A polynomial-time approximation algorithm for the number of $k$-matchings in bipartite graphs

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Abstract

We show that the number of $k$-matching in a given undirected graph $G$ is equal to the number of perfect matching of the corresponding graph $G_k$ on an even number of vertices divided by a suitable factor. If $G$ is bipartite then one can construct a bipartite $G_k$. For bipartite graphs this result implies that the number of $k$-matching has a polynomial-time approximation algorithm. The above results are extended to permanents and hafnians of corresponding matrices.

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1 Introduction

Let $G = (V, E)$ be an undirected graph, (with no self-loops), on the set of vertices $V$ and the set of edges $E$. A set of edges $M \subseteq E$ is called a matching if no two distinct edges $e_1, e_2 \in M$ have a common vertex. $M$ is called a $k$-matching if $\#M = k$. For $k \in \mathbb{N}$ let $\mathcal{M}_k(G)$ be the set of $k$-matchings in $G$. ($\mathcal{M}_k(G) = \emptyset$ for $k > \lceil \frac{\#V}{2} \rceil$.) If $\#V = 2n$ is even then an $n$-matching is called a perfect matching. $\phi(k, G) := \#\mathcal{M}_k(G)$ is number of $k$-matchings, and let $\phi(0, G) := 1$. Then $\Phi(x, G) := \sum_{k=0}^{\infty} \phi(k, G)x^k$ is the matching polynomial of $G$. It is known that a nonconstant matching polynomial of $G$ has only real negative roots [6].

Let $G$ be a bipartite graph, i.e., $V = V_1 \cup V_2$ and $E \subseteq V_1 \times V_2$. In the special case of a bipartite graph where $n = \#V_1 = \#V_2$, it is well known that $\phi(n, G)$ is given as $\text{perm} B(G)$, the permanent of the incidence matrix $B(G)$ of the bipartite graph $G$. It was shown by Valiant that the computation of the
permanent of a \((0,1)\) matrix is \(\#P\)-complete \([8]\). Hence, it is believed that the computation of the number of perfect matching in a general bipartite graph satisfying \(\#V_1 = \#V_2\) cannot be polynomial.

In a recent paper Jerrum, Sinclair and Vigoda gave a fully-polynomial randomized approximation scheme (fpras) to compute the permanent of a non-negative matrix \([7]\). (See also Barvinok \([1]\) for computing the permanents within a simply exponential factor, and Friedland, Rider and Zeitouni \([5]\) for concentration of permanent estimators for certain large positive matrices.)

\([7]\) yields the existence a fpras to compute the number of perfect matchings in a general bipartite graph satisfying \(\#V_1 = \#V_2\). The aim of this note is to show that there exists fpras to compute the number of \(k\)-matchings for any bipartite graph \(G\) and any integer \(k \in [1, \frac{\#V}{2}]\). In particular, the generating matching polynomial of any bipartite graph \(G\) has a fpras. This observation can be used to find a fast computable approximation to the pressure function, as discussed in \([4]\), for certain families of infinite graphs appearing in many models of statistical mechanics, like the integer lattice \(Z^d\).

More generally, there exists a fpras for computing \(\per_{k} B\), the sum of all \(k \times k\) subpermanents of an \(m \times n\) matrix \(B\), for any nonnegative \(B\). This is done by showing that \(\per_{k} B = \frac{\per_{B_k}}{(m-k)!(n-k)!}\) for a corresponding \((m + n - k) \times (m + n - k)\) matrix \(B_k\).

It is known that for a nonbipartite graph \(G\) on \(2n\) vertices, the number of perfect matchings is given by \(\haf A(G)\), the hafnian of the incidence matrix \(A(G)\) of \(G\). The existence of a fpras for computing the number of perfect matching for any undirected graph \(G\) on even number of vertices is an open problem. (The probabilistic algorithm suggested in \([7]\) applies to the computation of perfect matchings in \(G\), however it is not known if this algorithm is fpras.) The number of \(k\)-matchings in a graph \(G\) is equal to \(\haf_k A(G)\), the sum of the hafnians of all \(2k \times 2k\) principle submatrices of \(A(G)\). We show that that for any \(m \times m\) matrix \(A\) there exists a \((2m - 2k) \times (2m - 2k)\) matrix \(A_k\) such that \(\haf_k A = \frac{\haf_{A_k}}{(2m-k)!}\). Hence the computation of the number of \(k\)-matching in an arbitrary \(G\), where \(n = O(k)\), has fpras if and only if the number of perfect matching in \(G\) has fpras.

2 The equality \(\per_{k} B = \frac{\per_{B_k}}{(m-k)!(n-k)!}\)

Recall that for a square matrix \(A = [a_{ij}]_{i,j=1}^{n} \in \mathbb{R}^{n \times n}\), the permanent of \(A\) is given as \(\per A := \sum_{\sigma \in S_n} a_{1\sigma(1)} \cdots a_{n\sigma(n)}\), where \(S_n\) is the permutation group on \(\{1, \ldots, n\}\). Let \(Q_{k,m}\) denote the set of all subset of cardinality \(k\) of \(\{m\}\). Identify \(\alpha \in Q_{k,m}\) with the subset \(\{\alpha_1, \ldots, \alpha_k\}\) where \(1 \leq \alpha_1 < \ldots < \alpha_k \leq m\). Given an \(m \times n\) matrix \(B = [b_{ij}]_{i,j=1}^{m,n} \in \mathbb{R}^{m \times n}\) and \(\alpha \in Q_{k,m}, \beta \in Q_{l,n}\) we let \(B[\alpha, \beta] := [b_{\alpha_i, \beta_j}]_{i,j=1}^{k,l} \in \mathbb{R}^{k \times l}\) to be the corresponding \(k \times l\) submatrix of
Let $G = (V_1 \cup V_2, E)$ be a bipartite graph on two classes of vertices $V_1$ and $V_2$. For simplicity of notation we assume that $E \subset V_1 \times V_2$. It would be convenient to assume that $\#V_1 = m, \#V_2 = n$. So $G$ is presented by $(0, 1)$ matrix $B(G) \in \{0, 1\}^{m \times n}$. That is $B(G) = [b_{ij}]_{i,j=1}^{m,n}$ and $b_{ij} = 1 \iff (i, j) \in E$. Let $k \in [1, \min(m, n)]$ be an integer. Then $k$-matching is a choice of $k$ edges in $E_k := \{e_1, \ldots, e_k\} \subset E$ such that $E_k$ covers $2k$ vertices in $G$. That is, no two edges in $E_k$ have a common vertex. It is straightforward to show that $\text{perm}_k B(G)$ is the number of $k$-matching in $G$.

More generally, let $B = [b_{ij}] \in \mathbb{R}_{+}^{m \times n}, \mathbb{R}_{+} := [0, \infty)$ be an $m \times n$ non-negative matrix. We associate with $B$ the following bipartite graph $G(B) = (V_1(B) \cup V_2(B), E(B))$. Identify $V_1(B), V_2(B)$ with $\langle m \rangle, \langle n \rangle$ respectively. Then for $i \in \langle m \rangle, j \in \langle n \rangle$ the edge $(i, j)$ is in $E(B)$ if and only if $b_{ij} > 0$. Let $G_w := (V_1(B) \cup V_2(B), m(B))$ be the weighted graph corresponding to $B$. I.e., the weight of the edge $(i, j) \in E(B)$ is $b_{ij} > 0$. Hence $B(G_w)$, the representation matrix of the weighted bipartite graph $G_w$, is equal to $B$. Let $M \in \mathcal{M}_k(G(B))$. Then $\prod_{(i,j) \in M} b_{ij}$ is the weight of the matching $M$ in $G_w$. In particular, $\text{perm}_k B$ is the total weight of weighted $k$-matchings of $G_w$. The weighted matching polynomial corresponding to $B \in \mathbb{R}_+^{m \times n}$, or $G_w$ induced by $B$, is defined as:

$$\Phi(x, B) := \sum_{k=0}^{\min(m,n)} \text{perm}_k B \ x^k, \ B \in \mathbb{R}_+^{m \times n}, \ \text{perm}_0 B := 0.$$ 

$\Phi(x, B)$ can be viewed as the grand partition function for the monomer-dimer model in statistical mechanics [6]. (See §3 for the case of a nonbipartite graph.) In particular, all roots of $\Phi(x, B)$ are negative.

**Theorem 2.1** Let $B \in \mathbb{R}_+^{m \times n}$ and $k \in \langle \min(m,n) \rangle$. Let $B_k \in \mathbb{R}_+^{(m+n-k) \times (m+n-k)}$ be the following $2 \times 2$ block matrix

$$B_k := \begin{bmatrix} B & 1_{m,m-k} \\ 1_{n-k,n} & 0 \end{bmatrix}, \text{ where } 1_{p,q} \text{ is a } p \times q \text{ matrix whose all entries are equal to } 1. \text{ Then}$$

$$\text{perm}_k B = \frac{\text{perm} B_k}{(m-k)!(n-k)!}. \quad (2.1)$$

**Proof.** For simplicity of the exposition we assume that $k < \min(m, n)$. (In the case that $k = \min(m, n)$ then $B_k$ has one of the following block structure: $1 \times 1, 1 \times 2, 2 \times 1$.) Let $G_w := (V_1(B) \cup V_2(B), E_w(B)), G_w,k := (V_1(B_k) \cup V_2(B), E_w(B_k))$ be the weighted graphs corresponding to $B, B_k$ respectively. Note that $G_w$ is a weighted subgraph of $G_w,k$ induced by $V_1(B) = \langle m \rangle \subset \langle m+n-k \rangle = V_1(B_k), V_2(B) = \langle n \rangle \subset \langle n+m-k \rangle = V_2(B_k)$. Furthermore, each vertex in $V_1(B_k) \setminus V_1(B)$ is connected exactly to each vertex in $V_2(B)$, and
each vertex in \( V_2(B_k) \setminus V_2(B) \) is connected exactly to each vertex in \( V_1(B) \). The weights of each of these edges is 1. These are all edges in \( G(B_k) \). A perfect match in \( G(B_k) \) correspond to:

- An \( n - k \) match between the set of vertices \( V_1(B_k) \setminus V_1(B) \) and the set of vertices \( \beta' \in Q_{n-k,n} \), viewed as a subset of \( V_2(B) \).
- An \( m - k \) match between the set of vertices \( V_2(B_k) \setminus V_2(B) \) and the set of vertices \( \alpha' \in Q_{m-k,m} \), viewed as a subset of \( V_1(B) \).
- A \( k \) match between the set of vertices \( \alpha := \langle m \rangle \setminus \alpha' \subset V_1(B) \) and \( \beta := \langle n \rangle \setminus \beta' \subset V_2(B) \).

Fix \( \alpha \in Q_{k,m}, \beta \in Q_{k,n} \). Then the total weight of \( k \)-matchings in \( G_w(B_k) \) using the set of vertices \( \alpha \subset V_1(B_k), \beta \subset V_2(B_k) \) is given by \( \text{perm} B[\alpha, \beta] \). The total weight of \( n - k \) matchings using \( V_1(B_k) \setminus V_1(B) \) and \( \beta' \subset V_2(B_k) \) is \( (n-k)! \). The total weight of \( m - k \) matchings using \( V_2(B_k) \setminus V_2(B) \) and \( \alpha' \subset V_1(B) \) is \( (m-k)! \). Hence the total weight of perfect matchings in \( G_w(B_k) \), which matches the set of vertices \( \alpha \subset V_1(B_k) \) with the set \( \beta \subset V_2(B_k) \) is given by \( (m-k)!(n-k)! \text{perm} B[\alpha, \beta] \). Thus \( \text{perm} B_k = (m-k)!(n-k)! \text{perm}_k B \). □

We remark that the special case of Theorem 2.1 where \( m = n \) appears in an equivalent form in [2].

**Proposition 2.2** The complexity of computing the number of \( k \)-matchings in a bipartite graph \( G = (V_1 \cup V_2, E) \), where 
\[
\min(\# V_1, \# V_2) \geq k \geq c \max(\# V_1, \# V_2) \alpha \quad \text{and} \quad c, \alpha \in (0, 1] \text{, is polynomially equivalent to the complexity of computing the number of perfect matching in a bipartite graph } G' = (V_1' \cup V_2', E'), \text{ where } \# V_1' = \# V_2'.
\]

**Proof.** Assume first that \( G = (V_1 \cup V_2, E), m = \# V_1, n = \# V_2 \) and \( k \in [c \max(\# V_1, \# V_2) \alpha, \min(m, n)] \) are given. Let \( G' = (V_1' \cup V_2', E') \) be the bipartite graph constructed in the proof of Theorem 2.1. Theorem 2.1 yields that the number of perfect matching in \( G' \) determines the number of \( k \)-matching in \( G \). Note that \( n' := \# V_1' = \# V_2' = O(k^{1 \over 2}) \). So the \( k \)-matching problem is a special case of the perfect matching problem.

Assume second that \( G' = (V_1' \cup V_2', E') \) is a given bipartite graph with \( k = \# V_1 = \# V_2 \). Let \( m, n \geq k \) and denote by \( G = (V_1 \cup V_2, E'), \# V_1 = m, \# V_2 = n \) the graph obtained from \( G \) by adding \( m - k, n - k \) isolated vertices to \( V_1', V_2' \) respectively, \( (E' = E) \). Then a perfect matching in \( G' \) is a \( k \)-matching in \( G \), and the number of perfect matching in \( G' \) is equal to the number of \( k \)-matchings in \( G \). Furthermore if \( k \geq c \max(m, n) \alpha \) it follows that \( m, n = O(k^{1 \over 2}) \). □


**Corollary 2.3** Let \( B \in \mathbb{R}^{|m \times n|} \) and \( k \in \langle \min(m, n) \rangle \). Then there exists a fully-polynomial randomized approximation scheme to compute \( \text{perm}_k B \). Furthermore for each \( x \in \mathbb{R} \) there exists a fully-polynomial randomized approximation scheme to compute the matching polynomial \( \Phi(x, B) \).
3 Hafnians

Let $G = (V, E)$ be an undirected graph on $m := \#V$ vertices. Identify $V$ with $(m)$. Let $A(G) = [a_{ij}]_{i,j=1}^{m} \in \{0,1\}^{m \times m}$ be the incidence matrix of $G$, i.e. $a_{ij} = 1$ if and only if $(i,j) \in E$. Since we assume that $G$ is undirected and has no self-loops, $A(G)$ is a symmetric $(0,1)$ matrix with a zero diagonal. Denote by $S_m(T) \supset S_{m,0}(T)$ the set of symmetric matrices and the subset of symmetric matrices with zero diagonal respectively, whose nonzero entries are in the set $T \subseteq \mathbb{R}$. Thus any $A = [a_{ij}] \in S_{m,0}(\mathbb{R}^+)$ induces $G(A) = (V(A), E(A))$, where $V(A) = \langle m \rangle$ and $(i,j) \in E(A)$ if and only if $a_{ij} > 0$. Such an $A$ induces a general, the total weight of all weighted perfect matchings of $V, E$ with $\langle m \rangle$ and $(i,j) \in E(A)$ has the weight $a_{ij} > 0$. Let $M \in \mathcal{M}_k(G(A))$ be a $k$-matching in $G(A)$. Then the weight of $M$ in $G_w(A)$ is given by $\prod_{(i,j) \in M} a_{ij}$.

Assume that $m$ is even, i.e. $m = 2n$. It is well known that the number of perfect matchings in $G$ is given by $\text{haf}(A(G))$, the hafnian of $A(G)$. More general, the total weight of all weighted perfect matchings of $G_w(A), A \in S_{2n,0}(\mathbb{R}^+)$ is given by $\text{haf}(A)$, the hafnian of $A$.

Recall the definition of the hafnian of $2n \times 2n$ real symmetric matrix $A = [a_{ij}] \in \mathbb{R}^{2n \times 2n}$. Let $K_{2n}$ be the complete graph on $2n$ vertices, and denote by $\mathcal{M}(K_{2n})$ the set of all perfect matches in $K_{2n}$. Then $\alpha \in \mathcal{M}(K_{2n})$ can be represented as $\alpha = \{(i_1, j_1), (i_2, j_2), \ldots, (i_n, j_n)\}$ with $i_k < j_k$ for $k = 1, \ldots$. Denote $a_{\alpha} := \prod_{k=1}^{n} a_{i_k,j_k}$. Then $\text{haf}(A) := \sum_{\alpha \in \mathcal{M}(K_{2n})} a_{\alpha}$. Note that $\text{haf}(A)$ does not depend on the diagonal entries of $A$. Hafnian of $A$ is related to the pfaffian of the skew symmetric matrix $B = [b_{ij}] \in \mathbb{R}^{2n \times 2n}$, where $b_{ij} = a_{ij}$ if $i < j$, the same way the permanent of $C \in \mathbb{R}^{n \times n}$ is related to the determinant of $C$. Recall $\text{pfaf}(B) = \sum_{\alpha \in \mathcal{M}(K_{2n})} \text{sgn}(\alpha) b_{\alpha}$, where $\text{sgn}(\alpha)$ is the signature of the permutation $\alpha \in S_{2n}$ given by $\alpha = \begin{bmatrix} 1 & 2 & 3 & \ldots & 2n \\ i_1 & j_1 & i_2 & j_2 & \ldots & j_n \end{bmatrix}$.

Furthermore

$$\det B = \text{pfaf}(B)^2.$$

Let $A \in S_m(\mathbb{R})$. Then

$$\text{haf}_k A := \sum_{\alpha \in Q_{2k,m}} \text{haf}(A[\alpha, \alpha]), \ k = 1, \ldots, \lfloor \frac{m}{2} \rfloor.$$

For $A \in S_{m,0}(\mathbb{R}^+)$ $\text{haf}_k A$ is the total weight of all weighted $k$-matchings in $G_w(A)$. Let $\text{haf}_0(A) := 1$. Then the weighted matching polynomial of $G_w(A)$ is given by $\Phi(x, A) := \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \text{haf}_k A x^k$. It is known that a nonconstant $\Phi(x, A), A \in S_{m,0}(\mathbb{R}^+)$ has only real negative roots [6].

**Theorem 3.1** Let $A \in S_{m,0}(\mathbb{R}^+)$ and $k \in \{1, \ldots, \lfloor \frac{m}{2} \rfloor \}$. Let $A_k \in S_{2m-2k,0}(\mathbb{R}^+)$ be the following $2 \times 2$ block matrix

$$A_k := \begin{bmatrix} A & 1_{m,m-2k} \\ 1_{m-2k,m} & 0 \end{bmatrix},$$

Then

$$\text{haf}_k A = \frac{\text{haf}_k A_k}{(m-2k)!}. \quad (3.1)$$

5
An eigenvalue, if and only if all the eigenvalues of the computation of haf for an arbitrary undirected graph on an even number of vertices, or more generally in this section we offer an explanation, using the recent results in [3], why reducible. Then perm is a nicer function than haf as the computation of haf for an arbitrary polynomial. (See the definition in [3].) Assume that x := (x_1, ..., x_n) \top \in \mathbb{R}^n let

\[ p(x) := \prod_{i=1}^{n} (\sum_{j=1}^{n} b_{ij} x_j), \quad q(x) := \frac{1}{2} x \top A x. \]

Then perm B = \frac{\partial^n}{\partial x_1 \cdots \partial x_n} p(x) and haf A = ((\frac{1}{2} n)!)^{-1} \frac{\partial^n}{\partial x_1 \cdots \partial x_n} q(x)^2 if n is even. Assume that B \in \mathbb{R}_{+}^{n \times n} has no zero row. Then p(x) is a positive hyperbolic polynomial. (See the definition in [3].) Assume that A \in S_{2m,0}(\mathbb{R}_{+}) is irreducible. Then q(x), and hence any power q(x)^i, i \in \mathbb{N}, is positive hyperbolic if and only if all the eigenvalues of A, except the Perron-Frobenius eigenvalue, are nonpositive.

4 Remarks

In this section we offer an explanation, using the recent results in [3], why perm A is a nicer function than haf A. Let A = [a_{ij}] \in S_n(\mathbb{R}), B = [b_{ij}] \in \mathbb{R}^{n \times n}. For x := (x_1, ..., x_n) \top \in \mathbb{R}^n let

\[ p(x) := \prod_{i=1}^{n} (\sum_{j=1}^{n} b_{ij} x_j), \quad q(x) := \frac{1}{2} x \top A x. \]

Then perm B = \frac{\partial^n}{\partial x_1 \cdots \partial x_n} p(x) and haf A = ((\frac{1}{2} n)!)^{-1} \frac{\partial^n}{\partial x_1 \cdots \partial x_n} q(x)^2 if n is even. Assume that B \in \mathbb{R}_{+}^{n \times n} has no zero row. Then p(x) is a positive hyperbolic polynomial. (See the definition in [3].) Assume that A \in S_{2m,0}(\mathbb{R}_{+}) is irreducible. Then q(x), and hence any power q(x)^i, i \in \mathbb{N}, is positive hyperbolic if and only if all the eigenvalues of A, except the Perron-Frobenius eigenvalue, are nonpositive.
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