6.3). *Then $J(f)$-the Julia set of f is defined to be the closure of all periodic repelling points.*

It can be easily shown that $J(f)$ must be contained in the Julia set defined in [**Hei**]. It is nontrivial to show that $J(f)$ is an infinite set and $J(f) \subset \partial A(f,\infty)$. Using the structural stability results of §5 we will exhibit an open set of polynomial maps g for which $J(g)$ is a homeomorphic to $J(f)$ given by (6.5).

Fix an integer $d > 1$. Consider all polynomial maps f of $\mathbf{C}^2$ satisfying $f = (f_1, f_2), deg(f_1), deg(f_2) \le d$. Then the space of these polynomials is a linear space $L(d)$ isomorphic to $\mathbf{C}^{r(d)}$. Denote by $L_1(d) \subset L(d)$ the set of all polynomials f satisfying the conditions (6.3). It is straightforward to show that $L(d)\backslash L_1(d)$ is an algebraic variety of $L(d)$. Hence $L_1(d)$ is an open dense set in $L(d)$. Assume that $f \in L_1(d)$. Then by an f neighborhood $U \subset L_1(d)$ we will mean an open neighborhood of f in the standard topology of $\mathbf{C}^{r(d)}$.

Theorem 6.7. *Let $f_1, f_2 : \mathbf{C} \to \mathbf{C}$ be two polynomial maps of degree $d > 1$. Assume that $J(f_1), J(f_2)$ are hyperbolic. Consider $f(z_1, z_2) = (f_1(z_1), f_2(z_2)$. Then there exists a neighborhood $O \supset J(f_1) \times J(f_2)$ and $\epsilon_0 > 0$ so that the following conditions hold. For any $0 < \epsilon < \epsilon_0$ there exists an f-neighborhood $U_{f,\epsilon} \subset L_1(d)$ so that for any $g \in U_{f,\epsilon}$ there exists a unique closed set $X(g) \subset O, X \subset J(g)$, such that $g(X(g)) = g^{-1}(X(g)) = X(g)$. g is expanding on $J(g)$. Moreover, there is a homeomorphism $\phi : J(f) \to X(g)$ which commutes with $f|J(f), g|X(g)$ such that $d(x, \phi(x)) < \epsilon, x \in J(f)$.*

Suppose furthermore that

$$\Omega(f_i) = \cup_{j=0}^{l_i} \Lambda_j(f_i), \quad \Lambda_0(f_i) = J(f_i), \quad \Lambda_{l_i}(f_i) = \infty, \quad i = 1, 2,$$

such that none of $\Lambda_j(f_i), j = 1, ..., l_{i-1}, i = 1, 2,$ are super-attractive. Then $X(g) = J(g), g \in U_{f,\epsilon}$ and

$$\Omega(\hat{f}) = \{\infty\} \cup_{i,j=0}^{l_1-1, l_2-1} \Lambda_i(f_1) \times \Lambda_j(f_2).$$

For any $g \in U_{f,\epsilon}$, $\Omega(\hat{g})$ has $1 + l_1 \times l_2$ basic sets. $1 + (l_1 - 1) \times (l_2 - 1)$ attracting cycles (including ∞), one expanding set $J(g)$ and $l_1 + l_2 - 2$ orbit hyperbolic sets with one expanding and one contracting direction. Finally, there is a homeomorphism $\phi : \Omega(\hat{f})^{\hat{f}} \to \Omega(\hat{g})^{\hat{g}}$ such that $\mathrm{dist}(x, \phi(x)) < \epsilon$.

Proof. Observe first that $J(f) = J(f_1) \times J(f_2)$ is a compact forward and backward f-invariant set. Use Theorem 5.1 to deduce that $J(f)^f$ is orbit stable. Since $deg(g) = d^2, g \in L_1(d)$, it follows that for $g \in U_{f,\epsilon}$ the set $X(g)$ is backward and forward g-invariant. Use Theorem 5.2 to deduce the existence of the homeomorphism $\phi : J(f) \to X(g)$ so that $d(x, \phi(x)) < \epsilon, x \in J(f)$. As $J(f)$ is the closure of f-periodic points it follows that $X(g)$ is the closure of g-periodic points in $X(g)$. Clearly, every periodic point in $X(g)$ is repelling. Hence $X(g) \subset J(g)$.

Assume now that f_1, f_2 are hyperbolic polynomial maps. Then $\hat{f}$ has an atractive point at ∞ and $(l_1 - 1) \times (l_2 - 1)$ attracting cyles $\Lambda_i(f_1) \times \Lambda_j(f_2), i = 1, ..., l_1 - 1, j = 1, ..., l_2 - 1$. Moreover any $x \in \mathbf{C}^2 \backslash \cup_{i,j=0}^{l_1-1} \Lambda_i(f_1) \times \Lambda_j(f_2)$ is in the domain of the attraction of the attracting cycles. Hence there exists a neighborhood $U \subset L_1(d)$ so that any $g \in U$ the map $\hat{g}$ has an attracting fixed point ∞ and $(l_1 - 1) \times (l_2 - 1)$

attracting cycles which are the corresponding perturbations of the attracting cycles of $\hat{f}$ in $\mathbf{C}^2$. Thus, there exists open neighborhoods

$$\begin{aligned} &O_0 \supset J(f_1) \times J(f_2), \\ &O_{1,j} \supset J(f_1) \times \Lambda_j(f_2), \quad j = 1, ..., l_2 - 1, \\ &O_{2,j} \supset \Lambda_j(f_1) \times J(f_2), \quad j = 1, ..., l_1 - 1, \end{aligned} \tag{6.8}$$

so that any $x \in \mathbf{C}^2 \backslash O_0 \cup_{j=1}^{l_2-1} O_{1,j} \cup_{j=1}^{l_1-1} O_{2,j}$ is in the domain of the attraction of the above $1+(l_1-1)\times(l_2-1)$ attracting cylces of $\hat{g}$. Choosing U small enough we can make the neighborhoods (6.8) as small as needed. Suppose that all the finite attracting cycles of f_1, f_2 are not super attracting. Then all the closed sets $J(f_1) \times \Lambda_i(f_2), \Lambda_j(f_1) \times J(f_2) \subset \mathbf{C}^2$ are hyperbolic according to (3.1). (Df is invertible on these sets.) We then can apply the structural stability results of Theorem 5.1 for all $1 + (l_1 - 1) \times (l_2 - 1)$ hyperbolic sets for some neighborhoods given by (6.8). Hence $\Omega(\hat{f})$ is orbit structurally stable. As $X(g)$ is the unique expanding set of $\Omega(\hat{g})$ it follows that $J(g) = X(g)$. $\diamond$

Assume the conditions of Theorem 6.7. Then the set $J(f)$ is strongly hyperbolic if there exists $m \geq 1$ that if the following condition hold

$$\max_{z \in J(f_p)} |(f_p^m)'(z)| < \min_{z \in J(f_q)} |(f_q^m)(z)|, \quad \{p, q\} = \{1, 2\}. \tag{6.9}$$

In this case $X(g), g \in U_{f,\epsilon}$ is strongly hyperbolic.

We close our paper with another perturbation result. Consider a map f given by (6.3) where $h_1 = h_2 = 0$. That is f is a homogeneous map of degree d:

$$f(t(z_1, z_2)) = t^d f(z_1, z_2), \quad t, z_1, z_2 \in \mathbf{C}.$$

Observe first that 0 is a super attracting point of f. Let L be a line in $\mathbf{C}^2$ through the origin. This line is given by homogenuous coordinates $(z_1, z_2) \in \mathbf{CP}$. Note that $f(L)$ is another line L' whose homogeneous coordinates are $(g_1(z_1, z_2), g(z_1, z_2))$. That is, on the space of all lines through the origin in $\mathbf{C}^2$, which is identical to the Riemann sphere $\mathbf{CP}$, f acts a rational function q given by (6.4). On each line L the map $f : L \to L'$ is of the form $z \to K(L)z^d$. That is f is a twisted product of the q and z^d. In particular $J(f)$ is homeomorphic to $J(g) \times S^1$. Moreover $\partial A(f, \infty)$ is homeomorphic to $\mathbf{CP} \times S^1$ and is a backward and a forward f-invariant set separating the domain of attraction of 0 and ∞.

Theorem 6.10 *Let $f : \mathbf{C}^2 \to \mathbf{C}^2$ be a homogeneous map of degree d satifying the assumptions* (6.3). *Let q be the rational map given by* (6.4). *Assume that $J(q)$ is hyperbolic. Then there exists a neighborhood $O \supset J(f)$ and $\epsilon_0 > 0$ so that the following conditions hold. For any $0 < \epsilon < \epsilon_0$ there exists an f-neighborhood $U_{f,\epsilon} \subset L_1(d)$ so that for any $g \in U_{f,\epsilon}$ there exists a unique closed set $X(g) \subset O, X \subset J(g)$, such that $g(X(g)) = g^{-1}(X(g)) = X(g)$. g is expanding on $J(g)$. Moreover, there is a homeomorphism $\phi : J(f) \to X(g)$ which commutes with $f|J(f), g|X(g)$ such that $d(x, \phi(x)) < \epsilon, x \in J(f)$.*

Assume furthermore that none of the attracting cycles $\Lambda_1(q), ..., \Lambda_l(q)$ of q are super-attracting. Then $X(g) = J(g), g \in U_{f,\epsilon}$ and

$$\begin{aligned} &\Omega(\hat{f}) = \{0\} \cup \{\infty\} \cup J(f) \cup_{i=1}^{l} \Lambda_i(f), \\ &\Lambda_i(f) \approx \Lambda_i(q) \times S^1, \quad i = 1, ..., l. \end{aligned} \tag{6.11}$$

For any $g \in U_{f,\epsilon}$ $\Omega(\hat{g})$ has $3 + l$ basic sets. Two attracting points $z(g), \infty$, one expanding set $J(g)$ and l orbit hyperbolic sets with one expanding and one contracting direction. Finally, there is a homeomorphism $\phi : \Omega(\hat{f})^{\hat{f}} \to \Omega(\hat{g})^{\hat{g}}$ such that $\mathrm{dist}(x, \phi(x)) < \epsilon$.

Proof. As $J(q)$ is hyperbolic and the map $t \mapsto t^d$ is Axiom A rational map it follows that $J(f) \approx J(q) \times S^1$ is hyperbolic. Then the arguments of the proof of Theorem 6.7 imply the existence of $O \supset J(f)$ with the stated properties.

Let $z \in \partial A(f, \infty) \backslash J(f)$. That is, z is not on the line L corresponding to $J(q)$. Then $f^m(z), m = 1, ...,$ will converge to some $\Lambda_j(f) \approx \Lambda_j(q) \times S^1$. We then deduce (6.11). It is quite straightforward to show that the basic sets of $\Omega(\hat{f})$ satisfy the no periodicity condition. Assume that none of the attracting cycles of q are super-attracting. It follows that each $\Lambda_i(f), i = 1, ..., l$ is hyperbolic with one contracting and one expanding direction. Fix a neighborhood $N \supset \partial A(f, \infty)$. Then there exists $\epsilon_0 > 0$ so that for any $g \in U_{f,\epsilon}$ and any $z \in \mathbf{C}^2 \backslash N$, $f^m(z), m = 1, ...,$ will converge either to ∞ or to the unique fixed point $z(g)$ which is a perturbation of 0. That is,

$$\Omega(\hat{g}) = \{0\} \cup \{\infty\} \cup \Omega_1(\hat{g}), \quad \Omega_1(\hat{g}) \subset N.$$

By choosing N as small as we need we can use the arguments of Theorem 5.4 to deduce the orbit stability of $\Omega(\hat{f})^{\hat{f}}$. As the only expanding component of $\Omega(\hat{g})$ is in the neighborhood of $J(f)$ we deduce that $X(g) = J(g)$. $\diamond$

References

[**Bar**] L. Barreira, A non-additive thermodynamics formalism and applications to dimension theory of hyperbolic dynamical systems, *Ergod. Th. & Dynam. Sys.*, to appear.
[**B-P-S**] L. Barreira, Y. Pesin and J. Schmeling, Dimension of hyperbolic measures - a proof of the Eckmann-Ruelle conjecture, *preprint* 1996.
[**Bea**] A. F. Beardon, *Iteration of rational functions*, Springer 1991.
[**B-S1**] E. Bedford and J. Smillie, Polynomial diffeomorphisms of $\mathbf{C}^2$: Currents, equilibrium measure and hyperbolicity, *Invent. Math.* 103 (1991), 69-99.
[**B-S2**] E. Bedford and J. Smillie, Polynomial diffeomorphisms of $\mathbf{C}^2$. II: Stable manifolds and recurrence, *J. Am. Math. Soc.* 4 (1991), 657-679.
[**B-S3**] E. Bedford and J. Smillie, Polynomial diffeomorphisms of $\mathbf{C}^2$. III: Ergodicity, exponents of entropy of the equilibrium measure, *Math. Ann.* 294 (1992), 395-420.
[**B-M**] S. Besicovitch and P. A. Moran, The measure of product and cylindrical sets, *J. London Math. Soc.* 20 (1945), 110-120.
[**Bow1**] R. Bowen, *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms*, Lecture Notes Math. 470, Springer-Verlag, 1975.
[**Bow2**] R. Bowen, Hausdorff dimension of quasi-circles, *Publ. Math. IHES* 50 (1979), 11-25.
[**B-K**] M. Brin and A. Katok, On local entropy, in *Geometric Dynamics*, Springer Lect. Notes 1007 (1983).
[**C-H-Y**] Z.-P. Chen, L.-F. He and S.-L. Yang, Orbit shift Ω-stability, *Scientia Sinica* (Ser. A) 31 (1988), 512-520.
[**Fal**] K. Falconer, *Fractal Geometry, Mathematical Foundations and Applications*, Wiley, 1990.
[**F-H-Y**] A. Fathi, M. R. Herman and J. C. Yoccoz, A proof of the Pesin stable manifold theorem, in *Geometric Dynamics*, Springer Lecture Notes in Math. 1007 (1983), 177-215.
[**F-K**] H. M. Farkas and I. Kra, *Riemann Surfaces*, Springer-Verlag 1980.
[**F-S**] J. E. Fornaess and N. Sibony, Complex dynamics in higher dimension, in *Complex Potential Theory*, 131-186, P.M. Gauthier (ed.), Kluwer 1994.
[**Fri1**] S. Friedland, Computing Hausdorff dimension of subshifts using matrices, *Linear Algebra Appl.*, to appear.
[**Fri2**] S. Friedland, Discrete Lyapunov exponents and Hausdorff dimension, *preprint* 1997.
[**F-M**] S. Friedland and J. Milnor, Dynamical properties of plane polynomial diffeomorphisms, *Ergod. Th. & Dynam. Sys.* 9 (1989), 67-99.
[**G-H**] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Wiley 1978.
[**Hei**] S. M. Heinemann, Julia sets for holomorphic endomorphisms of $\mathbf{C}^n$, *Ergod. Th. & Dynam. Sys.*, to appear.
[**Hén1**] M. Hénon, Numerical study of area preserving mappings, *Q. Appl. Math.* 27 (1969), 291-312.
[**Hén2**] M. Hénon, A two-dimensional mapping with a strange attractor, *Comm. Math. Phys.* 50 (1976), 69-77.
[**H-P**] J. H. Hubbard and P. Papadopol, Supperattractive fixed points in $\mathbf{C}^n$, Indiana Math. J. 43 (1994), 321-365.

[**L-Y**] F. Ledrappier and L.-S. Young, The metric entropy of diffeomorphisms, I-II, *Annals Math.* 122 (1985), 509-574.
[**Liu**] P.-D. Liu, Stability of orbit spaces of endomorphisms, Rep. #22, Peking Univ., 1996.
[**Mañ**] R. Mañe, A proof of Pesin's formula, *Ergod. Th. & Dynam. Sys.* 1 (1981), 95-102.
[**M-P**] R. Mañe and C. Pugh, Stability of endomorphisms, *Lecture Notes in Math.* 468, pp. 175-184, Springer, 1975.
[**Man**] A. Manning, A relation between Lyapunov exponents, Hausdorff dimension and entropy, *Ergod. Th. & Dynam. Sys.* 1 (1981), 451-459.
[**M-M**] H. McCluskey and A. Manning, Hausdorff dimension for horseshoes, *Ergod. Th. & Dynam. Sys.* 3 (1983), 251-260.
[**Och**] G. Ochs, *Hyperbolische limesmengen bei iteration algebraischer funktionen*, Ph.D. thesis, University of Bremen, 1995.
[**Ose**] V. I. Oseledec, A multiplicative ergodic theorem: Lyapunov characteristic numbers for dynamical systems, *Trans. Moscow Math. Soc.* 19 (1968), 197-221.
[**Pes1**] Y. B. Pesin, Families of invariant manifolds correponding to non zero characteristic exponents, *USSR Izvestia* 10 (1976), 1261-1305.
[**Pes2**] Y. B. Pesin, Characteristic Lyapunov exponents and smooth ergodic theory, *Russian Math. Surveys* 32 (1977), 55-114.
[**Prz**] F. Przytycki, Anosov endomorphisms, *Studia Math.*, 58 (1976), 249-285.
[**Q-Z**] M. Qian and Z. Zhang, Ergodic theory for axiom A endomorphisms, *Ergod. Th. & Dynam. Sys.* 15 (1995), 161-174.
[**Rok**] V. A. Rokhlin, On the fundamental ideas in measure theory, *A.M.S. Trans. 1*, 10 (1962), 1-54.
[**Rue1**] D. Ruelle, Ergodic theory of differential dynamical systems, *Publ. Math. IHES* 50 (1979), 27-58.
[**Rue2**] D. Ruelle, Repellers for real analytic maps, *Ergod. Th. & Dynam. Sys.* 2 (1982), 99-107.
[**Rue3**] D. Ruelle, *Elements of Differentiable Dynamics and Bifurcation Theory*, Academic Press, San Diego, 1989.
[**Shu**] M. Shub, *Global Stability of Dynamical Systems*, Springer-Verlag 1987.
[**V-W**] A. Verjovsky and H. Wu, Hausdorff dimension of Julia sets of complex Hénon maps, *Ergod. Th. & Dynam. Sys.* 16 (1996), 849-861.
[**Wal**] P. Walters, *An Introduction to Ergodic Theory*, Springer-Verlag, 1982.
[**You**] L.-S. Young, Dimension, entropy and Lyapunov exponents, *Ergod. Th. & Dynam. Sys.* 2 (1982), 109-129.
[**Zha**] Z.-S. Zhang, Expanding invariant sets of endomorphisms, *Scientia Sinica* (Ser. A) 27 (1984), 1035-1045.