Positive entries of stable matrices

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Abstract

The question of how many elements of a real stable matrix must be positive is investigated. It is shown that any real stable matrix of order greater than 1 has at least two positive entries. Furthermore, for every stable spectrum of cardinality greater than 1 there exists a real matrix with that spectrum with exactly two positive elements, where all other elements of the matrix can be chosen to be negative.

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1 Introduction

For a square complex matrix $A$ let $\sigma(A)$ be the spectrum of $A$, that is, the set of eigenvalues of $A$ listed with their multiplicities. Recall that a (multi) set of complex numbers is called \textit{(positive) stable} if all the elements of the set have positive real parts, and that a square complex matrix $A$ is called \textit{stable} if $\sigma(A)$ is stable. In this paper we investigate the question of how many elements of a real stable matrix must be positive.

It is easy to show that a stable real matrix $A$ has either positive diagonal elements or it at least one positive diagonal element and one positive off-diagonal element. We then show that for any stable set $\zeta$ of $n$ complex numbers, $n > 1$, such that $\zeta$ is symmetric with respect to the real axis, there exists a real stable $n \times n$ matrix $A$ with exactly two positive entries such that $\sigma(A) = \zeta$. 
The stable $n \times n$ matrix with exactly two positive entries, whose existence is proven in Section 2, has $(n - 1)^2$ zeros in it. In Section 3 we prove that for any stable set $\zeta$ of $n$ complex numbers, $n > 1$, such that $\zeta$ is symmetric with respect to the real axis, there exists a real stable $n \times n$ matrix $A$ with two positive entries and all other entries negative such that $\sigma(A) = \zeta$.

In Section 4 we suggest some alternative approaches to obtain the results of Section 2.

2 Positive entries of stable matrices

Our aim in this Section is to show that for any stable set $\zeta$ of $n$ complex numbers, $n > 1$, consisting of real numbers and conjugate pairs, there exists a real stable $n \times n$ matrix $A$ with exactly two positive entries such that $\sigma(A) = \zeta$. We shall first show that every real stable matrix of order greater than 1 has at least two positive elements. In fact we show more than that, that is, that for a stable real matrix $A$ either all diagonal elements of $A$ are positive or $A$ must have at least one positive entry on the main diagonal and one off the main diagonal.

Notation 2.1 For a set $\zeta = \{\zeta_1, \ldots, \zeta_n\}$ of complex numbers we denote by $s_1(\zeta), \ldots, s_n(\zeta)$ the elementary symmetric functions of $\zeta$, that is,

$$s_k(\zeta) = \sum_{1 \leq i_1 < \ldots < i_k \leq n} \zeta_{i_1} \cdot \ldots \cdot \zeta_{i_k}, \quad k = 1, \ldots, n.$$ 

Also, we let $s_0(\zeta) = 1$ and $s_k(\zeta) = 0$ whenever $k > n$ or $k < 0$.

**Lemma 2.2** Let $\zeta = \{\zeta_1, \ldots, \zeta_n\} \subset \mathbb{C}$ have positive elementary symmetric functions. Then $\zeta$ contains no nonpositive real numbers.

**Proof.** Note that $\zeta$ has positive elementary symmetric functions if and only if the polynomial $p(x) = \prod_{i=1}^{n} (x + \zeta_i)$ has positive coefficients. It follows that $p(x)$ cannot have nonnegative roots, implying that none of the $\zeta_i$’s is a nonpositive real number.

**Notation 2.3** For $F = \mathbb{R}, \mathbb{C}$, the fields of real and complex numbers respectively, we denote by $M_n(F)$ the algebra of $n \times n$ matrices with entries in $F$. For $A = (a_{ij})_1^n \in M_n(F)$ we denote by $\text{tr} A$ the trace of $A$, that is, the sum $\sum_{i=1}^{n} a_{ii}$.

**Proposition 2.4** Let $A = (a_{ij})_1^n \in M_n(\mathbb{R})$, and assume that $\sigma(A)$ has positive elementary functions. Then either all the diagonal elements of $A$ are positive or $A$ has at least one positive diagonal element and one positive off-diagonal element.

**Proof.** As is well known, the trace of $A$ is equal to $s_1(\sigma(A))$, and so we have $\sum_{i=1}^{n} a_{ii} > 0$, and it follows that at least one diagonal element of $A$ is positive. Assume that that all off-diagonal elements of $A$ are nonpositive. Such a real matrix is called a $Z$-matrix. Since the elementary symmetric functions of $\sigma(A)$ are positive, it follows by Lemma 2.2 that $A$ has no nonpositive real eigenvalues. Since a $Z$-matrix has no nonpositive real eigenvalues if and only if all its principal minors are positive, e.g. [1, Theorem (6.2.3), page 134], it follows that all the diagonal elements of $A$ are positive.
Notation 2.5 For a set $\zeta = \{\zeta_1, \ldots, \zeta_n\}$ of complex numbers we denote by $\overline{\zeta}$ be the set $\{\overline{\zeta_1}, \ldots, \overline{\zeta_n}\}$.

Note that $\overline{\zeta} = \zeta$ if and only if all elementary symmetric functions of $\zeta$ are real.

The following result is well known, and we provide a proof for the sake of completeness.

Proposition 2.6 Let $\zeta$ be a stable set of complex numbers such that $\overline{\zeta} = \zeta$. Then $\zeta$ has positive elementary symmetric functions.

Proof. We prove our claim by induction on the cardinality $n$ of $\zeta$. For $n = 1, 2$ the result is trivial. Assume that the result holds for $n \leq m$ where $m \geq 2$, and let $n = m + 1$. Assume first that $\zeta$ contains a positive number $\lambda$. Note that the set $\zeta' = \zeta \setminus \{\lambda\}$ is stable and $\overline{\zeta'} = \zeta'$. By the inductive assumption we have $s_k(\zeta') > 0$, $k = 1, \ldots, n - 1$, and it follows that

$$s_k(\zeta) = s_k(\zeta') + \lambda s_{k-1}(\zeta') > 0, \quad k = 1, \ldots, n.$$  

If $\zeta$ does not contain a positive number then it contains a conjugate pair $\{\lambda, \overline{\lambda}\}$, where $\text{Re}(\lambda) > 0$. Note that the set $\zeta' = \zeta \setminus \{\lambda, \overline{\lambda}\}$ is stable and $\overline{\zeta'} = \zeta'$. By the inductive assumption we have $s_k(\zeta') > 0$, $k = 1, \ldots, n - 2$, and it follows that

$$s_k(\zeta) = s_k(\zeta') + 2\text{Re}(\lambda)s_{k-1}(\zeta') > 0 + |\lambda|^2 s_{k-2}(\zeta') > 0, \quad k = 1, \ldots, n,$$

proving our claim. $\square$

It is easy to show that the converse of Proposition 2.6 holds when the cardinality of $\zeta$ is less than or equal to 2. However, the converse does not hold for larger sets, as is demonstrated by the nonstable set $\zeta = \{3, -1 + 3i, -1 - 3i\}$, whose elementary symmetric functions are positive.

As a corollary of Propositions 2.4 and 2.6 we obtain

Corollary 2.7 Let $A$ be a stable real square matrix. Then either all the diagonal elements of $A$ are positive or $A$ has at least one positive diagonal element and one positive off-diagonal element.

In order to prove the existence of a real stable $n \times n$ matrix $A$ with exactly two positive entries, we introduce:

Notation 2.8 Let $n$ be a positive integer. For a set $\zeta$ of $n$ complex numbers we denote by $C_1(\zeta)$, $C_2(\zeta)$ and $C_3(\zeta)$ the matrices

$$C_1(\zeta) = \begin{pmatrix}
0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
1 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 1 & 0 & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 0 & 1 & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 & 1
\end{pmatrix},$$

$$C_2(\zeta) = \begin{pmatrix}
(-1)^{n-1}s_n(\zeta) \\
(-1)^{n-2}s_{n-1}(\zeta) \\
(-1)^{n-3}s_{n-2}(\zeta) \\
\vdots \\
-s_2(\zeta) \\
s_1(\zeta)
\end{pmatrix},$$

$$C_3(\zeta) = \begin{pmatrix}
\end{pmatrix},$$
Recall that $A \in \mathbb{M}_n(\mathbb{C})$ is called nonderogatory if for every eigenvalue $\lambda$ of $A$ the Jordan canonical form of $A$ has exactly one Jordan block corresponding to $\lambda$. Equivalently, the minimal polynomial of $A$ is equal to the characteristic polynomial of $A$.

**Lemma 2.9** Let $n$ be a positive integer, $n > 1$, and let $\zeta = \{\zeta_1, \ldots, \zeta_n\} \subset \mathbb{C}$. Then the matrices $C_1(\zeta)$, $C_2(\zeta)$ and $C_3(\zeta)$ are diagonally similar, are nonderogatory and share the spectrum $\zeta$.

**Proof.** The matrix $C_1(\zeta)$ is the companion matrix of the polynomial $q(x) = \prod_{i=1}^{n} (x - \zeta_i)$. Hence $\sigma(C_1(\zeta)) = \zeta$ and $C_1(\zeta)$ is nonderogatory. Clearly

$$C_2(\zeta) = D_1 C_1(\zeta) D_1,$$

where $D_1 = \text{diag}((-1)^1, (-1)^2, \ldots, (-1)^n)$, and

$$C_3(\zeta) = D_2 C_2(\zeta) D_2,$$

where $D_2 = \text{diag}(1, 1, \ldots, 1, -1)$.

Our claim follows. \hfill \Box

In view of Lemma 2.9, the claim of Proposition 2.6 on $C_3(\zeta)$ yields the following main result of this section.

**Theorem 2.10** Let $\zeta$ be a set of $n$ complex numbers, $n > 1$, such that $\overline{\zeta} = \zeta$. If $\zeta$ has positive elementary symmetric functions then there exists a matrix $A \in \mathbb{M}_n(\mathbb{R})$ such that $\sigma(A) = \zeta$ and $A$ has one positive diagonal entry and one positive off-diagonal entry, while all other entries of $A$ are nonpositive. In particular, every nonderogatory stable matrix $A \in \mathbb{M}_n(\mathbb{R})$ is similar to a real $n \times n$ matrix which has exactly two positive entries.

### 3 Eliminating the zero entries

The proof of Theorem 2.10 uses the matrix $C_3(\zeta)$ which has $(n - 1)^2$ zero entries. The aim of this section is to strengthen Theorem 2.10 by replacing $C_3(\zeta)$ with a real matrix $A$, having exactly two positive entries, all other entries being negative and $\sigma(A) = \zeta$. 
We start with a weaker result, which one gets easily using perturbation techniques. Let $A \in M_n(\mathbb{R})$ and let $\| \cdot \| : M_n(\mathbb{R}) \to [0, \infty)$ be the $l_2$ operator norm. Since the eigenvalues of a $A$ depend continuously on the entries of the $A$, it follows that if $\sigma(A)$ has positive elementary symmetric functions, then for $\varepsilon > 0$ sufficiently small, every matrix $A \in M_n(\mathbb{R})$ with $\| A - A \| < \varepsilon$ has a spectrum $\sigma(A)$ with positive elementary symmetric functions. Also, if $A$ is stable then for $\varepsilon > 0$ sufficiently small, every matrix $A \in M_n(\mathbb{R})$ with $\| A - A \| < \varepsilon$ is stable. Consequently, it follows immediately from Theorem 2.10 that

**Corollary 3.1** For a positive integer $n > 1$ there exists a matrix $A \in M_n(\mathbb{R})$ such that $\sigma(A)$ has positive elementary symmetric functions and $A$ has one positive diagonal entry and one positive off-diagonal entry, while all other entries of $A$ are negative. Furthermore, the matrix $A$ can be chosen to be stable.

In the rest of this section we prove that one can find such a matrix $A$ with any prescribed stable spectrum.

**Lemma 3.2** Let $\zeta$ be a set of $n$ complex numbers, $n > 1$, such that $\zeta = \zeta$, and assume that $\zeta$ has positive elementary symmetric functions. Suppose that there exists $X \in M_n(\mathbb{R})$ such that

$$(C_3(\zeta))_{ij} = 0 \implies (C_3(\zeta)X - XC_3(\zeta))_{ij} < 0, \quad i, j = 1, \ldots, n.$$  

Then there exist $A \in M_n(\mathbb{R})$ similar to $C_3(\zeta)$ such that $a_{nn}, a_{n,n-1} > 0$ and all other entries of $A$ are negative.

**Proof.** Assume the existence of such a matrix $X$. Define the matrix $T(t) = I - tX$, $t \in \mathbb{R}$. Let $r = \|X\|^{-1}$. Using the Neumann series expansion, e.g. [2, page 7], for $|t| < r$ we have $T(t)^{-1} = \sum_{i=0}^{\infty} t^iX^i$. The matrix $A(t) = T(t)C_3(\zeta)T(t)^{-1}$ thus satisfies

$$A(t) = C_3(\zeta) + t(C_3(\zeta)X - XC_3(\zeta)) + O(t^2).$$

Therefore, there exists $\varepsilon \in (0, r)$ such that for $t \in (0, \varepsilon)$ the matrix $A(t)$ has positive entries in the $(n, n-1)$ and $(n, n)$ positions, while all other entries of $A(t)$ are negative. \hfill $\square$

The following lemma is well known, and we provide a proof for the sake of completeness.

**Lemma 3.3** Let $A, B \in M_n(\mathbb{F})$ where $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$. The following are equivalent.

(i) The system $AX -XA = B$ is solvable over $\mathbb{F}$.

(ii) For every matrix $E \in M_n(\mathbb{F})$ that commutes with $A$ we have $\text{tr} BE = 0$.

**Proof.** (i)$\implies$(ii). Let $E \in M_n(\mathbb{F})$ commute with $A$. Then

$$\text{tr} BE = \text{tr}(AX -XA)E = \text{tr} AXE - \text{tr} XEA = \text{tr} XEA - \text{tr} XEA = 0.$$  

(i)$\implies$(ii). Consider the linear operator $L : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ defined by $L(X) = AX -XA$. Its kernel consists of all matrices in $M_n(\mathbb{F})$ commuting with $A$. By the previous implication, the image of $L$ is contained in the subspace $V$ of $M_n(\mathbb{F})$ consisting of all matrices $B$ such that $\text{tr} BE = 0$ whenever $E \in \text{kernel}(L)$. Since clearly $\dim(V) = n^2 - \dim(\text{kernel}(L)) = \dim(\text{image}(L))$, it follows that $\text{image}(L) = V$. \hfill $\square$
Theorem 3.4 Let $\zeta$ be a set of $n$ complex numbers, $n > 1$. Let $b_{ij}$, $i = 1, \ldots, n$, $j = 1, \ldots, n-1$ be given complex numbers, and let $C = C_k(\zeta)$ for some $k \in \{1, 2, 3\}$. Then there exists unique $b_n \in \mathbb{C}$, $i = 1, \ldots, n$ such that for the matrix $B = (b_{ij})^T \in M_n(\mathbb{C})$ the system $CX - XC = B$ is solvable. Furthermore, if $\xi = \zeta$ and $b_{ij}$ is real for $i = 1, \ldots, n, j = 1, \ldots, n-1$, then the matrix $B$ is real, and the solution $X$ can be chosen to be real.

Proof. Since $C_2(\zeta)$ and $C_3(\zeta)$ are diagonally similar to $C_1(\zeta)$, where the corresponding diagonal matrices are real, it is enough to prove the theorem for $C = C_1(\zeta)$. So, let $C = C_1(\zeta)$ and consider the system

$$CX - XC = B.$$  \hfill (3.1)

As is well known, e.g. [3, Corollary 1, page 222], since $C = C_1(\zeta)$ is nonderogatory, every matrix that commutes with $C$ is a polynomial in $C$. Therefore, it follows from Lemma 3.3 that the system (3.1) is solvable if and only if

$$\text{tr } BC^k = 0, \ k = 0, \ldots, n-1.$$  \hfill (3.2)

Denote $v_k = b_{n+1-k,n}, \ k = 1, \ldots, n$. Note that (3.2) is a system of $n$ equations in the variables $v_1, \ldots, v_n$. Furthermore, it is easy to verify that the first nonzero element in the $n$th row of $C^k$ is located at the position $(n, n-k)$ and its value is 1. It follows that if we write (3.2) as $Ev = f$, where $E \in M_n(\mathbb{C})$ and $v = (v_1, \ldots, n)^T$, then $E$ is a lower triangular matrix with 1's along the main diagonal. It follows that the matrix $B$ is uniquely determined by (3.2).

If $\xi = \zeta$ and $b_{ij}$ is real for $i = 1, \ldots, n, j = 1, \ldots, n-1$ then $C = C_1(\zeta)$ is real and hence the system (3.2) has real coefficients, and the uniquely determined $B$ is real. It follows that the system (3.1) is real, and so it has a real solution $X$. \hfill $\square$

If we choose the numbers $b_{ij}$, $i = 1, \ldots, n$, $j = 1, \ldots, n-1$, to be negative, then Lemma 3.2 and Theorem 3.4 yield

Corollary 3.5 Let $\zeta$ be a set of $n$ complex numbers, $n > 1$, and assume that the elementary symmetric functions of $\zeta$ are positive. Then there exists a matrix $A \in M_n(\mathbb{R})$ with $\sigma(A) = \zeta$ such that $A$ has one positive diagonal element, one positive off-diagonal element and all other entries of $A$ are negative. In particular, the above holds for stable sets $\zeta$ such that $\xi = \zeta$.

4 Other types of companion matrices

Another way to prove some of the results of Section 2 is to parameterize the companion matrices in Notation 2.8. Consider

$$C = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \gamma_0 \\
\beta_0 & 0 & 0 & \cdots & 0 & 0 & 0 & \gamma_1 \\
0 & \beta_1 & 0 & \cdots & 0 & 0 & 0 & \gamma_2 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & 0 & \beta_{n-3} & 0 & \gamma_{n-2} \\
0 & 0 & 0 & \cdots & 0 & \beta_{n-2} & 0 & \gamma_{n-1}
\end{pmatrix}$$
From looking at the directed graph of $C$ is

![Directed graph of $C$](image)

one can immediately see that there is exactly one simple cycle of length $k$ for $1 \leq k \leq n$, that is, $(n, \ldots, n+1-k)$. Therefore, the only nonzero principal minors of $C$ are those whose rows and columns are indexed by $\{k, \ldots, n\}$, $k = 1, \ldots, n$, and their respective values are $(-1)^{n-k} \gamma_{k-1} \beta_{k-1} \cdots \beta_{n-2}$ for $k < n$ and $\gamma_{n-1}$ for $k = n$. It follows that the characteristic polynomial $\chi_C(x)$ of $C$ is

$$
\chi_C(x) = x^n - \gamma_{n-1}x^{n-1} - \gamma_{n-2}\beta_{n-2}x^{n-2} - \gamma_{n-3}\beta_{n-3}\beta_{n-2}x^{n-3} - \cdots - \gamma_1\beta_1\beta_2 \cdots \beta_{n-2}x - \gamma_0\beta_0\beta_1 \cdots \beta_{n-2}.
$$

(4.1)

Using this explicit formula, one can prove directly the claim contained in Lemma 2.9 that the matrices $C_1(\zeta)$, $C_2(\zeta)$ and $C_3(\zeta)$ share the spectrum $\zeta$.

There are other possibilities to generate companion matrices. For example, consider the matrix

$$
L = \begin{pmatrix}
\gamma_{n-1} & 0 & 0 & \cdots & 0 & 0 & 0 & \beta_{n-2} \\
\beta_{n-3} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & \beta_{n-4} & 0 & \cdots & 0 & 0 & 0 & 0 \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
0 & 0 & 0 & \cdots & 0 & \beta_0 & 0 & 0 \\
\gamma_{n-2} & \gamma_{n-3} & \gamma_{n-4} & \cdots & \gamma_2 & \gamma_1 & \gamma_0 & 0
\end{pmatrix},
$$

The directed graph of $L$ is

![Directed graph of $L$](image)

Again it is clear that there is exactly one simple cycle of length $k$ for any $1 \leq k \leq n$, that is, $(1)$ for $k = 1$ and $(n, k-1, \ldots, 1)$ for $1 < k \leq n$. Therefore, the only nonzero $1 \times 1$ principal minor of $L$ is $l_{11} = \gamma_{n-1}$, and for $1 < k \leq n$ the only
nonzero $k \times k$ principal minor of $L$ is the one whose rows and columns are indexed by \{1, \ldots, k-1, n\}, and its value is $(-1)^{n-k} \gamma_{k-1} \beta_{k-1} \cdots \beta_{n-2}$. It follows that the characteristic polynomial $\chi_L(x)$ of $L$ is identical to $\chi_C(x)$. Note that there is no permutation matrix $P$ with $P^T CP = L$ or $P^T C^T P = L$.

Now, take the following specific choice of the parameters $\beta$ and $\gamma$

$$L_1 = \begin{pmatrix} -p_{n-1} & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ -p_{n-2} & -p_{n-3} & -p_{n-4} & \cdots & -p_2 & -p_1 & -p_0 & 0 \end{pmatrix}.$$  

By (4.1), the characteristic polynomial computes to

$$\chi_{L_1}(x) = \sum_{\nu=0}^{n} p_\nu x^\nu,$$

where $p_n = 1$.

So $L_1$ is another kind of companion matrix. Note that $L_1$ is almost lower triangular, with only one nonzero element above the main diagonal and one on the main diagonal.

Another specific choice of the parameters $\beta$ and $\gamma$ can be used to produce another direct proof of Theorem 2.10. For a set $\zeta$ of complex numbers with $\zeta = \overline{\zeta}$ and positive elementary symmetric functions, the polynomial $q(x) = \prod_{i=1}^{n} (x - \zeta_i) = \sum_{i=0}^{n} q_i x^i$ has coefficients $q_i$, $0 \leq i \leq n$ of alternating signs, where $q_n = 1$. By (4.1), the polynomial $q(x)$ is the characteristic polynomial of the matrix

$$L_2 = \begin{pmatrix} -q_{n-1} & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & -1 & 0 & 0 \\ -q_{n-2} & -q_{n-3} & -q_{n-4} & \cdots & (-1)^{n-3} q_2 & (-1)^{n-2} q_1 & (-1)^{n-1} q_0 & 0 \end{pmatrix}.$$  

which has exactly two positive entries, that is, $-q_{n-1}$ on the diagonal and 1 in the right upper corner.

**References**

