Available at
www.ElsevierMathematics.com
POWERED BY SCIENCE DIRECT
Journal of Functional Analysis 210 (2004) 436–464
JOURNAL OF Functional Analysis
http://www.elsevier.com/locate/jfa

The limit of the product of the parameterized exponentials of two operators

Shmuel Friedland[a,*] and Gaspar Porta[b]

[a] Department of Mathematics, Statistics and Computer Science, University of Illinois at Chicago, 322 Science and Engineering Offices, 851 S. Morgan St., Chicago, IL 60607-7045, USA
[b] Department of Mathematics, Colgate University, Hamilton, NY 13346, USA

Received 31 March 2003; accepted 25 July 2003

Communicated by L. Gross

Abstract

Given two self-adjoint, positive, compact operators A, B on a separable Hilbert space, we show that there exists a self-adjoint, positive, compact operator C commuting with B such that $\lim_{t\to\infty} ||(e^{\frac{Bt}{2}} e^{At} e^{\frac{Bt}{2}})^{\frac{1}{t}} - e^C|| = 0$.

MSC: Primary: 47B06, 47B07, 47B25, 47B65

Keywords: Golden–Thompson inequality; Log-convexity of spectral radius; Limit operator

1. Introduction

Recall the fundamental inequality of Golden [8], Thompson [14] arising in statistical mechanics. Let A, B be $n \times n$ Hermitian. Then

$$\mathrm{Trace}\, e^A e^B \geqslant \mathrm{Trace}\, e^{A+B}. \tag{1.1}$$

This inequality was generalized by Kostant [9] to general finite-dimensional Lie groups. In a paper by Cohen et al. [3] (1.1) was parameterized as follows: the function $f(t) = \mathrm{Trace}(e^{At}e^{Bt})^{\frac{1}{t}}$ is increasing on $[0, \infty)$. The Lie–Trotter formula

*Corresponding author. Fax: 1-312-996-1491.
E-mail addresses: friedlan@uic.edu (S. Friedland), gporta@mail.colgate.edu (G. Porta).

0022-1236/$ - see front matter
doi:10.1016/j.jfa.2003.07.007

yields $f(0) = \text{Trace } e^{A+B}$. Hence (1.1) follows from $f(0) \leqslant f(1) = \text{Trace } e^A e^B$. It is of interest to study $\lim_{t\to\infty} f(t)$. It is convenient to consider the parameterized family $e^{\frac{Bt}{2}} e^{At} e^{\frac{Bt}{2}}$, $t \in \mathbb{R}$, which is Hermitian and positive definite, instead of $e^{At} e^{Bt}$ as both expressions have the same eigenvalues. Friedland and So [5] showed

$$\lim_{t\to\infty} (e^{\frac{Bt}{2}} e^{At} e^{\frac{Bt}{2}})^{\frac{1}{t}} = e^C, \tag{1.2}$$

for some Hermitian matrix C.

The object of this paper is to generalize this result to the infinite-dimensional case. Let $\mathbb{R}^* := \mathbb{R}\backslash\{0\}$. For a self-adjoint, positive, compact operator A on an infinite-dimensional separable Hilbert space $\mathbf{H}$ we denote by $\alpha_1 \geqslant \cdots \geqslant \alpha_n \geqslant \cdots > 0$ the eigenvalue sequences of A, where each eigenvalue is counted with its multiplicity.

Theorem 1.1. *Let $A, B: \mathbf{H} \to \mathbf{H}$ be linear, self-adjoint, positive, compact operators on an infinite-dimensional separable Hilbert space $\mathbf{H}$ with the eigenvalue sequences $\alpha_1 \geqslant \cdots \geqslant \alpha_n \geqslant \cdots > 0$, $\beta_1 \geqslant \cdots \geqslant \beta_n \geqslant \cdots > 0$ respectively. Then for each $t \in \mathbb{R}^*$ $e^{\frac{Bt}{2}} e^{At} e^{\frac{Bt}{2}} = e^{C(t)t}$, where $C(t) = C(-t)$ is a self-adjoint, positive, compact operator $C(t)$ with the eigenvalue sequence $\omega_1(t) \geqslant \cdots \geqslant \omega_i(t) \geqslant \cdots > 0$. Furthermore*

(a) $\sum_{i=1}^n \omega_i(t)$ *is a nondecreasing function on* $(0, \infty)$ *bounded from above by* $\sum_{i=1}^n (\alpha_i + \beta_i)$ *for each* $n \in \mathbb{N}$. *Hence*

$$\lim_{t\to\infty} \omega_i(t) = \omega_i, i = 1, \ldots, \omega_1 \geqslant \cdots \geqslant \omega_n \geqslant \cdots \geqslant 0, \lim_{n\to\infty} \omega_n = 0. \tag{1.3}$$

(b) $\lim_{t\to 0} C(t) = A + B$.
(c) *There exists a self-adjoint, positive, compact operator* C *such that* $\lim_{t\to\infty} ||C(t) - C|| = 0$.
(d) *There exist two bijections* $\Phi, \Psi : \mathbb{N} \to \mathbb{N}$ *and an orthonormal basis of* $\mathbf{H}$ $\mathbf{g}_1, \ldots, \mathbf{g}_n, \ldots,$ *consisting of eigenvectors of* B *such that*

$$C\mathbf{g}_i = \omega_i \mathbf{g}_i,\ \omega_i = \alpha_{\Phi(i)} + \beta_{\Psi(i)}, \quad i = 1, \ldots, . \tag{1.4}$$

In particular $BC = CB$, $\omega_i > 0$, $i = 1, \ldots,$ *and* (1.2) *holds in the norm topology.*

We now survey briefly the contents of this paper. In this paper we assume that $\mathbf{H}$ is an infinite-dimensional separable Hilbert space over $\mathbb{C}$, unless stated otherwise, with the inner product $<\cdot,\cdot>$ and the norm $||\mathbf{x}|| = \sqrt{<\mathbf{x},\mathbf{x}>}$. Denote by $\mathscr{L} \supset \mathscr{S}$ the algebra of bounded linear operators on $\mathbf{H}$ and the real linear subspace of self-adjoint operators, respectively. For $T \in \mathscr{L}$ let T^*, $||T||$, $\sigma(T) \subset \mathbb{C}$ and $\rho(T)$ be the adjoint, the norm, the spectrum and its spectral radius of T, respectively. $\rho(T) \leqslant ||T||$ and equality holds for any self-adjoint T.

We always assume that $A, B \in \mathscr{S}$. Then one associates with A an increasing family of commuting projections $P(A, (-\infty, \lambda))$ which enable to define a projection $P(A, \mathscr{O})$

for any Borel set $\mathcal{O} \subset \mathbb{R}$. We then define $e_{\mathcal{O}}^{At} := e^{At} P(A, \mathcal{O}) \in \mathcal{S}$. In Section 2 we consider the function $\phi(t, \tau) = \ln \rho(e_{\mathcal{O}_1}^{At} e_{\mathcal{O}_2}^{B\tau})$ on $\mathbb{R}^2$. The main result of Section 2 is Theorem 2.1, which characterizes some properties of $\phi(t, \tau)$. Namely if $P(A, \mathcal{O}_1)P(B, \mathcal{O}_2) = 0$ then $\phi(t, \tau) \equiv -\infty$. Otherwise $\phi(t, \tau)$ is a continuous convex function on $\mathbb{R}^2$. Then the function

$$g(t) := \frac{\phi(t,t) - \ln \rho(P(A, \mathcal{O}_1)P(B, \mathcal{O}_2))}{t}$$

is a nondecreasing function on $(0, \infty)$. We characterize the limits $\lim_{t \searrow 0} g(t)$ and $\lim_{t \to \infty} g(t)$. (The characterization of the second limit is proved only for $\mathcal{O}_1, \mathcal{O}_2$ which are a finite union of half open intervals $(a, b]$.) In Section 3, we introduce the operator $\Lambda(t) = e^{\frac{Bt}{2}} e^{At} e^{\frac{Bt}{2}}$ for $t \in \mathbb{R}^*$. Then $\Lambda(t) = e^{tC(t)}$ and $C(t) = C(-t) \in \mathcal{S}$. If A, B are also compact then $C(t)$ is compact. A is called diagonal if $\mathbf{H}$ has an orthonormal basis consisting of eigenvalues of A. A belongs to the class $\mathcal{D}$ if A is diagonal, each eigenvalue λ of A is an isolated point of $\sigma(A)$, $P(A, \{\lambda\})$ has a finite-dimensional range, and for any $t \in \sigma(A)$ there exists $\varepsilon(t) > 0$ such that $(t - \varepsilon, t) \cap \sigma(A) = \emptyset$. The main result of Section 3 is Lemma 3.4, which shows that for $A, B \in \mathcal{D}$ there exists $t_0 > 0$ such that for $t \geqslant t_0$ the $\rho(e^{C(t)})$ is an isolated point of $\sigma(e^{C(t)})$ and $\dim P(e^{C(t)}, \{\rho(e^{C(t)})\})$ is a constant integer. Moreover, there is a fixed gap between the next point in the spectrum of $e^{C(t)}$ for $t \geqslant t_0$. In Section 4, we discuss the tensor spaces and exterior spaces. We give natural examples of operators in class $\mathcal{D}$ which are not compact. Section 5 is devoted to the proof of Theorem 1.1. Section 6 is devoted to some connections between $C(t)$ and A, B under the assumptions of Theorem 1.1. We also raise the problem of extending our results to more general operators $A, B \in \mathcal{S}$.

Parts of this paper are based on the dissertation of the second author [12], written under the supervision of the first author.

2. The limit of spectral radius of a parameterized family

For a closed subspace $\mathbf{G} \subset \mathbf{H}$ we denote by $P(\mathbf{G})$ the orthogonal projection on $\mathbf{G}$. For any orthogonal projection $P : \mathbf{H} \to \mathbf{H}$ we denote by $\dim P$ the dimension of $P\mathbf{H}$, which is either a nonnegative integer or ∞. Recall the spectral decomposition of $T \in \mathcal{S}$, e.g. [13, VII.3]. T determines an increasing family of commuting orthogonal projections $P(T, (-\infty, \lambda)) \in \mathcal{S}$, $\lambda \in \mathbb{R}$:

$$P(T, (-\infty, \lambda_1))P(T, (-\infty, \lambda_2)) = P(T, (-\infty, \min(\lambda_1, \lambda_2))), \quad \text{for any } \lambda_1, \lambda_2 \in \mathbb{R},$$

$$P(T, (-\infty, \lambda_1)) = 0, \quad P(T, (-\infty, \lambda_2)) = I, \quad \text{for any } \lambda_1 \leqslant -||T||, \lambda_2 > ||T||,$$

$$P(T, (-\infty, \lambda_1)) + (I - P(T, (-\infty, \lambda_2))) \neq 0 \quad \text{if } \lambda_1 > -||T||, \lambda_2 < ||T||.$$

Then $T = \int_{\mathbb{R}} \lambda \, dP(T, (-\infty, \lambda))$. $P(T, (-\infty, \lambda))$ is called the spectral projection on $(-\infty, \lambda)$. Given a Borel function $g : \mathbb{R} \to \mathbb{R}$, which is bounded on $[-||T||, ||T||]$, we get a bounded operator $g(T) = \int_{\mathbb{R}} g(\lambda) \, dP(T, (-\infty, \lambda))$, which commutes with T. Let $\mathcal{O} \subset \mathbb{R}$ be a Borel set and denote by $\chi_{\mathcal{O}}$ the characteristic function of $\mathcal{O}$. Define

$$P(T, \mathcal{O}) = \int_{\mathbb{R}} \chi_{\mathcal{O}}(\lambda) \, dP(T, (-\infty, \lambda)) = \int_{\mathcal{O}} dP(T, (-\infty, \lambda)),$$

$$\begin{aligned} e_{\mathcal{O}}^{tT} &= \int_{\mathbb{R}} \chi_{\mathcal{O}}(\lambda) e^{t\lambda} \, dP(T, (-\infty, \lambda)) = \int_{\mathcal{O}} e^{t\lambda} \, dP(T, (-\infty, \lambda)) \\ &= e^{tT} P(T, \mathcal{O}) = P(T, \mathcal{O}) e^{tT} = P(T, \mathcal{O}) e^{tT} P(T, \mathcal{O}), \quad t \in \mathbb{R}. \end{aligned} \tag{2.1}$$

Let $\mathcal{A}$ be an algebra of sets, where each $\mathcal{O} \in \mathcal{A}$ is a finite union of intervals of the form $(a, b]$, where $-\infty \leqslant a \leqslant b \leqslant \infty$. In what follows we use the convention $\ln 0 = -\infty$.

The following theorem is the main result of this section.

Theorem 2.1. *Let $A, B \in \mathcal{S}$ and assume that $\mathcal{O}_1, \mathcal{O}_2$ are Borel sets. Define*

$$\phi(t, \tau) = \ln \rho \left(e_{\mathcal{O}_1}^{At} e_{\mathcal{O}_2}^{B\tau} \right), \quad t, \tau \in \mathbf{R}. \tag{2.2}$$

Then $\phi(t, \tau) \equiv -\infty$ if and only if $P(A, \mathcal{O}_1) P(B, \mathcal{O}_2) = 0$. Assume that $P(A, \mathcal{O}_1) P(B, \mathcal{O}_2) \neq 0$. Denote

$$K := P(B, \mathcal{O}_2) P(A, \mathcal{O}_1) P(B, \mathcal{O}_2) \geqslant 0, \tag{2.3}$$

which is a nonzero operator. Then the function ϕ is a real, continuous, convex function on $\mathbb{R}^2$. The function

$$g(t) := \frac{\phi(t, t) - \ln \rho(P(A, \mathcal{O}_1) P(B, \mathcal{O}_2))}{t}$$

is a bounded continuous increasing function on $(0, \infty)$. Furthermore,

$$\begin{aligned} &\lim_{t \searrow 0} g(t) \\ &= \lim_{a \nearrow ||K||} \sup_{0 \neq \mathbf{x} \in P(K, (a, \infty)) \mathbf{H}} \frac{<((A + B) P(A, \mathcal{O}_1) + P(A, \mathcal{O}_1)(A + B)) \mathbf{x}, \mathbf{x}>}{2 ||K|| <\mathbf{x}, \mathbf{x}>}. \end{aligned} \tag{2.4}$$

Assume that $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{A}$. Then

$$\lim_{t \to \infty} g(t) = \sup_{\lambda \in \mathcal{O}_1, \mu \in \mathcal{O}_2, P(A, [\lambda, \infty) \cap \mathcal{O}_1)) P(B, [\mu, \infty) \cap \mathcal{O}_2))) \neq 0} \lambda + \mu. \tag{2.5}$$

In particular

$$\rho(e^{At}_{\mathcal{O}_1} e^{Bt}_{\mathcal{O}_2}) \leqslant e^{(\lambda+\mu)t} \quad \text{for} \quad t > 0. \tag{2.6}$$

To prove the theorem we need a few auxiliary results. $A \in \mathcal{S}$ is called positive (res. nonnegative) if $(A\mathbf{x}, \mathbf{x}) > 0, \quad \forall \mathbf{x} \neq 0$ (res. $(A\mathbf{x}, \mathbf{x}) \geqslant 0, \quad \forall \mathbf{x}$). A positive A is denoted by $A > 0$ and a nonnegative A is denoted by $A \geqslant 0$. For $A, B \in \mathcal{S}$ we let $A > B$ (res. $A \geqslant B$) if $A - B > 0$ (res. $A - B \geqslant 0$). Whenever we write $A > B$ ($A \geqslant B$) we assume that $A, B \in \mathcal{S}$.

Let $A \in \mathcal{S}$. Then $A \geqslant 0 \Leftrightarrow \sigma(A) \subset \mathbb{R}_+ := [0, \infty)$. Furthermore, $A > 0 \Leftrightarrow \sigma(A) \subset \mathbb{R}_+$ and $0 \notin \sigma_p(A)$. Here $\sigma_p(A)$ denotes the point spectrum of A. For $\alpha > 0$ let $g_\alpha(t) = (\max(0, t))^\alpha$. Let $g_0(t) = \lim_{\alpha \searrow 0} g_\alpha(t)$, i.e. $g_0(t) = 0$ for $t \leqslant 0$ and $g_0(t) = 1$ for $t > 0$.

Let $A \geqslant 0$. Define $A^\alpha := g_\alpha(A) \geqslant 0$. Note that $A^0 = P(A, (0, \infty))$. For $\alpha, \beta, t \in \mathbb{R}_+$ we have the standard relations $A^\alpha A^\beta = A^{\alpha+\beta}$ and $(A^\alpha)^t = A^{\alpha t}$. Furthermore $A > 0 \Leftrightarrow A^\alpha > 0$ for all $\alpha > 0$.

Let $A \geqslant B \geqslant 0$. Let $\alpha > 1$. Then there are examples of finite-dimensional $\mathbf{H}$, i.e. A, B are hermitian matrices, such that $A^\alpha \not\geqslant B^\alpha$, e.g. [11, Chapter 16, Section E]. Hence there exist compact self-adjoint A, B such that $A \geqslant B \geqslant 0$ and $A^\alpha \not\geqslant B^\alpha$, e.g. [1, Lemma 7]. Let $\alpha \in (0, 1)$. A remarkable result due to Loewner [10] for hermitian matrices, which extends to self-adjoint operators [1, Lemma 5] claims that $A^\alpha \geqslant B^\alpha$. Since A^α converge in the strong topology to A^0 as $\alpha \searrow 0$ we deduce $A^0 \geqslant B^0$. Note

$$A^0 \geqslant B^0 \Leftrightarrow P(A, (0, \infty)) \geqslant P(B, (0, \infty)) \Leftrightarrow P(A, (0, \infty))\mathbf{H} \supset P(B, (0, \infty))\mathbf{H}.$$

Although we are not using these results, they may be useful when trying to answer the open problems raised in Section 6.

The following results are well known to the experts and we state their proofs for completeness.

Proposition 2.2. *Let* $A_1 \geqslant A_2$, $B_1 \geqslant B_2 \geqslant 0$, $C \geqslant 0$, $D \in \mathcal{L}$. *Assume that* P, Q *are orthogonal projections. Then*

(a) $DA_1D^* \geqslant DA_2D^*$,
(b) $\rho(B_2C) = \rho(CB_2) = \rho(C^{\frac{1}{2}} B_2 C^{\frac{1}{2}}) = ||C^{\frac{1}{2}} B_2 C^{\frac{1}{2}}|| \leqslant \rho(B_1C) = \rho(CB_1)$,
(c) $PQ = 0 \Leftrightarrow PQP = 0 \Leftrightarrow QPQ = 0 \Leftrightarrow QP = 0$.

Proof. Clearly

$$A_1 \geqslant A_2 \Leftrightarrow \langle A_1\mathbf{x}, \mathbf{x} \rangle \geqslant \langle A_2\mathbf{x}, \mathbf{x} \rangle \quad \text{for any } \mathbf{x} \in \mathbf{H}.$$

Let $\mathbf{x} = D^*\mathbf{y}$ for any $\mathbf{y} \in \mathbf{H}$ and deduce (a).

To prove (b) we first assume that in addition to $B_1 \geqslant B_2 \geqslant 0$, $C \geqslant 0$ that C is invertible. That is there exists $\varepsilon > 0$ such that $\sigma(C) \subset [\varepsilon, \infty)$. Note that $\sigma(C^{\frac{1}{2}}) \subset [\varepsilon^{\frac{1}{2}}, \infty)$, i.e. $C^{\frac{1}{2}}$ is also invertible. Then

$$CB_2 = C(B_2C)C^{-1},$$

$$C^{\frac{1}{2}}B_2C^{\frac{1}{2}} = C^{\frac{1}{2}}(B_2C)C^{-\frac{1}{2}}$$

$$\Rightarrow \rho(B_2C) = \rho(CB_2) = \rho(C^{\frac{1}{2}}B_2C^{\frac{1}{2}}).$$

As $C^{\frac{1}{2}}B_2C^{\frac{1}{2}} \in \mathscr{S}$ it follows that $\rho(C^{\frac{1}{2}}B_2C^{\frac{1}{2}}) = ||C^{\frac{1}{2}}B_2C^{\frac{1}{2}}||$. The first part of the proposition yield that $C^{\frac{1}{2}}B_1C^{\frac{1}{2}} \geqslant C^{\frac{1}{2}}B_2C^{\frac{1}{2}} \geqslant 0$. Hence $||C^{\frac{1}{2}}B_1C^{\frac{1}{2}}|| \geqslant ||C^{\frac{1}{2}}B_2C^{\frac{1}{2}}||$. This proves (b) for $C > 0$ and invertible.

Assume now that $C \geqslant 0$. Let $\varepsilon > 0$ and define $C(\varepsilon) = C + \varepsilon I$. Then $C(\varepsilon) \geqslant \varepsilon I$ and the second inequality of the proposition holds for $C(\varepsilon)$. Let $\varepsilon \searrow 0$ to deduce (b) for any $C \geqslant 0$.

Let P, Q be orthogonal projections. Then

$$PQ(PQ)^* = PQ^2P = PQP, \quad (PQ)^*(PQ) = QP^2Q = QPQ$$

and (c) follows. $\square$

Definition 2.3. $A \in \mathscr{L}$ is called simple if A is self-adjoint and $\sigma(A)$ is a finite set.

Let A be simple. Then

$$\sigma(A) = \{\lambda_1, \dots, \lambda_n\} \subset \mathbb{R}, \quad \lambda_1 > \lambda_2 > \cdots > \lambda_n,$$

$$A = \sum_{i=1}^{n} \lambda_i P_i,$$

$$P(A, \{\lambda_i\}) = P_i \neq 0, \quad i = 1, \dots, n, \quad \sum_{i=1}^{n} P_i = I, \quad P_iP_j = \delta_{ij}P_i, \quad i,j = 1, \dots, n. \tag{2.7}$$

Vice versa, if $P_1, \dots, P_n$ are n orthogonal projections which satisfy $P_iP_j = 0$ for $i \neq j$ and are a decomposition of the identity: $\sum_{i=1}^{n} P_i = I$, then $A = \sum_{i=1}^{n} \lambda_i P_i$ is a simple operator if $\lambda_1, \dots, \lambda_n$ are real. Furthermore $\sigma(A) \subset \{\lambda_1, \dots, \lambda_n\}$. If each $P_i \neq 0$ then $\sigma(A) = \{\lambda_1, \dots, \lambda_n\}$.

Proposition 2.4. *Let $A \in \mathscr{S}$, $n > 1$ and assume that*

$$v_1 = ||A|| + 1 > v_2 > \cdots > v_n = -||A|| - 1.$$

Denote

$$\overline{A}_n = \sum_{i=1}^{n-1} v_i P(A, (v_{i+1}, v_i]),$$
$$\underline{A}_n = \sum_{i=1}^{n-1} v_{i+1} P(A, (v_{i+1}, v_i]).$$

Then $\underline{A}_n, \overline{A}_n, A$ *are commuting operators which satisfy*

$$\underline{A}_n \leqslant A \leqslant \overline{A}_n,$$
$$||A - \underline{A}_n||, ||A - \overline{A}_n|| \leqslant ||\overline{A}_n - \underline{A}_n|| \leqslant \max_{1 \leqslant i \leqslant n-1} (v_i - v_{i+1}).$$

Let $\mathcal{O} \in \mathscr{A}$ *be a disjoint union of k intervals* $\mathcal{O}_1, \dots, \mathcal{O}_k$ *such that each of the intervals* $\mathcal{O}_j$ *is of the form* $(v_{i+1}, v_i]$ *for some* $i = i(j) \in [1, n-1]$. *Then for any and* $t \in \mathbb{R}_+$

$$0 \leqslant e_{\mathcal{O}}^{\underline{A}_n t} \leqslant e_{\mathcal{O}}^{At} \leqslant e_{\mathcal{O}}^{\overline{A}_n t}. \tag{2.8}$$

Proof. The first claim of the proposition follows straightforward from the integral representation of $A \in \mathscr{S}$. To prove (2.8) observe first that $e_{\mathcal{O}}^{At} = \sum_{j=1}^{k} e_{\mathcal{O}_j}^{At}$. Thus it is enough to show (2.8) for $O_j = (v_{i+1}, v_j]$. Then

$$0 \leqslant e_{(v_{i+1}, v_i]}^{\underline{A}_n t} = e^{v_{i+1} t} P(A, (v_{i+1}, v_i]) \leqslant \int_{(v_{i+1}, v_i]} e^{\lambda t} dP(A, (-\infty, \lambda)) = e_{(v_{i+1}, v_i]}^{At}$$
$$\leqslant e^{v_i t} P(A, (v_{i+1}, v_i]) = e_{(v_{i+1}, v_i]}^{\overline{A}_n t}. \quad \square$$

Proof of Theorem 2.1. Suppose first that $\mathcal{O}_1, \mathcal{O}_2$ are Borel sets. Let

$$P(1) := P(A, \mathcal{O}_1), \quad P(2) = P(B, \mathcal{O}_2). \tag{2.9}$$

In view of (2.1)

$$C(t, \tau) := e_{\mathcal{O}_1}^{At} e_{\mathcal{O}_2}^{B\tau} = e^{At} P(1) P(2) e^{B\tau}.$$

Using the invertibility of e^{At} and $e^{B\tau}$ we obtain

$$C(t, \tau) = 0 \Leftrightarrow P(1)P(2) = 0 \Leftrightarrow \phi(t, \tau) = -\infty.$$

We now assume that $P(1)P(2) \neq 0$. Use (2.1) to obtain

$$e_{\mathcal{O}_1}^{A(t_1+t_2)} = e^{A(t_1+t_2)} P(1) = e^{At_1} e^{At_2} P(1) = e^{At_1} P(1) e^{At_2} P(1) = e_{\mathcal{O}_1}^{At_1} e_{\mathcal{O}_1}^{At_2}. \tag{2.10}$$

Since $e^{At} > 0$ it follows from (2.1) that $e_{\mathcal{O}_1}^{At} \geqslant 0$. Similar results hold for B. Use Proposition 2.2 and the above identities to deduce $e^{\phi(t,\tau)} = \rho(e_{\mathcal{O}_2}^{\frac{B\tau}{2}} e_{\mathcal{O}_1}^{At} e_{\mathcal{O}}^{\frac{B\tau}{2}})$. Clearly,

$D(t,\tau) := e_{\mathcal{O}_2}^{\frac{B\tau}{2}} e_{\mathcal{O}_1}^{At} e_{\mathcal{O}_2}^{\frac{B\tau}{2}}$ is a self-adjoint nonnegative operator family, which is continuous in (t,τ). The maximal characterization:

$$\rho(D(t,\tau)) = \sup_{\mathbf{x},\langle \mathbf{x},\mathbf{x}\rangle = 1} \langle D(t,\tau)x, x\rangle,$$

and the assumption that $P(1)P(2)\neq 0 \Rightarrow P(2)P(1)P(2)\neq 0$ implies straightforward that $\rho(D(t,\tau))$ is a positive continuous function on $\mathbf{R}^2$. Hence $\phi(t,\tau)$ is a continuous function on $\mathbf{R}^2$. Clearly,

$$\phi(0,0) = \ln \rho(P(1)P(2)) \leqslant \ln||P(1)P(2)|| \leqslant \ln\,(||P(1)||\,||P(2)||) = 0.$$

To show the convexity of ϕ on $\mathbb{R}^2$ it is enough to show

$$\phi\Big(\frac{t_1+t_2}{2}, \frac{\tau_1+\tau_2}{2}\Big) \leqslant \frac{\phi(t_1,\tau_1)+\phi(t_2,\tau_2)}{2}.$$

Use (2.10) and Proposition 2.2 to obtain

$$\begin{aligned}\ln\rho\left(e_{\mathcal{O}_1}^{A\frac{t_1+t_2}{2}} e_{\mathcal{O}_2}^{B\frac{\tau_1+\tau_2}{2}}\right) &= \ln\rho\left(e_{\mathcal{O}_1}^{A\frac{t_1}{2}} e_{\mathcal{O}_1}^{A\frac{t_2}{2}} e_{\mathcal{O}_2}^{B\frac{\tau_2}{2}} e_{\mathcal{O}_2}^{B\frac{\tau_1}{2}}\right)\\ &= \ln\rho\left(e_{\mathcal{O}_2}^{B\frac{\tau_1}{2}} e_{\mathcal{O}_1}^{A\frac{t_1}{2}} e_{\mathcal{O}_1}^{A\frac{t_2}{2}} e_{\mathcal{O}_2}^{B\frac{\tau_2}{2}}\right) = \ln\rho(F_1F_2),\end{aligned}$$

where $F_1 = e_{\mathcal{O}_2}^{B\frac{\tau_1}{2}} e_{\mathcal{O}_1}^{A\frac{t_1}{2}}$ and $F_2 = e_{\mathcal{O}_1}^{A\frac{t_2}{2}} e_{\mathcal{O}_2}^{B\frac{\tau_2}{2}}$. Then, since $\rho(X)\leqslant||X||$ for a bounded operator X, it follows

$$\ln||F_1F_2|| \leqslant \ln\,(||F_1||||F_2||).$$

Since $||X||^2 = \rho(XX^*)$ we deduce

$$\begin{aligned}\ln\,(\rho(F_1F_1^*)^{\frac{1}{2}}\rho(F_2F_2^*)^{\frac{1}{2}}) &= \frac{1}{2}(\ln\rho(F_1F_1^*) + \ln\rho(F_2F_2^*))\\ &= \frac{1}{2}\left(\ln\rho\left(e_{\mathcal{O}_2}^{B\frac{\tau_1}{2}} e_{\mathcal{O}_1}^{A\frac{t_1}{2}}\left(e_{\mathcal{O}_2}^{B\frac{\tau_1}{2}} e_{\mathcal{O}_1}^{A\frac{t_1}{2}}\right)^*\right) + \ln\rho\left(e_{\mathcal{O}_1}^{A\frac{t_2}{2}} e_{\mathcal{O}_2}^{B\frac{\tau_2}{2}}\left(e_{\mathcal{O}_1}^{A\frac{t_2}{2}} e_{\mathcal{O}_2}^{B\frac{\tau_2}{2}}\right)^*\right)\right)\\ &= \frac{1}{2}\left(\ln\rho\left(e_{\mathcal{O}_2}^{B\frac{\tau_1}{2}} e_{\mathcal{O}_1}^{A\frac{t_1}{2}} e_{\mathcal{O}_1}^{A\frac{t_1}{2}} e_{\mathcal{O}_2}^{B\frac{\tau_1}{2}}\right) + \ln\rho\left(e_{\mathcal{O}_1}^{A\frac{t_2}{2}} e_{\mathcal{O}_2}^{B\frac{\tau_2}{2}} e_{\mathcal{O}_2}^{B\frac{\tau_2}{2}} e_{\mathcal{O}_1}^{A\frac{t_2}{2}}\right)\right)\\ &= \frac{1}{2}\left(\ln\rho\left(e_{\mathcal{O}_1}^{At_1} e_{\mathcal{O}_2}^{B\tau_1}\right) + \ln\rho\left(e_{\mathcal{O}_1}^{At_2} e_{\mathcal{O}_2}^{B\tau_2}\right)\right),\end{aligned}$$

establishing the convexity.

Let $f(t) := \phi(t,t) - \phi(0,0)$, $t\in\mathbf{R}$. Then $f(t)$ is a continuous convex function on $\mathbf{R}$. Furthermore, $f(0) = 0$. Hence $g(t)$ is nondecreasing on $(0,\infty)$, e.g. [3, Lemma 10].

We first show (2.4). Recall that

$$e^{\phi(t,t)} = \rho(D(t,t)) = ||D(t,t)|| = \sup_{\mathbf{x}\neq 0} \frac{\langle D(t,t)\mathbf{x}, \mathbf{x}\rangle}{\langle \mathbf{x}, \mathbf{x}\rangle}.$$

Thus $h(t) := \rho(D(t,t))$ is a continuous function with $h(0) = ||K||$. Use (2.1) to obtain

$$D(t,t) = P(2)e^{\frac{Bt}{2}}\, P(1)e^{At}P(1)e^{\frac{Bt}{2}}P(2) = K + tE + t^2R(t),$$

$$E = \frac{P(2)BP(1)P(2) + P(2)P(1)BP(2)}{2} + P(2)P(1)AP(1)P(2),$$

$$||R(t)|| \leqslant r, \quad \text{for } t \in [0,1] \text{ and some } r > 0.$$

In what follows we always assume that $t \in [0,1]$. Thus

$$h(t) = \sup_{0\neq \mathbf{x}\in \mathbf{H}} \frac{\langle (K + tE + t^2R(t))\mathbf{x}, \mathbf{x}\rangle}{\langle \mathbf{x}, \mathbf{x}\rangle}.$$

Let

$$s(a) := \sup_{0\neq \mathbf{x}\in P(K,(a,\infty))\mathbf{H}} \frac{\langle ((A+B)P(1) + P(1)(A+B))\mathbf{x}, \mathbf{x}\rangle}{2\langle \mathbf{x}, \mathbf{x}\rangle}, \text{ for } a \in (0, ||K||).$$

Then $s(a)$ is a nonincreasing function on $(0, ||K||)$. Let $s := \lim_{a\nearrow ||K||} s(a)$. Note that $P(1)AP(1) = AP(1) = P(1)A$ and for $a > 0$ $P(K,(a,\infty))\mathbf{H} \subset P(2)\mathbf{H}$. Hence $\langle ((A + B)P(1) + P(1)(A + B))\mathbf{x}, \mathbf{x}\rangle = \langle 2E\mathbf{x}, \mathbf{x}\rangle$ for any $\mathbf{x} \in P(K,(a,\infty))\mathbf{H}$. Since $t \in [0,1]$, for any $a \in (0, ||K||)$

$$h(t) \geqslant \sup_{0\neq \mathbf{x}\in P(K,(a,\infty))\mathbf{H}} \frac{\langle (K + tE + t^2R(t))\mathbf{x}, \mathbf{x}\rangle}{\langle \mathbf{x}, \mathbf{x}\rangle} \geqslant a + s(a)t - rt^2.$$

Let $a \nearrow ||K||$ and deduce $h(t) \geqslant ||K|| + st - rt^2$. Thus $\lim_{t\searrow 0} g(t) \geqslant \frac{s}{||K||}$. Fix $a \in (0, ||K||)$ and consider the ratio

$$b(t,\mathbf{x}) = \frac{\langle (K + tE + t^2R(t))\mathbf{x}, \mathbf{x}\rangle}{\langle \mathbf{x}, \mathbf{x}\rangle} \quad \text{for } \mathbf{x} \neq 0.$$

For $\mathbf{x} \in P(K,(a,\infty))\mathbf{H}$ we get

$$b(t,\mathbf{x}) \leqslant ||K|| + ts(a) + rt^2. \tag{2.11}$$

For $\mathbf{x} \in P(K,(-\infty, a])\mathbf{H}$ we have

$$b(t,\mathbf{x}) \leqslant a + t||E|| + rt^2.$$

Let

$$t(a) = \min\left(1, \frac{||K|| - a}{||E|| - s(a)}, \frac{||K|| - a}{4||E||}\right).$$

Then (2.11) holds for $\mathbf{x} \in P(K, (-\infty, a])\mathbf{H}$ and $t \in [0, t(a)]$. Assume now that $\mathbf{x} \notin P(K, (-\infty, a])\mathbf{H} \cup P(K, (a, \infty)\mathbf{H}$. Let

$$\mathbf{e}_1 = \frac{P(K, (-\infty, a])\mathbf{x}}{||P(K, (-\infty, a])\mathbf{x}||}, \quad \mathbf{e}_2 = \frac{P(K, (a, \infty))\mathbf{x}}{||P(K, (a, \infty))\mathbf{x}||}, \quad \mathbf{X} = \mathrm{span}(\mathbf{e}_1, \mathbf{e}_2).$$

As $\mathbf{e}_1, \mathbf{e}_2$ is an orthonormal basis in $\mathbf{X}$ it follows

$$\begin{aligned}
\max_{0 \neq \mathbf{x} \in \mathbf{X}} \frac{\langle ((K + tE)\mathbf{x}, \mathbf{x} \rangle}{\langle \mathbf{x}, \mathbf{x} \rangle} &= \lambda_1(K_1 + tE_1), \\
K_1 &:= \begin{pmatrix} \langle K\mathbf{e}_1, \mathbf{e}_1 \rangle & \langle K\mathbf{e}_1, \mathbf{e}_2 \rangle \\ \langle K\mathbf{e}_2, \mathbf{e}_1 \rangle & \langle K\mathbf{e}_2, \mathbf{e}_2 \rangle \end{pmatrix} = \begin{pmatrix} \langle K\mathbf{e}_1, \mathbf{e}_1 \rangle & 0 \\ 0 & \langle K\mathbf{e}_2, \mathbf{e}_2 \rangle \end{pmatrix} \\
\leqslant K_2 &:= \begin{pmatrix} a & 0 \\ 0 & ||K|| \end{pmatrix}, \\
E_1 &:= \begin{pmatrix} \langle E\mathbf{e}_1, \mathbf{e}_1 \rangle & \langle E\mathbf{e}_1, \mathbf{e}_2 \rangle \\ \langle E\mathbf{e}_2, \mathbf{e}_1 \rangle & \langle E\mathbf{e}_2, \mathbf{e}_2 \rangle \end{pmatrix},
\end{aligned}$$

where $\lambda_1(C)$ is the maximal eigenvalue of any 2×2 hermitian matrix C. Then for $t \in (0, t(a))$

$$\begin{aligned}
\lambda_1(K_1 + tE_1) &\leqslant \lambda_1(K_2 + tE_1) \\
&= \frac{1}{2}(||K|| + a + t(\langle E\mathbf{e}_2, \mathbf{e}_2 \rangle + \langle E\mathbf{e}_1, \mathbf{e}_1 \rangle) \\
&\quad + \frac{1}{2}\sqrt{(||K|| - a + t(\langle E\mathbf{e}_2, \mathbf{e}_2 \rangle - \langle E\mathbf{e}_1, \mathbf{e}_1 \rangle))^2 + 4t^2|\langle E\mathbf{e}_1, \mathbf{e}_2 \rangle|^2} \\
&\leqslant \frac{1}{2}(||K|| + a + t(\langle E\mathbf{e}_2, \mathbf{e}_2 \rangle + \langle E\mathbf{e}_1, \mathbf{e}_1 \rangle) \\
&\quad + \frac{1}{2}(||K|| - a + t(\langle E\mathbf{e}_2, \mathbf{e}_2 \rangle - \langle E\mathbf{e}_1, \mathbf{e}_1 \rangle)) \\
&\quad + \frac{t^2|\langle E\mathbf{e}_1, \mathbf{e}_2 \rangle|^2}{||K|| - a + t(\langle E\mathbf{e}_2, \mathbf{e}_2 \rangle - \langle E\mathbf{e}_1, \mathbf{e}_1 \rangle)} \\
&\leqslant ||K|| + t\langle E\mathbf{e}_2, \mathbf{e}_2 \rangle + \frac{2t^2|\langle E\mathbf{e}_1, \mathbf{e}_2 \rangle|^2}{||K|| - a} \\
&\leqslant ||K|| + ts(a) + \frac{2t^2||E||}{||K|| - a}.
\end{aligned}$$

Then for $t \in (0, t(a))$

$$\begin{aligned} b(t, \mathbf{x}) &\leqslant ||K|| + ts(a) + t^2\left(\frac{2||E||}{||K|| - a} + r\right) \\ &\Rightarrow h(t) \leqslant ||K|| + ts(a) + t^2\left(\frac{2||E||}{||K|| - a} + r\right) \\ &\Rightarrow g(t) \leqslant \frac{1}{t}\left(\ln\left(||K|| + ts(a) + t^2\left(\frac{2||E||}{||K|| - a} + r\right)\right) - \ln||K||\right) \\ &\Rightarrow \lim_{t \searrow 0} g(t) \leqslant \frac{s(a)}{||K||}. \end{aligned}$$

Let $a \nearrow ||K||$ to deduce $\lim_{t \searrow 0} g(t) \leqslant \frac{s}{||K||}$. The proof of (2.4) is completed.

We now prove (2.5) for $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{A}$. We first establish (2.5) in the case where A and B are simple operators. Let A be of form (2.7) and $B = \sum_{j=1}^{m} \mu_j Q_j$ be of similar form. In particular

$$\sigma(B) = \{\mu_1, \ldots, \mu_m\}, \quad \mu_1 > \cdots > \mu_m.$$

Let

$$\sigma'(A) := \sigma(A) \cap \mathcal{O}_1 = \{\lambda_{i_1}, \ldots, \lambda_{i_p}\},\ 1 \leqslant i_1 < \cdots < i_p \leqslant n,$$

$$\sigma'(B) := \sigma(B) \cap \mathcal{O}_2 = \{\mu_{j_1}, \ldots, \mu_{j_q}\},\ 1 \leqslant j_1 < \cdots < i_q \leqslant m.$$

The assumption $P(1)P(2) \neq 0$ yields that $\sigma'(A) \neq \emptyset, \sigma'(B) \neq \emptyset$. Clearly

$$e^{At}_{\mathcal{O}_1} = \sum_{k=1}^{p} e^{\lambda_{i_k} t} P_{i_k}, \quad e^{Bt}_{\mathcal{O}_2} = \sum_{\ell=1}^{q} e^{\mu_{j_\ell} t} Q_{j_\ell}.$$

In view of Proposition 2.2 it is enough to consider the spectral radius of

$$\begin{aligned} D(t, t) = e^{\frac{Bt}{2}}_{\mathcal{O}_2} e^{At}_{\mathcal{O}_1} e^{\frac{Bt}{2}}_{\mathcal{O}_2} &= \sum_{k, \ell, \ell' = 1}^{p,q,q} e^{\left(\lambda_{i_k} + \frac{\mu_{j_\ell} + \mu_{j_{\ell'}}}{2}\right) t} Q_{j_\ell} P_{i_k} Q_{j_{\ell'}} \\ &= \sum_{r=1}^{s} e^{\theta_r t} R_r, \quad \theta_1 > \theta_2 > \cdots > \theta_s. \end{aligned} \tag{2.12}$$

In the above first sum we deleted the zero terms corresponding $Q_{j_\ell} P_{i_k} Q_{j_{\ell'}} = 0$ and rearranged the nonzero terms in decreasing order of the exponentials ($t > 0$). (Recall that the left-hand side of the above equality is not identically zero.) Proposition 2.2 claims that $Q_{j_\ell} P_{i_k} = 0 \Leftrightarrow P_{i_k} Q_{j_\ell} = 0$. So

$$\begin{aligned} \theta_1 &= \max_{P_{i_k} Q_{j_\ell} \neq 0} \lambda_{i_k} + \mu_{j_\ell}, \\ 0 \neq R_1 &= \sum_{k, \ell, P_{i_k} Q_{j_\ell} \neq 0} Q_{j_\ell} P_{i_k} Q_{j_\ell} \geqslant 0. \end{aligned} \tag{2.13}$$

As $\lim_{t\to\infty} e^{-\theta_1 t}D(t,t) = R_1$ we deduce that

$$\lim_{t\to\infty} \frac{\ln \rho((D(t,t))}{t} = \lim_{t\to\infty} \theta_1 + \frac{\ln \rho(e^{-\theta_1 t}D(t,t))}{t} = \theta_1 \Rightarrow \lim_{t\to\infty} \frac{g(t)}{t} = \theta_1.$$

It is straightforward to check in this case that

$$\theta_1 = \sup_{\lambda\in\mathcal{O}_1,\mu\in\mathcal{O}_2,P(A,[\lambda,\infty)\cap\mathcal{O}_1))P(B,[\mu,\infty)\cap\mathcal{O}_2))\neq 0} \lambda+\mu.$$

Hence (2.5) holds for simple A and B.

To prove (2.5) for arbitrary $A, B\in\mathcal{S}$ we use Proposition 2.4. Recall that $\mathcal{O}_1, \mathcal{O}_2\in\mathcal{A}$. Clearly, it is enough to assume that $\mathcal{O}_1\subset(-||A||-1, ||A||+1]$, $\mathcal{O}_2\subset(-||B||-1, ||B||+1]$. (Otherwise replace $\mathcal{O}_1, \mathcal{O}_2$ by $\mathcal{O}_1\cap(-||A||-1, ||A||+1]$, $\mathcal{O}_2\cap(-||B||-1, ||B||+1]$ respectively.) Let $\varepsilon>0$ be given. Choose n big enough so the following conditions hold. We subdivide the interval $(-||A||-1, ||A||+1]$ to n subintervals $(v_{i+1}, v_i]$, $i=1,\dots,n-1$ such that $v_i - v_{i+1}<\varepsilon$, $i=1,\dots,n-1$ and $\mathcal{O}_1$ is a union of some of the intervals $(v_{i+1}, v_i]$. (That is, the end points of the disjoint intervals in $\mathcal{O}_1$ appear in the set $\{v_1,\dots,v_n\}$.) Let $\underline{A}_n, \overline{A}_n$ be defined as in Proposition 2.4. Repeat the same construction for B. That is, subdivide the interval $(-||B||-1, ||B||+1]$ to n subintervals $(v'_{i+1}, v'_i]$, $i=1,\dots,n-1$ such that $v'_i - v'_{i+1}<\varepsilon$, $i=1,\dots,n-1$ and $\mathcal{O}_2$ is a union of some of the intervals $(v_{i+1}, v_i]$. Then

$$\overline{B}_n = \sum_{i=1}^{n-1} v'_i P(B,(v'_{i+1}, v'_i]),$$
$$\underline{B}_n = \sum_{i=1}^{n-1} v'_{i+1} P(B,(v'_{i+1}, v'_i]).$$

Combine (2.8) for the operators A and B with Proposition 2.2 to obtain for $t\in\mathbb{R}_+$:

$$\rho(e^{\underline{A}_n t}_{\mathcal{O}_1} e^{\underline{B}_n t}_{\mathcal{O}_2}) \leqslant \rho(e^{At}_{\mathcal{O}_1} e^{\underline{B}_n t}_{\mathcal{O}_2}) \leqslant \rho(e^{At}_{\mathcal{O}_1} e^{Bt}_{\mathcal{O}_2}) \leqslant \rho(e^{\overline{A}_n t}_{\mathcal{O}_1} e^{Bt}_{\mathcal{O}_2}) \leqslant \rho(e^{\overline{A}_n t}_{\mathcal{O}_1} e^{\overline{B}_n t}_{\mathcal{O}_2}). \tag{2.14}$$

Let

$$\theta = \sup_{\lambda\in\mathcal{O}_1,\mu\in\mathcal{O}_2,P(A,[\lambda,\infty)\cap\mathcal{O}_1)P(B,[\mu,\infty)\cap\mathcal{O}_2))\neq 0} \lambda+\mu,$$
$$\overline{\theta}_n = \sup_{\lambda\in\mathcal{O}_1,\mu\in\mathcal{O}_2,P(\overline{A}_n,[\lambda,\infty)\cap\mathcal{O}_1)P(\overline{B}_n,[\mu,\infty)\cap\mathcal{O}_2))\neq 0} \lambda+\mu,$$
$$\underline{\theta}_n = \sup_{\lambda\in\mathcal{O}_1,\mu\in\mathcal{O}_2,P(\underline{A}_n,[\lambda,\infty)\cap\mathcal{O}_1)P(\underline{B}_n,[\mu,\infty)\cap\mathcal{O}_2))\neq 0} \lambda+\mu.$$

It is straightforward to show

$$\underline{\theta}_n \leqslant \theta \leqslant \overline{\theta}_n, \quad \underline{\theta}_n > \theta - 2\varepsilon, \quad \theta > \overline{\theta}_n - 2\varepsilon.$$

Combine the above inequalities with (2.14) and with the fact that (2.5) holds for simple operators to obtain that

$$\theta - 2\varepsilon < \lim_{t\to\infty} g(t) < \theta + 2\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary we deduce (2.5) for arbitrary $A, B \in \mathscr{S}$.

To show (2.6) observe that for $t > 0$

$$\rho(e^{At}_{\mathcal{O}_1} e^{Bt}_{\mathcal{O}_2}) = e^{tg(t)}\rho(P(A,\mathcal{O}_1)P(B,\mathcal{O}_2)) \leqslant e^{(\lambda+\mu)t}||P(A,\mathcal{O}_1)P(B,\mathcal{O}_2)|| \leqslant e^{(\lambda+\mu)t}. \quad \square$$

Corollary 2.5. *Let $A, B \in \mathscr{S}$. Then $\ln \rho(e^{At} e^{B\tau})$ is a continuous convex function on $\mathbb{R}^2$. The function $g(t) = \frac{\ln \rho(e^{At}e^{Bt})}{t}$ is a nondecreasing function on $(0, \infty)$ such that*

$$\lim_{t\searrow 0} g(t) = \ln \rho(e^{A+B}) \leqslant g(1) = \ln \rho(e^A e^B)$$
$$\leqslant \lim_{t\to\infty} g(t) = \sup_{\lambda\in\mathbb{R},\mu\in\mathbb{R},P(A,[\lambda,\infty))P(B,[\mu,\infty))\neq 0} \lambda + \mu. \qquad (2.15)$$

Proof. One needs only to show the first limit in (2.15). This follows from the Lie–Trotter formula [13, VIII.8]

$$\lim_{t\searrow 0} (e^{At}e^{Bt})^{\frac{1}{t}} = e^{A+B}.$$

One can also deduce the first equality of (2.15) from (2.4). $\square$

3. The main lemma

For $T \in \mathscr{L}$ let $\sigma_p(T) \subset \sigma(T)$ be the point spectrum of T, i.e. the set of eigenvalues of T. $T \in \mathscr{L}$ is called *diagonable* if **H** has an orthonormal basis consisting of eigenvectors of T. Assume that T is diagonable and $\sigma(T) \subset \mathbb{R}$. Then T is self-adjoint and $\sigma(T)$ is the closure of $\sigma_p(T)$. Since **H** is separable $\sigma_p(T)$ is countable.

Definition 3.1. Let $\mathscr{D} \subset \mathscr{S}$ be the family of all diagonable A satisfying the following conditions:

(a) $\sigma_p(A)$ is a discrete set ($\sigma_p(A)$ does not have an accumulation point);
(b) for each $\alpha \in \sigma_p(A)$ $\text{ind}(A, \alpha) \coloneqq \dim\ P(A, \{\alpha\})$ is finite;
(c) for any $t \in \sigma \backslash \sigma_p(A)$ there exists $\varepsilon(t) > 0$ such that $(t - \varepsilon(t), t) \cap \sigma(A) = \emptyset$.

Let $\mathscr{D}_+ \coloneqq \{A \in \mathscr{D} : \quad A > 0\}$.

It is straightforward to show that for any $A \in \mathscr{D}$, $\sigma(A)$ is a countable set. (We are not going to use this fact.) Denote by $\mathscr{C} \subset \mathscr{L}$ the ideal of compact operators. Let $A \in \mathscr{C}$. Then $\sigma(A)$ is a countable set with 0 the only possible accumulation point. Furthermore $A \in \mathscr{C} \cap \mathscr{S}$ is diagonable, also for each $\lambda \in \sigma(A) \backslash \{0\}$ the subspace $P(A, \{\lambda\})\mathbf{H}$ is finite dimensional.

Assume that $A \in \mathscr{C} \cap \mathscr{S}$ and $A \geqslant 0\ (\neq 0)$. Then

$$\sigma(A)\backslash\{0\} = \bigcup_{i \in \mathscr{I}} \tilde{\alpha}_i, \quad \tilde{\alpha}_1 > \cdots > \tilde{\alpha}_i > 0 \text{ for each } i \in \mathscr{I}. \tag{3.1}$$

The *eigenvalue sequence* of A, which is a nonnegative nonincreasing sequence $\{\alpha_i\}_1^\infty$ converging to 0, is defined as follows:

$$\begin{aligned} &\mathrm{ind}_0(A) = 0,\ \mathrm{ind}_j(A) = \dim\ P\left(A, \bigcup_{k=1}^{j} \{\tilde{\alpha}_k\}\right), \quad \text{for each } j \in \mathscr{I} \\ &\alpha_k = \tilde{\alpha}_j \text{ for } k = \mathrm{ind}_{j-1}(A) + 1, \ldots, \mathrm{ind}_j(A), \\ &\text{if } \#\mathscr{I} = n < \infty \text{ then } \alpha_k = 0 \text{ for each } k > \mathrm{ind}_n(A), \end{aligned} \tag{3.2}$$

where $\#\mathscr{I}$ is the cardinality of $\mathscr{I}$. Note that $A \in \mathscr{D}_+ \Leftrightarrow A > 0$. We will give later other natural examples of $A \in \mathscr{D}_+$.

Let

$$\mathscr{K} := \{B = I + A : \quad A \in \mathscr{C}\}.$$

Then $\mathscr{K}$ is a closed convex set in $\mathscr{L}$. Furthermore $\mathscr{K}$ is a semigroup: $\mathscr{K}\mathscr{K} \subset \mathscr{K}$. For $A, B \in \mathscr{S}$ and $t \in \mathbb{R}^*(= \mathbb{R}\backslash\{0\})$ let

$$\Lambda(t) := e^{\frac{Bt}{2}}\, e^{At}\, e^{\frac{Bt}{2}},$$

$$\Omega(t) := (e^{\frac{Bt}{2}}\, e^{At}\, e^{\frac{Bt}{2}})^{\frac{1}{t}} = \Lambda(t)^{\frac{1}{t}}.$$

Proposition 3.2. *Let $A, B \in \mathscr{S}$. Then for each $t \in \mathbb{R}^*$ there exists a unique $C(t) = C(A, B, t) \in \mathscr{S}$ such that*

$$\Lambda(t) = e^{tC(t)}, \quad \Omega(t) = e^{C(t)}, \quad C(-t) = C(t), \quad ||C(t)|| \leqslant ||A|| + ||B||,$$

and

$$\lim_{t \to 0} C(t) = A + B.$$

If $A, B \geqslant 0$ then $C(t) \geqslant 0$. If either $A > 0$ and $B \geqslant 0$ or $A \geqslant 0$ and $B > 0$ then $C(t) > 0$. If $A, B \in \mathscr{S} \cap \mathscr{C}$ then $C(t) \in \mathscr{S} \cap \mathscr{C}$ for each $t \in \mathbb{R}^*$.

Proof. Clearly $\Lambda(-t) = \Lambda(t)^{-1}$ for $t \in \mathbb{R}^*$. Hence $C(-t) = C(t)$ for $t \in \mathbb{R}^*$. To prove the rest of the proposition we assume that $t > 0$. Let $A \in \mathscr{S}$. Then $e^{t||A||}I \geqslant e^{At} \geqslant e^{-t||A||}I$. Assume that $B \in \mathscr{S}$. Use Proposition 2.2 to obtain

$$\Lambda(t) \leqslant e^{\frac{Bt}{2}}(e^{t||A||}I)\, e^{\frac{Bt}{2}} = e^{t||A||}e^{Bt} \leqslant e^{t||A||}e^{t||B||}I = e^{t(||A||+||B||)}I,$$

$$\Lambda(t) \geqslant e^{\frac{Bt}{2}}(e^{-t||A||}I)e^{\frac{Bt}{2}} = e^{-t||A||}e^{Bt} \geqslant e^{-t||A||}e^{-t||B||}I = e^{-t(||A||+||B||)}I.$$

Hence $C(t) = \frac{1}{t}\ln \Lambda(t) \in \mathscr{S}$ and $||C(t)|| \leqslant ||A|| + ||B||$. The equality $\lim_{t\to 0} C(t) = A + B$ follows from the Lie–Trotter formula [13, VIII.8]. Assume that $A > 0$, $B \geqslant 0$. Then $e^{At} > I$, $e^{Bt} \geqslant I$ and $\Lambda(t) > e^{\frac{Bt}{2}}\, I e^{\frac{Bt}{2}} = e^{Bt} \geqslant I$. Hence $C(t) > 0$. Similarly if $A \geqslant 0$ and $B > 0$ then $C(t) > 0$. For the case $A, B \geqslant 0$ it is clear that $C(t) \geqslant 0$.

Assume that $A, B \in \mathscr{S} \cap \mathscr{C}$. Then $e^{At}, e^{Bt} \in \mathscr{K} \Rightarrow \Lambda(t) \in \mathscr{K}$. Hence $\Lambda(t) = I + D(t)$ where $D(t) \in \mathscr{S} \cap \mathscr{C}$. Then $\Lambda(t) \geqslant e^{-t(||A||+||B||)} I$ is diagonable and 1 is the only possible accumulation point of $\sigma(\Lambda(t))$. Thus $C(t)$ is diagonable with 0 the only possible accumulation point in $\sigma(C(t))$. Hence $C(t) \in \mathscr{C}$. $\square$

Combine the above Proposition with Corollary 2.5 to obtain:

Corollary 3.3. *Let $A, B \in \mathscr{S}$, $t \in (0, \infty)$ and assume that $C(t)$ is defined as in Proposition 3.2. Then*

$$g(t) = \frac{\ln \rho(e^{At}\, e^{Bt})}{t} = \frac{\ln||\Lambda(t)||}{t} = \sup_{\mathbf{x} \in \mathbf{H}, ||\mathbf{x}||=1} \langle C(t)\mathbf{x}, \mathbf{x} \rangle$$

is a nondecreasing function on $(0, \infty)$ which converges to

$$\omega_1 := \sup_{\lambda \in \mathbb{R}, \mu \in \mathbb{R}, P(A,[\lambda,\infty))P(B,[\mu,\infty)) \neq 0} \lambda + \mu \tag{3.3}$$

as $t \to \infty$. If $A, B \geqslant 0$ then

$$g(t) = ||C(t)||$$

is a nondecreasing function on $(0, \infty)$ and $\lim_{t\to\infty} ||C(t)|| = \omega_1$.

The main result of this section is:

Lemma 3.4. *Let $A, B \in \mathscr{D}$. Let $t \in (0, \infty)$ and assume that $C(t)$ is defined as in Proposition 3.2. Let ω_1 be defined by (3.3). Then there exists $\lambda_1, \ldots, \lambda_m \in \sigma_p(A)$, $\mu_1, \ldots, \mu_m \in \sigma_p(B)$ with the following properties:*

$$\omega_1 = \max_{\lambda \in \sigma_p(A), \mu \in \sigma_p(B), P(A,\{\lambda\})P(B,\{\mu\}) \neq 0} \lambda + \mu,$$

$$\lambda_1 > \cdots > \lambda_m,\ \mu_1 > \cdots > \mu_m,\quad \bigcup_{i=1}^{m} \{\lambda_i\} \subset \sigma_p(A),\ \bigcup_{i=1}^{m} \{\mu_i\} \subset \sigma_p(B),$$

$$\lambda_i + \mu_{m-i+1} = \omega_1,\ P(A, \{\lambda_i\})P(B, \{\mu_i\}) \neq 0, \quad i = 1, \ldots, m,$$

$$\theta_1 := \sup_{\lambda \in \sigma_p(A)\setminus \cup_1^m \{\lambda_i\}, \mu \in \sigma_p(B), P(A,\{\lambda\})P(B,\{\mu\}) \neq 0} \lambda + \mu,$$

$$\theta_2 := \sup_{\lambda \in \sigma_p(A), \mu \in \sigma_p(B)\setminus \cup_1^m \{\mu_i\}, P(A,\{\lambda\})P(B,\{\mu\}) \neq 0} \lambda + \mu,$$

$$\theta := \max(\theta_1, \theta_2) < \omega_1. \tag{3.4}$$

Let $\varepsilon := \frac{\omega_1 - \theta}{4} > 0$

and

$$P_{1,i} := P(P(B,\{\mu_i\})P(A,\{\lambda_{m-i+1}\})P(B,\{\mu_i\})\mathbf{H}),$$

$$n_{1,i} = \dim\ P_{1,i} > 0,\ i = 1, \dots, m,$$

$$P_1 := \sum_{i=1}^{m} P_{1,i}. \tag{3.5}$$

Then $P_{1,i}P_{1,j} = \delta_{ij}P_{1,i},\ i,j = 1, \dots, m$. *Hence* P_1 *is an orthogonal projection commuting with* B *and* $n_1 := \dim\ P_1 = \sum_{i=1}^{m} n_{1,i}$. *There exists* $t_0 \gg 1$ *such that*

$$\sigma(C(t)) \subset \left(-\infty, \frac{\omega_1 + \theta + \varepsilon}{2}\right) \cup (\omega_1 - \varepsilon, \omega_1]$$

and

$$\dim\ P(C(t), (\omega_1 - \varepsilon, \infty)) = n_1 \text{ for } t > t_0,$$

$$\lim_{t \to \infty} C(t)|_{P(C(t),(\omega_1-\varepsilon,\infty))} = \omega_1 I_{n_1}, \quad \lim_{t \to \infty} P(C(t), (\omega_1 - \varepsilon, \infty)) = P_1, \tag{3.6}$$

where I_{n_1} *is* $n_1 \times n_1$ *identity matrix.*

Proof. Let $\mathcal{O} \subset \mathbb{R}$ be a Borel set. Then for any $D \in \mathcal{D}$ we have

$$P(D, \mathcal{O}) = \sum_{\lambda \in \sigma_p(D) \cap \mathcal{O}} P(D, \{\lambda\}). \tag{3.7}$$

Here the countable summation should be understood as a limit in the strong operator topology. Thus

$$P(A, [s, \infty))P(B, [t, \infty)) \neq 0 \Rightarrow P(A, \{\lambda\})P(B, \{\mu\}) \neq 0,$$

$$\text{for some } \lambda \in \sigma_p(A) \cap [s, \infty),\ \mu \in \sigma_p(B) \cap [t, \infty).$$

Clearly $s + t \leqslant \lambda + v$. Hence (3.3) implies the characterization

$$\omega_1 = \sup_{\lambda \in \sigma_p(A), \mu \in \sigma_p(B), P(A,\{\lambda\})P(B,\{\mu\}) \neq 0} \lambda + \mu.$$

Assume to the contrary that the supremum is not achieved. Then we have two bounded sequences $\{\lambda_i\}_1^\infty \subset \sigma_p(A)$, $\{\mu_i\}_1^\infty \subset \sigma_p(B)$ such that $\{\lambda_i + \mu_i\}_1^\infty$ is a strictly increasing sequence whose limit is ω_1. By taking subsequences we may assume that $\{\lambda_i\}_1^\infty$, $\{\mu_i\}_1^\infty$ are two converging monotonic sequences. By taking subsequences again we may assume that at least one of the sequences is strictly monotonic and the other sequence is either a constant sequence or strictly monotonic. If one of the sequences is a constant sequence than the other one must be a strictly increasing sequence. If both are strict monotonic then one of them must be strictly increasing. Assume for simplicity of notation $\{\lambda_i\}_1^\infty$ is strictly increasing. Then $\lim_{i\to\infty}\lambda_i = a\in\sigma(A)$. Since $A\in\mathscr{D}$ there exists $\delta>0$ such that $(a-\delta,a)\cap\sigma(A)=\emptyset$. This contradicts the assumption that $\{\lambda_i\}_1^\infty\subset\sigma_p(A)$ is a strictly increasing sequence which converges to a. That is, $\omega=\lambda+v$ and $P(A,\{\lambda\})P(B,\{\mu\})\neq 0$. The above argument also shows that there is a finite number $\lambda\in\sigma_p(A)$ and $\mu\in\sigma_p(B)$ which satisfy this property. The same argument shows that the suprema for θ_1 and θ_2 must be achieved. Clearly these maxima cannot be equal to ω_1. Hence $\theta<\omega_1$.

Let

$$K_i := P(B,\{\mu_i\})P(A,\{\lambda_{m-i+1}\})P(B,\{\mu_i\}),\ i=1,\dots,m,\quad K := \sum_{i=1}^m K_i.$$

As $P(B,\{\mu_i\})P(B,\{\mu_j\})=0$ for $i\neq j$ we deduce that $K_iK_j=0$ for $i\neq j$. Clearly $\mathbf{H}_i := K_i\mathbf{H}\subset P(B,\{\mu_i\})\mathbf{H}$. As $K_i\neq 0$ it follows that $n_{1,i}=\dim\,\mathbf{H}_i>0$. Then $P_{1,i}$ is the projection on $\mathbf{H}_i$. In particular for any $\mathbf{x}\in\mathbf{H}_i$ $B\mathbf{x}=\mu_i\mathbf{x}$. Hence $P_{1,i}$ commutes with B and each $P(B,\{\mu\})$. More precisely

$$P_{1,i}P(B,\{\mu\}) = P(B,\{\mu\})P_{1,i} = 0 \text{ if } \mu\neq\mu_i,$$

$$P_{1,i}P(B,\{\mu_i\}) = P(B,\{\mu_i\})P_{1,i} = P_{1,i}.$$

Hence $P_{1,i}P_{1,j}=\delta_{ij}P_{1,i}$ and $K\mathbf{H}=P_1\mathbf{H}$.

We claim that

$$A(t) = e^{\omega_1 t}K + R(t),\quad ||R(t)||\leqslant (m(m^2-1)+7)e^{\frac{(\omega_1+\theta)t}{2}} \text{ for } t>0. \tag{3.8}$$

Since $\sigma_p(A)$ and $\sigma_p(B)$ are discrete sets there exist bounded $\mathcal{O}_{1,1},\mathcal{O}_{2,1}\in\mathscr{A}$ with the following properties:

$$\mathcal{O}_{1,1}\cap\sigma_p(A) = \bigcup_{i=1}^m\{\lambda_i\},\quad \mathcal{O}_{2,1}\cap\sigma_p(B) = \bigcup_{i=1}^m\{\mu_i\}.$$

Let $\mathcal{O}_{i,2} := \mathbb{R}\backslash\mathcal{O}_{i,1}$ for $i=1,2$ and $t>0$. Then

$$\begin{aligned} A(t) &= \left(\sum_{i=1}^2 e^{\frac{Bt}{2}}_{\mathcal{O}_{2,i}}\right)\left(\sum_{j=1}^2 e^{At}_{\mathcal{O}_{1,j}}\right)\left(\sum_{k=1}^2 e^{\frac{Bt}{2}}_{\mathcal{O}_{2,k}}\right) \\ &= e^{\frac{Bt}{2}}_{\mathcal{O}_{2,1}} e^{At}_{\mathcal{O}_{1,1}} e^{\frac{Bt}{2}}_{\mathcal{O}_{2,1}} + \sum_{1\leqslant i,j,k\leqslant 2,(i,j,k)\neq(1,1,1)} e^{\frac{Bt}{2}}_{\mathcal{O}_{2,i}} e^{At}_{\mathcal{O}_{1,j}} e^{\frac{Bt}{2}}_{\mathcal{O}_{2,k}}. \end{aligned}$$

Observe next

$$\begin{aligned} ||e^{\frac{Bt}{2}}_{\mathcal{O}_{2,i}} e^{At}_{\mathcal{O}_{1,j}} e^{\frac{Bt}{2}}_{\mathcal{O}_{2,k}}|| &= ||e^{\frac{Bt}{2}}_{\mathcal{O}_{2,i}} e^{\frac{At}{2}}_{\mathcal{O}_{1,j}} e^{\frac{At}{2}}_{\mathcal{O}_{1,j}} e^{\frac{Bt}{2}}_{\mathcal{O}_{2,k}}|| \leqslant ||e^{\frac{Bt}{2}}_{\mathcal{O}_{2,i}} e^{\frac{At}{2}}_{\mathcal{O}_{1,j}}||\,||e^{\frac{At}{2}}_{\mathcal{O}_{1,j}} e^{\frac{Bt}{2}}_{\mathcal{O}_{2,k}}|| \\ &= \rho(e^{At}_{\mathcal{O}_{1,j}} e^{Bt}_{\mathcal{O}_{2,i}})^{\frac{1}{2}}\rho(e^{At}_{\mathcal{O}_{1,j}} e^{Bt}_{\mathcal{O}_{2,k}})^{\frac{1}{2}} \leqslant e^{\frac{(\omega_1+\theta)t}{2}} \text{ if } (i,j,k)\neq(1,1,1). \end{aligned}$$

In the last inequality we used Theorem 2.1, (2.6) and (3.4). Hence

$$\left\| \sum_{1\leqslant i,j,k\leqslant 2,(i,j,k)\neq(1,1,1)} e^{\frac{Bt}{2}}_{\mathcal{O}_{2,i}}\ e^{At}_{\mathcal{O}_{1,j}} e^{\frac{Bt}{2}}_{\mathcal{O}_{2,k}} \right\| \leqslant 7\, e^{\frac{(\omega_1+\theta)t}{2}}.$$

Clearly,

$$e^{\frac{Bt}{2}}_{\mathcal{O}_{2,1}}\ e^{At}_{\mathcal{O}_{1,1}}\ e^{\frac{Bt}{2}}_{\mathcal{O}_{2,1}}$$

$$= e^{\omega_1 t}K + \sum_{1\leqslant i,j,k\leqslant m,(i,j,k)\neq(i,m-i+1,i)} e^{\frac{(2\lambda_j+\mu_i+\mu_k)t}{2}} P(B,\{\mu_i\})P(A,\{\lambda_j\})P(B,\{\mu_k\}).$$

The definitions of ω_1, θ yield

$$\begin{aligned} &||e^{\frac{(2\lambda_j+\mu_i+\mu_k)t}{2}} P(B,\{\mu_i\})P(A,\{\lambda_j\})P(B,\{\mu_k\})|| \\ &\leqslant e^{\frac{(\lambda_j+\mu_i)t}{2}} ||P(B,\{\mu_i\})P(A,\{\lambda_j\})|| e^{\frac{(\lambda_j+\mu_k)t}{2}} ||P(A,\{\lambda_j\})P(B,\{\mu_k\})|| \\ &\leqslant e^{\frac{(\omega_1+\theta)t}{2}} \text{ for } (i,j,k)\neq(i,m-i+1,i). \end{aligned}$$

This establishes (3.8).

Note that if we replace A by $A_1 = A - \omega_1 I$ then $\Lambda(t)$ and $C(t)$ are replaced by $\Lambda_1(t) = e^{-\omega_1 t}\Lambda(t)$ and $C_1(t) = C(t) - \omega_1 I$, respectively. Thus it is enough to prove the theorem in the case $\omega_1 = 0$. Then $\varepsilon = -\theta > 0$. Eq. (3.8) yields that $\lim_{t\to\infty}\Lambda(t) = K$. Hence $K|_{K\mathbf{H}} = K|_{P_1\mathbf{H}}$ has n_1 positive eigenvalues $\kappa_1 \geqslant \cdots \kappa_{n_1} > 0$. Let $\tilde{\kappa}_1 = \kappa_1 > \cdots > \tilde{\kappa}_{\tilde{n}_1} = \kappa_{n_1}$ be all distinct eigenvalues of $K|_{P_1\mathbf{H}}$. That is $\sigma(K) = \bigcup_{i=1}^{\tilde{n}_1}\{\tilde{\kappa}_i\}\cup\{0\}$. Since each $||K_i||\leqslant 1$ it follows that $\kappa_1 \leqslant 1$. We claim that

$$\begin{aligned} &\sigma(\Lambda(t)) \subset [0, (m^2(m-1)+7)e^{\frac{\theta t}{2}}] \cup \left(\frac{2\kappa_{n_1}}{3}, 1\right], \\ &\dim\ P\left(\Lambda(t), \left(\frac{2\kappa_{n_1}}{3}, \infty\right)\right) = n_1, \\ &\text{for } t > t_1 := \frac{2(\ln\kappa_{n_1} - \ln 3(m^2(m-1)+7))}{\theta}. \end{aligned} \tag{3.9}$$

In what follows we use the results of [4, Section 5]. For $E\in\mathcal{S}$ and $\mathbf{X}\subset\mathbf{H}$ a subspace denote

$$v(E,\mathbf{X}) := \inf_{\mathbf{x}\in\mathbf{X},||\mathbf{x}||=1} \langle E\mathbf{x},\mathbf{x}\rangle.$$

Then for $i\in\mathbb{N}$ the ith *width* of E is given by

$$v_i(E) = \sup_{\mathbf{X},\dim\ \mathbf{X}=i} v(E,\mathbf{X}).$$

The sequence $\{v_i(E)\}_1^\infty$ is a nonincreasing sequence. Furthermore, if $v_i(E) > v_{i+1}(E)$ then

(a) $\sigma(E) \subset (-\infty, v_{i+1}] \cup [v_i(E), v_1(E)]$;
(b) $\dim P(E, (\frac{v_{i+1}(E)+v_i(E)}{2}, \infty)) = i$;
(c) $E|_{P(E,(\frac{v_{i+1}(E)+v_i(E)}{2},\infty))\mathbf{H}}$ has eigenvalues $v_1(E) \geqslant \cdots \geqslant v_i(E)$ counted with their multiplicities.

In particular, if $E \in \mathscr{S} \cap \mathscr{C}$ and $E \geqslant 0$ then $\{v_i(E)\}_1^\infty$ is the eigenvalue sequence of E.

Let $\dim \mathbf{X} = n_1 + 1$. Then $\dim(\mathbf{X} \cap (P_1\mathbf{H})^\perp) \geqslant 1$. Choose $\mathbf{x} \in \mathbf{X} \cap (P_1\mathbf{H})^\perp$ of length 1. Then $\langle K\mathbf{x}, \mathbf{x} \rangle = 0$ and

$$\begin{aligned}
&\langle \Lambda(t)\mathbf{x}, \mathbf{x} \rangle = \langle R(t)\mathbf{x}, \mathbf{x} \rangle \leqslant (m^2(m-1)+7)e^{\frac{\theta t}{2}} \\
&\quad \Rightarrow v(\Lambda(t), \mathbf{X}) \leqslant (m^2(m-1)+7)e^{\frac{\theta t}{2}} \\
&\quad \Rightarrow \sup_{X, \dim\ X = n_1+1} v(\Lambda(t), X) = v_{n_1+1}(\Lambda(t)) \leqslant (m^2(m-1)+7)e^{\frac{\theta t}{2}}.
\end{aligned}$$

On the other hand for $||\mathbf{x}|| = 1$

$$\begin{aligned}
&\langle \Lambda(t)\mathbf{x}, \mathbf{x} \rangle = \langle K\mathbf{x}, \mathbf{x} \rangle + \langle R(t)\mathbf{x}, \mathbf{x} \rangle \geqslant \langle K\mathbf{x}, \mathbf{x} \rangle - (m^2(m-1)+7)e^{\frac{\theta t}{2}} \\
&\quad \Rightarrow v_i(\Lambda(t)) \geqslant v_i(K) - (m^2(m-1)+7)e^{\frac{\theta t}{2}}.
\end{aligned}$$

Therefore, for $t > t_1$ and $i = n_1$

$$v_{n_1}(\Lambda(t)) > \frac{2\kappa_{n_1}}{3} > \frac{\kappa_{n_1}}{3} \geqslant (m^2(m-1)+7)e^{\frac{\theta t}{2}} \geqslant v_{n_1+1}(\Lambda(t)),$$

which implies (3.9). (Note that $\Lambda(t) \geqslant 0$ and in view of the assumptions $\omega_1 = 0$ $||\Lambda(t)|| \leqslant 1$.)

Use (3.9) and the countour integration on $\mathbb{C}$ to obtain

$$P\left(\Lambda(t), \left(\frac{2\kappa_{n_1}}{3}, \infty\right)\right) = \frac{1}{2\pi\sqrt{-1}} \int_{|z-1|=1-\frac{\kappa_{n_1}}{2}} (zI - \Lambda(t))^{-1} dz, \quad \text{for } t > t_1.$$

As $\lim_{t\to\infty} \Lambda(t) = K$ it follows that $\lim_{t\to\infty} P(\Lambda(t), (\frac{2\kappa_{n_1}}{3}, \infty)) = P_1$. Observe next that for

$$\sigma(C(t)) \subset \left(-\infty, \frac{\ln(m^2(m-1)+7)e^{\frac{\theta t}{2}}}{t}\right] \cup \left(\frac{\ln 2\kappa_{n_1} - \ln 3}{t}, 0\right],$$
$$P\left(C(t), \left(\frac{\ln 2\kappa_{n_1} - \ln 3}{t}, \infty\right)\right) = P\left(\Lambda(t), \left(\frac{2\kappa_{n_1}}{3}, \infty\right)\right), \quad \text{for } t > t_1.$$

Letting $t \to \infty$ we obtain (3.6). □

Combine Lemma 3.4 and its proof, Proposition 3.2 and Theorem 2.1 to deduce:

Corollary 3.5. *Let $A, B \in \mathscr{S} \cap \mathscr{C}$ and assume that $A, B > 0$, $t > 0$. Let $C(t) > 0$ be a compact operator defined as Proposition* 3.2. *Let $\{\alpha_i\}_1^\infty$, $\{\beta_i\}_1^\infty$, $\{\omega_i(t)\}_1^\infty$ the eigenvalue sequences of A, B, $C(t)$ respectively. Let ω_1, θ, ε, P_1, n_1 be defined as in Lemma* 3.4. *Then*

$$\omega_1(t) \text{ is a nondecreasing function on } (0, \infty),$$
$$\lim_{t\to\infty} \omega_i(t) = \omega_1 \text{ for } i = 1, \dots, n_1,$$
$$\omega_1 = \alpha_{i_k} + \beta_{j_{n_1+1-k}},\ P(A, \{\alpha_{i_k}\})P(B, \{\beta_{j_{n_1+1-k}}\}) \neq 0,\ k = 1, \dots, n_1,$$
$$\text{for appropriate } 1 \leqslant i_1 < \dots < i_{n_1},\ 1 \leqslant j_1 < \dots < j_{n_1},$$
$$\limsup_{t\to\infty} \omega_{n_1+1}(t) \leqslant \frac{\omega_1 + \theta}{2},$$
$$\lim_{t\to\infty} P(C(t), (\omega_1 - \varepsilon, \infty)) = P_1.$$

4. Exterior spaces

To prove Theorem 1.1 using Corollary 3.5 we need to pass to the exterior spaces over $\mathbf{H}$ and exterior powers of operators. Let $\mathbf{H}^n$ be the *n-fold* tensor product $\mathbf{H} \otimes \mathbf{H} \otimes \dots \otimes \mathbf{H}$. An element $\mathbf{u} \in \mathbf{H}^n$ is called a tensor product (of $\mathbf{u}_1, \dots, \mathbf{u}_n$) if $\mathbf{u}$ is of the form

$$\mathbf{u} = \mathbf{u}_1 \otimes \mathbf{u}_2 \otimes \dots \otimes \mathbf{u}_n, \quad \mathbf{u}_i \in \mathbf{H}, \quad i = 1, \dots, n.$$

Let $\mathbf{v}$ be a tensor product of $\mathbf{v}_1, \dots, \mathbf{v}_n$. The inner product $(\mathbf{u}, \mathbf{v})$ is given by

$$(\mathbf{u}, \mathbf{v}) := \langle \mathbf{u}_1, \mathbf{v}_1 \rangle \langle \mathbf{u}_2, \mathbf{v}_2 \rangle \cdots \langle \mathbf{u}_n, \mathbf{v}_n \rangle.$$

Then $\mathbf{H}^n$ is the linear closure of the subspace spanned by the tensor product elements. See for example [13, II.4]. Denote by $\mathscr{L}^n, \mathscr{S}^n, \mathscr{C}^n$ the set of linear operators which are bounded, self-adjoint and compact on $\mathbf{H}^n$, respectively.

Let $A_i \in \mathscr{L}, i = 1, \dots, n$. Then $T := A_1 \otimes A_2 \otimes \cdots \otimes A_n \in \mathscr{L}^n$ is given by the following action on the tensor product element $\mathbf{u}$:

$$T\mathbf{u} := (A_1\mathbf{u}_1) \otimes (A_2\mathbf{u}_2) \otimes \cdots \otimes (A_n\mathbf{u}_n).$$

If $A_1, \dots, A_n$ are self-adjoint (res. compact) then T is self-adjoint (res. compact). Furthermore

$$(A_1 \otimes A_2 \otimes \cdots \otimes A_n)(B_1 \otimes B_2 \otimes \cdots \otimes B_n) = A_1B_1 \otimes A_2B_2 \otimes \cdots \otimes A_nB_n,$$

$$A_i, B_i \in \mathscr{L}, \quad i = 1, \dots, n.$$

It is straightforward to show that

$$\rho(T) = \rho(A_1)\rho(A_2)\cdots\rho(A_n).$$

If $A_i = A, i = 1, \dots, n$ we consider $T = \otimes^n A$. For $A \in \mathscr{L}$ define

$$F_n(A) = I \otimes \cdots \otimes I \otimes A + I \otimes \cdots \otimes A \otimes I + \cdots + A \otimes I \otimes \cdots \otimes I \in \mathscr{L}^n. \quad (4.1)$$

It is a straightforward computation

$$e^{tF_n(A)} = \otimes^n e^{tA}, \quad t \in \mathbb{C}.$$

We now recall the definition of nth exterior space over $\mathbf{H}$. (It is called nth *fold antisymmetric Fock space* over $\mathbf{H}$ in [13, II.4]). We will use some elementary facts of nth exterior spaces which can be found (for finite-dimensional inner product spaces) in [2, Section 3]. Let Π_n be the nth symmetric group which acts as a group of permutation on the set $\{1, \dots, n\}$. Let $\phi : \Pi_n \to \{-1, 1\}$ be the group homomorphism which maps any odd permutation onto -1 and any even permutation onto 1. Let $O_n : \mathbf{H}^n \to \mathbf{H}^n$ be the orthogonal projection on $\mathbf{H}^n$ given on the tensor product elements as follows:

$$\mathbf{u}_1 \wedge \mathbf{u}_2 \wedge \cdots \wedge \mathbf{u}_n := O_n(\mathbf{u}_1 \otimes \mathbf{u}_2 \otimes \cdots \otimes \mathbf{u}_n) = \frac{1}{\sqrt{n!}} \sum_{\theta \in \Pi_n} \phi(\theta)\mathbf{u}_{\theta(1)} \otimes \mathbf{u}_{\theta(2)} \otimes \cdots \otimes \mathbf{u}_{\theta(n)}.$$

(In [2] the factor $\frac{1}{\sqrt{n!}}$ is omitted.) Then $O_n(\mathbf{H}^n) \subset \mathbf{H}^n$ is the nth exterior space over $\mathbf{H}$. It is denoted by $\bigwedge^n \mathbf{H}$. We call $\mathbf{u}_1 \wedge \mathbf{u}_2 \wedge \cdots \wedge \mathbf{u}_n$ an exterior product in $\bigwedge^n \mathbf{H}$. Let $\mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \cdots \wedge \mathbf{v}_n$ be another exterior product in $\bigwedge^n \mathbf{H}$. Then

$$[\mathbf{u}_1 \wedge \mathbf{u}_2 \wedge \cdots \wedge \mathbf{u}_n, \mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \cdots \wedge \mathbf{v}_n] := \det(\langle \mathbf{u}_i, \mathbf{v}_j \rangle)_{i,j=1}^n. \quad (4.2)$$

It is well known that a nonzero exterior product $\tilde{\mathbf{u}} := \mathbf{u}_1 \wedge \mathbf{u}_2 \wedge \cdots \wedge \mathbf{u}_n$ determines a unique n-dimensional subspace $\mathbf{U} = \mathrm{span}(\mathbf{u}_1, \dots, \mathbf{u}_n) \subset \mathbf{H}$. Vice versa, any n-dimensional subspace $\mathbf{U} \subset \mathbf{H}$ determines a unique one-dimensional subspace $\tilde{\mathbf{U}} \subset \bigwedge^n \mathbf{H}$ spanned by $\tilde{\mathbf{u}} = \mathbf{u}_1 \wedge \mathbf{u}_2 \wedge \cdots \wedge \mathbf{u}_n$ for some basis $\mathbf{u}_1, \dots, \mathbf{u}_n$ of $\mathbf{U}$. Given two subspaces $\mathbf{U}, \mathbf{V}$ of the same dimension, say n, we define $[\mathbf{U}, \mathbf{V}]$ by the inner product given in (4.2), where $\mathbf{u}_1, \dots, \mathbf{u}_n$ and $\mathbf{v}_1, \dots, \mathbf{v}_n$ are any orthonormal bases of $\mathbf{U}$ and $\mathbf{V}$, respectively. Then $|[\mathbf{U},\mathbf{V}]|$ is independent of the choices of orthonormal bases in $\mathbf{U},\mathbf{V}$ and is called the cosine angle between $\mathbf{U}$ and $\mathbf{V}$. $\mathbf{U}$ and $\mathbf{V}$ are called orthogonal if $[\mathbf{U},\mathbf{V}]=0$. In the next section will use the following fact [2, Lemma 3.1]:

Proposition 4.1. *Let* $\mathbf{U},\mathbf{V}\subset\mathbf{H}$ *be two finite-dimensional subspaces of* $\mathbf{H}$ *of the same dimension. Then the following are equivalent*:

(a) $[\mathbf{U},\mathbf{V}]\neq 0$,
(b) $\mathbf{U}^{\perp}\cap\mathbf{V}=\{0\}$,
(c) $\mathbf{U}\cap\mathbf{V}^{\perp}=\{0\}$.

It is straightforward to show that $\bigwedge^n\mathbf{H}$ is an invariant subspace for any $\otimes^n A, A\in\mathscr{L}$. We define the nth exterior power of A as the restriction of $\otimes^n A$ to $\bigwedge^n\mathbf{H}$. We denote the nth exterior power of A by $\bigwedge^n A$. For $A,B\in\mathscr{L}$ we have $\wedge^n AB=\wedge^n A\wedge^n B$. It is straightforward to show that $\bigwedge^n\mathbf{H}$ is an invariant subspace of $F_n(A)$ for any $A\in\mathscr{L}$. Denote by $D_n(A)$ the restriction of $F_n(A)$ to $\bigwedge^n\mathbf{H}$:

$$D_n(A)=F_n(A)|_{\wedge^n\mathbf{H}}. \tag{4.3}$$

Hence $\bigwedge^n e^{tA}=e^{tD_n(A)}$. The following proposition gives natural examples of operators in class $\mathscr{D}_+$.

Proposition 4.2. *Let* $A>0$ *be a compact operator. Let* $\{\alpha_i\}_1^\infty$ *be the eigenvalues sequence of* A. *Let* $\{\mathbf{e}_1,\ldots,\mathbf{e}_i,\ldots\}$ *be an orthonormal basis of* $\mathbf{H}$ *consisting of eigenvectors of* A:

$$A\mathbf{e}_i=\alpha_i\mathbf{e}_i,\ \alpha_1\geqslant\alpha_2\geqslant\cdots\geqslant\alpha_i>0,\quad i=1,\ldots,.$$

Let $n>1$. *Then* $F_n(A)$ *and* $D_n(A)$ *are noncompact operators on* $\mathbf{H}^n$ *and* $\bigwedge^n\mathbf{H}$ *in class* $\mathscr{D}_+$. *More precisely*

$$F_n(A)(\mathbf{e}_{i_1}\otimes\cdots\otimes\mathbf{e}_{i_n})=\left(\sum_{k=1}^n\alpha_{i_k}\right)(\mathbf{e}_{i_1}\otimes\cdots\otimes\mathbf{e}_{i_n}),\quad i_1,\ldots,i_n\in\mathbb{N},$$

$$D_n(A)(\mathbf{e}_{i_1}\wedge\cdots\wedge\mathbf{e}_{i_n})=\left(\sum_{k=1}^n\alpha_{i_k}\right)(\mathbf{e}_{i_1}\wedge\cdots\wedge\mathbf{e}_{i_n}),\quad 1\leqslant i_1<\cdots<i_n.$$

Furthermore,

$$\sigma(F_n(A))\backslash\sigma_p(F_n(A))=\sigma(F_{n-1}(A)),\quad \sigma(D_n(A))\backslash\sigma_p(D_n(A))=\sigma(D_{n-1}(A)).$$

In particular, *the eigenvalue sequences of* $\otimes^n e^{tA}$ *and* $\bigwedge^n e^{tA}$ *consist of the elements*

$$\prod_{k=1}^n e^{t\alpha_{i_k}},\quad i_1,\ldots,i_n\in\mathbb{N},$$

$$\prod_{k=1}^n e^{t\alpha_{i_k}},\quad 1\leqslant i_1<\cdots<i_n$$

respectively. Hence

$$\rho(\otimes^n e^{tA}) = e^{tn\alpha_1}, \quad \rho\left(\bigwedge^n e^{tA}\right) = e^{t\sum_{k=1}^n \alpha_i}.$$

If $\alpha_n > \alpha_{n+1}$ *then* $\rho(\bigwedge^n e^{tA})$ *is a simple eigenvalue of* $\bigwedge^n e^{tA}$:

$$\dim\ P\left(\bigwedge^n e^{tA}, \left\{\rho\left(\bigwedge^n e^{tA}\right)\right\}\right) = 1.$$

The proof of the proposition is a basic excercise.

5. Proof of Theorem 1.1

Let $C(t)$ be defined as in Theorem 1.1. Proposition 3.2 yields that $C(-t) = C(t)$, $C(t)$ is compact and $C(t) > 0$ for $t \in \mathbb{R}^*$. Hence $||C(t)|| = \omega_1(t) \leqslant ||A|| + ||B|| = \alpha_1 + \beta_1$. We now assume that $t > 0$.

We first prove (a) Lemma 3.4 yields that $\omega_1(t)$ is a nondecreasing function which converges to $\omega_1 \leqslant \alpha_1 + \beta_1$. Consider now

$$\Lambda_n(t) := \wedge^n \Lambda(t) = e^{\frac{tD_n(B)}{2}} e^{tD_n(A)} e^{\frac{tD_n(B)}{2}} = e^{tE_n(t)},\ E_n(t) := D_n(C(t)).$$

Proposition 4.2 yields $\rho(\Lambda_n(t)) = e^{t\sum_{i=1}^n \omega_i(t)}$. Corollary 3.3 implies that $\rho(E_n(t)) = \sum_{i=1}^n \omega_i(t)$ is a nondecreasing function of t on $(0, \infty)$ bounded above by $||D_n(A)|| + ||D_n(B)|| = \sum_{i=1}^n (\alpha_i + \beta_i)$. Let $r_n := \lim_{t\to\infty} \sum_{i=1}^n \omega_i(t),\ n \in \mathbb{N}$. For $n \geqslant 2$ define $\omega_n := r_n - r_{n-1}$. Thus $\lim_{t\to\infty} \omega_n(t) = \omega_n$ for each $n \in \mathbb{N}$. As $\{\omega_i(t)\}_1^\infty$ is a nonincreasing positive sequence it follows that $\{\omega_i\}_1^\infty$ is a nonincreasing nonnegative sequence converging to $\omega \geqslant 0$. Observe next that

$$\omega = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n \omega_i \leqslant \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n (\alpha_i + \beta_i) = 0,$$

as $\lim_{n\to\infty} \alpha_i = \lim_{n\to\infty} \beta_i = 0$. Hence $\lim_{n\to\infty} \omega_i = 0$. This completes the proof of (a).

Corollary 3.5 implies that if $n_1 > 1$ then $\omega_i = \omega_1$ for $i = 2, \dots, n_1$, and $\omega_{n_1+1} \leqslant \frac{\omega_1 + \theta}{2} < \omega_1$. If $n_1 = 1$ then $\omega_2 \leqslant \frac{\omega_1 + \theta}{2} < \omega_1$. There are two cases: either each $\omega_i > 0$ or $\omega_i = 0$ for $i \geqslant N$. We first show that the second possibility cannot hold.

Assume to the contrary that $\omega_{N-1} > \omega_N = 0$. Apply Lemma 3.4 to $D_N(A), D_N(B)$. Hence there exists an integer $k \geqslant 1$ and $t_0 > 0$ such that

$$\dim\ P(E_N(t), (r_N - \delta, \infty)) = k \text{ for } t > t_0.$$

That is for any $t > t_0$ the interval $(r_N - \delta, \infty)$ contains exactly k eigenvalues of $E_N(t)$ (counted with their multiplicities). On the other hand our assumption that $\omega_\ell = 0$ for any $\ell \geqslant N$ yields that $\sum_{i=1}^{N-1} \omega_i(t) + \omega_\ell(t)$ converges to $r_{N-1} = r_N$. Let $\ell = N + k$.

Then for $t > T_0 > t_0$ at least $k+1$ eigenvalues of $E_N(t)$ (counted with their multiplicities) satisfy

$$r_N - \delta < \sum_{i=1}^{N-1} \omega_i(t) + \omega_{N+k}(t) \leqslant \cdots \leqslant \sum_{i=1}^{N-1} \omega_i(t) + \omega_N(t).$$

This is impossible in view of Lemma 3.4. Hence each $\omega_i > 0$.

For $j > 1$ define n_j recursively:

$$\omega_{n_{j-1}} > \omega_{n_{j-1}+1} = \cdots = \omega_{n_j} > \omega_{n_j+1}, \quad j = 2, 3, \ldots, .$$

We claim that for each $j \geqslant 1$ there exists $\delta_j > 0$ with the following properties: $\omega_{n_j} - \delta_j > \omega_{n_j+1} + \delta_j$ and $P(C(t), (\omega_{n_j} - \delta_j, \infty))$ converges in the norm topology to an orthogonal projection P_j as $t \to \infty$, where dim $P_j = n_j$. Furthermore $BP_j = P_j B$, i.e. the finite-dimensional subspace $P_j\mathbf{H}$ has an orthonormal basis spanned by eigenvectors of B. For $j = 1$ this claim follows from Lemma 3.4.

Let $j > 1$. Since each $\omega_i(t)$ converges to ω_i and $\omega_{n_j} > \omega_{n_j+1}$ it follows that there exists $\delta_j \in (0, \frac{\omega_{n_j} - \omega_{n_j}+1}{2})$ and $t_j > 0$ such that

$$\omega_{n_j}(t) > \omega_{n_j} - \delta_j \text{ and } \omega_{n_j+1}(t) < \omega_{n_j+1} + \delta_j \text{ for } t > t_j$$
$$\Rightarrow \omega_i(t) > \omega_{n_j} - \delta_j \ i = 1, \ldots, n_j, \ \omega_k(t) < \omega_{n_j+1} + \delta_j \ k = n_j + 1, \ldots, \text{ for } t > t_j.$$

Let $P_j(t) \coloneqq P(C(t), (\omega_{n_j} - \delta_j, \infty))$. Then $\dim P_j(t) = n_j$ for $t > t_j$. Consider $E_n(t)$ as an operator in $\bigwedge^n \mathbf{H}$ for each $n \in \mathbb{N}$. Let $\tilde{P}_j(t) \coloneqq P(E_{n_j}(t), \{\rho(E_{n_j}(t)\})$. Assume that $t > t_j$. Combine the above inequalities with Proposition 4.2 to deduce that $\dim \tilde{P}_j(t) = 1$. It is straightforward to show using the properties of exterior products that $\tilde{P}_j(t) = \wedge^{n_j} P_j(t)$. Since $D_n(A), D_n(B)$ are in the class $\mathscr{D}_+$ we can apply Lemma 3.4. Let $\tilde{P}_j \coloneqq \lim_{t\to\infty} \tilde{P}_j(t)$. As $\dim \tilde{P}_j(t) = 1$ for $t > t_j$ it follows that $\dim \tilde{P}_j = 1$. That is $\tilde{P}_j \bigwedge^n \mathbf{H} = \mathrm{span}(\tilde{\mathbf{v}})$ for some $\tilde{\mathbf{v}}$ with $||\tilde{\mathbf{v}}|| = 1$. Let $\mathbf{g}_1(t), \ldots, \mathbf{g}_{n_j}(t)$ be an orthonormal basis of $P_j(t)\mathbf{H}$. Then

$$\bigwedge^{n_j} P_j(t)\mathbf{H} = \mathrm{span}(\tilde{\mathbf{v}}(t)), \quad \tilde{\mathbf{v}}(t) \coloneqq \mathbf{g}_1(t) \wedge \cdots \wedge \mathbf{g}_{n_j}(t).$$

Since $\lim_{t\to\infty} \wedge^{n_j} P_j(t) = \tilde{P}_j$ it follows that up to a multiple of $s(t)$, $|s(t)| = 1$ $\tilde{\mathbf{v}}(t)$ converges to $\tilde{\mathbf{v}}$. That is, by replacing $\mathbf{g}_1(t)$ with $s(t)\mathbf{g}_1(t)$ we deduce that $\lim_{t\to\infty} ||\tilde{\mathbf{v}}(t) - \tilde{\mathbf{v}}|| = 0$. Hence $\tilde{\mathbf{v}} \coloneqq \mathbf{g}_1 \wedge \cdots \wedge \mathbf{g}_{n_j}$, for some orthonormal vectors $\mathbf{g}_1, \ldots, \mathbf{g}_{n_j}$. Let $\mathbf{G}_j \coloneqq \mathrm{span}(\mathbf{g}_1, \ldots, \mathbf{g}_{n_j})$ and $P_j \coloneqq P(\mathbf{G}_j)$, the orthogonal projection on $\mathbf{G}_j$. It is straightforward to show $\lim_{t\to\infty} ||P_j(t) - P_j|| = 0$. Lemma 3.4 yields that $\tilde{\mathbf{v}}$ is an eigenvector of $D_{n_j}(B)$. As $B \in \mathscr{D}_+$ it is diagonable, it is straightforward to show that $\mathbf{G}_j$ is an invariant subspace of B. Therefore, there exist an orthonormal basis of $\mathbf{H}$ consisting of eigenvectors of B: $\mathbf{g}_1, \dots,$ such that

$$(P_j - P_{j-1})\mathbf{H} = \mathrm{span}(\mathbf{g}_{n_{j-1}+1}, \dots, \mathbf{g}_{n_j}), \quad j = 1, \dots, \text{ and } n_0 = 0,$$

$$B\mathbf{g}_i = \mu_i \mathbf{g}_i, \ \ i = 1, \dots$$

We now apply the same arguments to A. To be precise interchange the roles of A and B. That is consider

$$\hat{\Lambda}(t) := e^{\frac{At}{2}} e^{Bt} e^{\frac{At}{2}} = e^{t\hat{C}(t)} = W(t)^{-1}\Lambda(t)W(t),$$

$$\hat{C}(t) = W(t)^{-1}C(t)W(t), \ W(t) := e^{\frac{Bt}{2}} \ e^{\frac{At}{2}}. \tag{5.1}$$

Then $\hat{C}(t) > 0$ is compact operator with the eigenvalue sequence $\{\omega_i(t)\}_1^{\infty}$. Let $Q_j(t) := P(\hat{C}(t), (\omega_{n_j} - \delta_j, \infty)), j = 1, \dots,$. Then $\lim_{t\to\infty} Q_j(t) = Q_j$ and $\dim \ Q_j = n_j$. Therefore there exist an orthonormal set of eigenvectors of A: $\mathbf{f}_1, \dots,$ such that

$$(Q_j - Q_{j-1})\mathbf{H} = \mathrm{span}(\mathbf{f}_{n_{j-1}+1}, \dots, \mathbf{f}_{n_j}), \quad j = 1, \dots, \ \text{ and } n_0 = 0,$$

$$A\mathbf{f}_i = \lambda_i \mathbf{f}_i, \ i = 1, \dots$$

Let $\tilde{\mathbf{u}} := \mathbf{f}_1 \wedge \cdots \wedge \mathbf{f}_{n_j}$. Lemma 3.4 implies that there exist unique $\tilde{\lambda}, \tilde{\mu} \in \mathbb{R}_+$ such that

$$r_{n_j} = \tilde{\lambda} + \tilde{\mu}, \ D_{n_j}(A)\tilde{\mathbf{u}} = \tilde{\lambda}\tilde{\mathbf{u}}, \ D_{n_j}(B)\tilde{\mathbf{v}} = \tilde{\mu}\tilde{\mathbf{v}}, \text{ and } (\tilde{\mathbf{u}}, \tilde{\mathbf{v}}) \neq 0.$$

(The last condition is equivalent to the condition $P(D_{n_j}(A), \{\tilde{\lambda}\})P(D_{n_j}(B), \{\tilde{\mu}\}) \neq 0$.) Hence there exist $\{\lambda_i\}_{i=1}^{n_j} \subset \sigma_p(A)$ and $\{\mu_i\}_{i=1}^{n_j} \subset \sigma_p(B)$ so that $r_{n_j} = \sum_{i=1}^{n_j} \lambda_i + \mu_i$. Apply Lemma 3.4 for $D_{n_{j-1}+1}(A)$ and $D_{n_{j-1}+1}(B)$. Observe now that $r_{n_{j-1}+1} = \sum_{i=1}^{n_{j-1}+1} \omega_i$. Furthermore for $t > \max(t_{j-1}, t_j)$

$$P_j(t) - P_{j-1}(t) = P(C(t), (\omega_{n_j} - \delta_j, \omega_{n_{j-1}+1} + \delta_{j-1})),$$

$$Q_j(t) - Q_{j-1}(t) = P(\hat{C}(t), (\omega_{n_j} - \delta_j, \omega_{n_{j-1}+1} + \delta_{j-1})),$$

$$\dim \ (P_j(t) - P_{j-1}(t)) = \dim \ (Q_j(t) - Q_{j-1}(t)) = n_j - n_{j-1}.$$

That is there exists $\delta > 0$ small enough so that

$$\lim_{t\to\infty} \dim \ P(D_{n_{j-1}+1}(C(t)), (r_{n_{j-1}+1} - \delta, \infty)) = n_j - n_{j-1}.$$

This construction gives an ordering of subsets of the eigenvalues sequences of A and B so that

$$\omega_i = \lambda_i + \mu_i, \quad i = 1, \dots, n_j. \tag{5.2}$$

Equivalently there are two injections $\Phi, \Theta : \mathbb{N} \to \mathbb{N}$ such that

$$\lambda_i = \alpha_{\Phi(i)},\ \mu_i = \beta_{\Psi(i)},\ \omega_i = \alpha_{\Phi(i)} + \beta_{\Psi(i)}, \quad i = 1, \ldots, .$$

Let $\mathbf{H}_1$ and $\mathbf{H}_2$ be the closed subspaces with orthonormal bases $\{\mathbf{f}_1, \ldots.\}$ and $\{\mathbf{g}_1, \ldots\}$ respectively. Then Φ (Ψ) is a bijection if and only if $\mathbf{H}_1$ $(\mathbf{H}_2)$ are equal to $\mathbf{H}$.

Assume first that $\mathbf{H}_1$ is a strict subspace of $\mathbf{H}$. Then $\mathbf{H}_1^\perp$ has an orthonormal basis consisting of eigenvectors of A. Hence there exists $\mathbf{u}_p \in \mathbf{H}_1^\perp$ such that $A\mathbf{u}_p = \alpha_p \mathbf{u}_p$, $||\mathbf{u}_p|| = 1$. Recalling that the eigenvalue α_p is of multiplicity dim $P(A, \{\alpha_p\})$ it is possible to choose a p such that $p \notin \Phi(\mathbb{N})$. Let n_j be the maximal number such that $\omega_{n_j} \geqslant \alpha_p$. Hence $\omega_{n_j+1} < \alpha_p$. Recall that

$$D_{n_j}(A)\tilde{\mathbf{u}} = \left(\sum_{i=1}^{n_j} \alpha_{\Phi(i)}\right)\tilde{\mathbf{u}},\ \tilde{\mathbf{u}} = \mathbf{f}_1 \wedge \cdots \wedge \mathbf{f}_{n_j},$$

$$D_{n_j}(B)\tilde{\mathbf{v}} = \left(\sum_{i=1}^{n_j} \beta_{\Psi(i)}\right)\tilde{\mathbf{v}},\ \tilde{\mathbf{v}} = \mathbf{g}_1 \wedge \cdots \wedge \mathbf{g}_{n_j},$$

$$(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}) \neq 0 \Leftrightarrow [\mathbf{F}_j, \mathbf{G}_j] \neq 0,\ \mathbf{F}_j = \mathrm{span}(\mathbf{f}_1, \ldots, \mathbf{f}_{n_j}),\ \mathbf{G}_j = \mathrm{span}(\mathbf{g}_1, \ldots, \mathbf{g}_{n_j}).$$

Proposition 4.1 yields that $\mathbf{F}_j \cap \mathbf{G}_j^\perp = \{0\}$. That is the n_j linear functionals

$$\phi_i : \mathbf{H} \to \mathbf{C}, \quad \phi_i(\mathbf{x}) = \langle \mathbf{x}, \mathbf{f}_i \rangle,\ i = 1, \ldots, n_j,$$

are linearly independent linear functionals on $\mathbf{G}_j$. Since $\mathbf{u}_p \in \mathbf{H}_1^\perp \subset \mathbf{F}_j^\perp$ it follows that $\mathbf{u}_p \notin \mathbf{F}_j$. Hence there exists a vector $\mathbf{w} \in \mathbf{F}_j$ such that $\langle \mathbf{x}, \mathbf{u}_p \rangle = \langle \mathbf{x}, \mathbf{w} \rangle$ for any $\mathbf{x} \in \mathbf{G}_j$, i.e. $0 \neq \mathbf{u}_p - \mathbf{w} \in \mathbf{G}_j^\perp$. Let $\mathbf{F} = \mathrm{span}(\mathbf{F}_j, \mathbf{u}_p)$. Clearly dim $\mathbf{F} = n_j + 1$. Since B is diagonable, there exists an eigenvector $\mathbf{v}_q$ of B such that $\langle \mathbf{u}_p - \mathbf{w}, \mathbf{v}_q \rangle \neq 0$. Clearly $\mathbf{v}_q \notin \mathbf{G}_j$. Let $\mathbf{G} = \mathrm{span}(\mathbf{G}_j, \mathbf{v}_q)$. Then dim $\mathbf{G} = n_j + 1$. We claim that $[\mathbf{F}, \mathbf{G}] \neq 0$. Observe that the exterior products $\hat{\mathbf{u}} := \tilde{\mathbf{u}} \wedge (\mathbf{u}_p - \mathbf{w})$ and $\hat{\mathbf{v}} := \tilde{\mathbf{v}} \wedge \mathbf{v}_q$ represent the subspaces $\mathbf{F}$ and $\mathbf{G}$, respectively. Use (4.2) and the assumption that $\mathbf{u}_p - \mathbf{w} \in \mathbf{G}_j^\perp$ to deduce

$$[\hat{\mathbf{u}}, \hat{\mathbf{v}}] = [\tilde{\mathbf{u}}, \tilde{\mathbf{v}}] \langle \mathbf{u}_p - \mathbf{w}, \mathbf{v}_q \rangle.$$

As $[\mathbf{F}_j, \mathbf{G}_j] \neq 0$ we get that $[\hat{\mathbf{u}}, \hat{\mathbf{v}}] \neq 0$. Hence the above inner product of two exterior products in $\bigwedge^{n_j+1} \mathbf{H}$ is nonzero, which is equivalent to $[\mathbf{F}, \mathbf{G}] \neq 0$. Lemma 3.4 yields that $r_{n_j+1} \geqslant (\alpha_p + \beta_q) + r_{n_j} \Rightarrow \omega_{n_j+1} \geqslant \alpha_p + \beta_q$ which contradicts the choice of n_j. Hence $\mathbf{H}_1 = \mathbf{H}$. Similarly $\mathbf{H}_2 = \mathbf{H}$. Thus Φ and Ψ are bijections.

Let

$$C := \sum_{j=1}^{\infty} \omega_{n_j}(P_j - P_{j-1}), \quad P_0 = 0. \tag{5.3}$$

Since each projection P_j is finite-dimensional and $\lim_{j\to\infty} \omega_{n_j} = 0$ it follows that $C \in \mathscr{S} \cap \mathscr{C}$ and the convergence of the above infinite sum is in norm topology. Since each $\omega_i > 0$ and $\mathbf{H}_2 = \mathbf{H}$ it follows that $C > 0$. As $P_j\mathbf{H} = \mathbf{G}_j$ is a finite-dimensional invariant subspace of B it follows that $P_jB = BP_j$, $j \in \mathbb{N}$ hence $BC = CB$. We claim that $\lim_{t\to\infty} ||C(t) - C|| = 0$. Clearly

$$C = C_i + (C - C_i),\; C_i = \sum_{j=1}^{i} \omega_{n_j}(P_j - P_{j-1}),\; ||C - C_i|| = \omega_{n_{i+1}},$$

$$C(t) = C_i(t) + (C(t) - C_i(t)),$$

$$C_i(t) = C(t)P_i(t),\; ||C(t) - C_i(t)|| \langle \omega_{n_i} \text{ for } t \rangle t_i.$$

Let $\varepsilon > 0$ be given. Choose $i \gg 1$ such that $\omega_{n_i} < \frac{\varepsilon}{3}$. The definition of $P_1, \ldots, P_i$ yields that the finite-dimensional operator $C_i(t)$ converges in norm topology to C_i as $t \to \infty$. Hence for $t \gg 1$

$$||C(t) - C|| \leqslant ||C(t) - C_i(t)|| + ||C_i(t) - C_i|| + ||C_i - C|| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Hence $\lim_{t\to\infty} ||C(t) - C|| = 0$.

Note that $e^{C(t)} - I, e^C - I \in \mathscr{S} \cap \mathscr{C}$ with the eigenvalue sequences $\{e^{\omega_i(t)} - 1\}_{i=1}^{\infty}$, $\{e^{\omega_i} - 1\}_{i=1}^{\infty}$ respectively. Clearly

$$e^{C(t)} - I = e^{C_i(t)} - I + e^{C(t) - C_i(t)} - I, \text{ for } t > t_i$$

$$e^C - I = \sum_{j=1}^{\infty} (e^{\omega_{n_j}} - 1)(P_j - P_{j-1}).$$

Hence the above arguments imply $\lim_{t\to\infty} ||e^{C(t)} - e^C|| = 0$.

Finally, part (b) of the Theorem 1.1 follows from Proposition 3.2. The proof of Theorem 1.1 is completed.

6. Additional results and open problems

Let the assumptions of Theorem 1.1 hold. Theorem 1.1 yields that

$$C = U^*AU + B, \quad U \text{ is unitary operator.} \tag{6.1}$$

U transfers the orthonormal basis $\mathbf{g}_1, \ldots$ of $\mathbf{H}$ to the orthonormal bases $\mathbf{f}_1, \ldots$ the two bases constructed in the proof of Theorem 1.1. It is natural to ask if

$$C(t) = U(t)^*AU(t) + V(t)^*BV(t), \quad U(t), V(t) \text{ are unitary for any } t \in \mathbb{R}^*. \tag{6.2}$$

Note that if we agree that $C(0) = A + B$ then the above equalities hold for $t = 0, \pm\infty$. We now show that (6.2) holds under additional assumptions. Recall that a compact, self-adjoint nonnegative operator A with a eigenvalues sequence $\{\alpha_i\}_1^\infty$ is called of trace class if Trace $A := \sum_{i=1}^{\infty} \alpha_i < \infty$.

Theorem 6.1. *Let the assumptions of Theorem* 1.1 *hold. Then for each* $t \in \mathbb{R}^*$ *there exist unitary operators* $U(t), V(t)$ *such that*

$$C(t) \leqslant U(t)^* A U(t) + V(t)^* B V(t). \tag{6.3}$$

If A *and* B *are in trace class then* (6.2) *holds for each* $t \in \mathbb{R}^*$.

Proof. Since $C(-t) = C(t)$ it is enough to assume that $t > 0$. Recall that $C(t) > 0$ and compact with the eigenvalue sequence $\{\omega_i(t)\}_1^\infty$. So $\Lambda(t) = e^{tC(t)}$ is a diagonable operator with the eigenvalue sequence $\{e^{t\omega_i(t)}\}_1^\infty$. Note that $\Lambda(t) = \Gamma(t)\Gamma(t)^*$, where $\Gamma(t) = e^{\frac{tB}{2}} e^{\frac{tA}{2}}$. That is the eigenvalue sequence of $e^{\frac{tC(t)}{2}}$ is the singular value sequence if $\Gamma(t)$. We now apply Schubert calculus as in [6] to singular values of $\Gamma(t)$, combined with the convoy principle as in [4] to obtain that the eigenvalue inequalities [4, (3′)] for the eigenvalues sequences $\{\alpha_i\}_1^\infty, \{\beta_i\}_1^\infty, \{\omega_i(t)\}_1^\infty$. That is for any three sets finite sets $J, K, L \subset \mathbb{N}$ of the same cardinality obtained from Schubert calculus we have the inequalities

$$\sum_{i \in L} \omega_i(t) \leqslant \sum_{i \in J} \alpha_i + \sum_{i \in K} \beta_i. \tag{6.4}$$

Combine the results of Fulton [7] with [4, Theorem 2], (see Note added in proof), to deduce (6.3).

Assume that A, B are in the trace class. Then Trace $C(t) \leqslant$ Trace $A +$ Trace $B =$ Trace $A + B < \infty$, i.e. $C(t)$ is in trace class, e.g. [4]. Let $\{\gamma_i\}_1^\infty$ be the eigenvalue sequence of $A + B$. Part (b) of Theorem 1.1 yields that $\sum_{i=1}^n \gamma_i = \lim_{t \searrow 0} \sum_{i=1}^n \omega_i(t)$. As $\sum_{i=1}^n \omega_i(t)$ increase on $(0, \infty)$ we get

$$\sum_{i=1}^n \gamma_i \leqslant \sum_{i=1}^n \omega_i(t) \leqslant \text{Trace } C(t)$$
$$\Rightarrow \text{Trace } A + B \leqslant \text{Trace } C(t) \Rightarrow \text{Trace } A + B = \text{Trace } C(t).$$

[4, Theorem 3] yields (6.2). □

It is an interesting problem to find out which parts of Theorem 1.1 and this section hold for the following cases:

(a) $A, B \in \mathscr{S} \cap \mathscr{C}$ and A, B are neither both positive definite nor both negative definite.
(b) $A, B \in \mathscr{D}_+$ and at least one of the operators is not compact.
(c) A, B are simple operators.
(d) $A, B \in \mathscr{S}$.

References

[1] K.V. Bhagwat, R. Subramanian, Inequalities between means of positive operators, Math. Proc. Cambridge Philos. Soc. 83 (1978) 393–401.
[2] R.A. Brualdi, S. Friedland, A. Pothen, The sparse basis problem and multilinear algebra, SIAM J. Matrix Anal. Appl. 16 (1995) 1–20.
[3] J.E. Cohen, S. Friedland, T. Kato, F.P. Kelley, Eigenvalue inequalities for product of matrix exponentials, Linear Algebra Appl. 45 (1982) 55–95.
[4] S. Friedland, Finite and infinite dimensional generalizations of Klyachko's theorem, Linear Algebra Appl. 319 (2000) 3–22.
[5] S. Friedland, W. So, Product of matrix exponentials, Linear Algebra Appl. 196 (1994) 193–205.
[6] W. Fulton, Eigenvalues, invariant factors, highest weights, and Schubert calculus, Bull. Amer. Math. Soc. 37 (2000) 209–249.
[7] W. Fulton, Eigenvalue of majorized Hermitian matrices and Littlewood–Richardson coefficients, Linear Algebra Appl. 319 (2000) 23–36.
[8] S. Golden, Lower bounds for Helmholtz function, Phys. Rev. 137 (1965) B1127–B1128.
[9] B. Kostant, On convexity, the Weyl group and Iwasawa decomposition, Ann. Sci. École Normal Sup. 6 (1973) 413–455.
[10] C. Loewner (K. Löwner), Über monotone Matrixfunktionen, Math. Z. 38 (1934) 177–216.
[11] A.W. Marshall, I. Olkin, Inequalities: Theory of Majorization and its Applications, Academic Press, Orlando, 1979.
[12] G. Porta, The limit of the product of the parameterized exponentials of two operators, Ph.D. Thesis, University of Illinois, Chicago, 1999.
[13] M. Reed, B. Simon, Functional Analysis, Academic Press, New York, 1998.
[14] C.J. Thompson, Inequality with applications in statistical mechanics, J. Math. Phys. 6 (1965) 1812–1813.