MULTI-DIMENSIONAL CAPACITY, PRESSURE AND HAUSDORFF DIMENSION

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Abstract. This paper surveys the major techniques and results for multi-dimensional capacity (entropy), topological pressure and Hausdorff dimension for $\mathbb{Z}^d$-subshifts of finite type.

1. Introduction. Consider an alphabet on n letters denoted by $< n >:= \{1, ..., n\}$. Suppose one has a long linear storage place of length $m >> 1$. Then one can store n^m different messages of length m. Hence the (unrestricted) capacity of our storage device is $\frac{\log_2 n^m}{m} = \log_2 n$. Consider now the storage problem with some restrictions. For example $n = 2$ and each 1 has to be followed by $2 \equiv 0$. (Such a code is called 1-dimensional $(0,1)$ limited channel.) Then the number messages of length m that can be stored is u_m and u_m, $m = 1, 2, ...$ is the Fibonacci sequence (starting from 2). Hence the capacity of this channel is $\log_2 \frac{1+\sqrt{5}}{2} = 0.694241914$.

Let $\mathbb{Z} \supset \mathbb{Z}_+ \supset \mathbb{N}$ be the set of integers, nonnegative integers and positive integers respectively. Let $1 < d \in \mathbb{N}$. Denote by $\mathbf{m}$ the point $(m_1, ..., m_d)$ in $\mathbb{Z}^d$. Let $\mathbf{m} \in \mathbb{N}^d$ and denote by $< \mathbf{m} >$ all lattice points $\mathbf{i} = (i_1, i_2, .., i_d), i_j \in < m_j >, j = 1, ..., d$. Assume that our storage device given by $< \mathbf{m} >$. The number of storage places in $< \mathbf{m} >$ is $| < \mathbf{m} > |_{pr} := m_1 m_2 ... m_d$. We fill this storage place with the elements of $< n >$. Without any constraints our storage device can hold $n^{|<\mathbf{m}>|_{pr}}$ messages. Hence the (unrestricted) capacity is again $\frac{\log_2 n^{|<\mathbf{m}>|_{pr}}}{|\mathbf{m}|_{pr}} = \log_2 n$. Assume now that we have the restriction as above for the alphabet $< 2 >$, i.e. we have d-dimensional $(0,1)$ limited channel. That is two distinct points $\mathbf{i}, \mathbf{j} \in \mathbb{Z}^d$ are called neighbors if exactly one of the coordinates of $\mathbf{i} - \mathbf{j}$ is ± 1 and the other coordinates of $\mathbf{i} - \mathbf{j}$ are 0. Then no two 1's stored in $< \mathbf{m} >$ are neighbors. This time nobody knows the exact formula for the capacity of this channel. For $d = 2$ this capacity is known up to the precision of 9 digits 0.587891162. For $d = 3$ this capacity is known up to the precision of 2 digits 0.524 ± 0.003. See §6.

The aim of this paper to introduce the reader to the theory of multi-dimensional capacity (mdc) in the broad sense. This subject arose first in statistical mechanics under the name Ising model in 1920's. Since then it was studied extensively in physics literature. In mathematics this subject goes by the name $\mathbb{Z}^d$-SOFT (subshifts of finite type). This paper is divided to three parts. The first part §2–§6 deal with the basic notions of $\mathbb{Z}^d$-SOFT and the computational aspects of mdc (entropy in physcis and mathematics). The second part of the paper §7–§9 deals with the notions

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of (topological) pressure and Hausdorff dimension. Pressure is a more general notion than entropy, as the pressure of the potential $q = 0$ reduces to entropy. A maximal characterization for the pressure shows the probabilistic aspect of the pressure. In particular the (Kolmogorov-Sinai) entropy of an invariant measure, which can be traced back to Shannon, appears in this maximal characterization. The Hausdorff dimension is intimately connected with the discrete version of Lyapunov exponent. The last part of the paper §10 consists of brief collection of needed facts about nonnegative matrices that are very useful in this area.

This paper is somewhat in between a survey paper and a research article. We wanted to give a clear exposition of main ideas and results in this area. In doing so we also improved and generalized some known results. The reader interested only in mcd can read only §2–§6 and §10. The more thorough reader is invited to look through §7–§9. When reading the first time about $\mathbb{Z}^d$-SOFT one feels that the notation is quite heavy and cumbersome. It seems that this feeling is part of the subject since it is indeed complicated.

We now give a brief nontechnical summary of the contents of this paper. We point out the new results of this paper. §2 summarize briefly the main results of 1-dimensional SOFT, referred in the literature as $\mathbb{Z}$-SOFT (biinifinite) and $\mathbb{Z}_+$-SOFT (infinite in one direction). It is well known that ($\mathbb{Z}$) $\mathbb{Z}_+$-SOFT can be coded as (bi)infinite walk on a given directed graph Γ. Then the combinatorial entropy h_{com} (one dimensional capacity) is the exponential growth of paths of length m on Γ. (In statistical mechanics sometimes $e^{h_{com}}$ is called the entropy of the system.) The periodic entropy h_{per} is the limsup of the density of the periodic paths on Γ of length m. The mathematical entropy is the density of the paths of length m on Γ which can be extended to (bi)infinite paths on Γ. Clearly $h_{per} \le h \le h_{com}$. The main result in 1-dimensional SOFT is Theorem 2.8: $h_{per} = h = h_{com} = \log \rho(\Gamma)$, where $\rho(\Gamma)$ is the spectral radius of Γ.

§3 introduces the reader to the theory of $\mathbb{Z}^d$-SOFT type. A SOFT $\mathcal{S} \subset < n >^{\mathbb{Z}^d}$ is called decidable if there exists a box $< \mathbf{m} > \subset \mathbb{N}^d$ such that either one can not fill $< \mathbf{m} >$ with any allowable configuration ($\iff$ $\mathcal{S} = \emptyset$) or there exists a periodic allowable configuration on $< \mathbf{m} >$. (A periodic configuration can be considered as an allowable configuration on the d-dimensional torus $T^{\mathbf{m}} = \mathbb{Z}^d/(m_1\mathbb{Z} \times \ldots \times m_d\mathbb{Z})$. Note that any periodic configuration can be extended to an allowable configuration in $\mathbb{Z}^d$.) The important result of Berger [Ber] claims that there are $\mathbb{Z}^2$-SOFT which are not decidable. This is the first result that demonstrates the intrinsic difference between 1-dimensional and multidimensional theory of SOFT. Next we recall the result of [Fr2] that $h = h_{com}$. We also observe that any $\mathbb{Z}^d$-SOFT can be coded as a matrix SOFT. That is there exist directed graphs $\Gamma_1, ..., \Gamma_d$ such that in each axis direction in $\mathbb{Z}^d$ the 1-dimensional SOFT describes an infinite walk on the graph Γ_k. In the rest of the paper we assume that $\mathcal{S}$ is given as a matrix SOFT.

In §4 we use this idea to imitate the one dimensional case by considering a long strip in direction k while the other coordinates are allowed to vary in some fixed "box" of dimension $k-1$. Then one obtains the transfer matrix, and an upper bound of the entropy in terms of the spectral radius the corresponding transfer matrix. In view of [Ber] there are no lower bounds unless we assume additional conditions. We first consider the case $d = 2$. Then the main result of §4 is Theorem 4.3. It shows that if either Γ_1 or Γ_2 are symmetric graph then $h_{per} = h$ and h is computable. That is, we give computable lower and upper bounds on h in terms of spectral radii of transfer matrices which converge to h. Theorem 4.3 is an improvement of [Fr2] following the techniques of Calkin-Wilf [CaW] for a special $\mathcal{S} \subset < 2 >^{\mathbb{Z}^2}$.

In §5 we continue the study of mcd mainly for $d > 2$. Theorem 5.2 (new result) shows that one can improve the upper bounds on the entropy if we assume that some of the graphs of $\Gamma_1, ..., \Gamma_d$ are symmetric. Theorem 5.5 (new) gives lower bounds on h under the condition that Γ_i is symmetric. In particular we reprove the result in [Fr2] that if all but one graphs in $\Gamma_1, ..., \Gamma_d$ are symmetric than $h_{per} = h$ and h is computable.

In §6 we give three examples of SOFT occuring in statistical mechanics and information theory. The first one is the residual entropy of the square ice. The second example is d-dimensional $(0, 1)$ limited channel discussed in the beginning of the Introduction. The third example is the d-dimensional dimer problem.

§7 discusses the invariant measures, the subadditive functions on $\mathbb{Z}_+^d$-SOFT and Kingman's [Kin] subergodic theorem, which is very useful in this area.

§8 we introduce a nonadditive topological pressure. It is a generalization of the standard topological pressure, which was introduced by Ruelle [Rue]. Here we follow the ideas and results in [Fal] and [Fr3]. In particular we state our version of the maximum principle for the topological pressure [Rue] and [Mis], which involve the entropy of invariant measure.

§9 discusses the Young formula [You] for the Hausdorff dimension using a version of a discrete Lyapunov exponent. §7–§9 is $\mathbb{Z}_+^d$ version of our results for $\mathbb{Z}_+$-SOFT in [Fr3] and most of the major results are stated here for the first time.

§10 consists of brief collection of needed facts about nonnegative matrices that are very useful in this area.

Finally let us mention that we omitted the algebraic part of the subject: abelian Markov groups. This subject is well described in by K. Schmidt [Sc].

2. One dimensional capacity. Let $< n >= \{1, ..., n\}$ be an alphabet on n letters. A word of length m in this alphabet is of the form $a = a_1...a_m$, where $a_i \in < n >$, $i = 1, ..., m$. We will identify a with a sequence $(a_i)_1^m$. Equivalently a can be viewed as map from the set $< m >= \{1, ..., m\}$ to the alphabet $< n >$, i.e. $a(i) = a_i,\ i = 1, ..., m$. Let

$$< n >^{<m>} := \{a : \quad a :< m > \to < n >\},$$

be the set of all maps from $< m >$ to $< n >$, i.e. the set of all words of length m in the alphabet $< n >$. Clearly $\# < n >^{<m>} = n^m$. Hence the m-th capacity (density) of the *all* words of length m in this alphabet is

$$h = \log n \tag{2.1}$$

(Here we assume that log is the logarithm on basis e.) Let

$$\delta_m := \log \# < n >^{<m>} = m \log n, \quad m = 1, ...,.$$

Then δ_m is an additive sequence:

$$\delta_{p+q} = \delta_p + \delta_q, \quad p, q = 1, 2, ...$$

Hence $\frac{\delta_m}{m}$ is the m-th capacity.

It is convenient to consider an infinite word $a = a_1 a_2 ... = (a_i)_{i \in \mathbb{N}}$ or a biinfnite word $a = ...a_{-1} a_0 a_1 ... = (a_i)_{i \in \mathbb{Z}}$ in the alphabet $< n >$, i.e. $a_i \in < n >$ for each i. Equivalently each a can be viewed as maps $a : \mathbb{N} \to < n >$ and $a : \mathbb{Z} \to < n >$ respectively. Let

$$< n >^{\mathbb{N}} := \{a : \quad a : \mathbb{N} \to < n >\},$$
$$< n >^{\mathbb{Z}} := \{a : \quad a : \mathbb{Z} \to < n >\},$$

be the set of infinite and biinfinite words on the alphabet $< n >$ respectively. In mathematical terminology $< n >^{\mathbb{N}}$ and $< n >^{\mathbb{Z}}$ are called *one sided shift* and *two sided shift* (on n letters) respectively. Shift σ is the following simple transformation of the given sequence, which is obtained by shifting to the "left" the given sequence:

$$\sigma :< n >^{\mathbb{N}} \to < n >^{\mathbb{N}}, \quad \sigma((a_i)_{i \in \mathbb{N}}) = (a_{i+1})_{i \in \mathbb{N}},$$
$$\sigma :< n >^{\mathbb{Z}} \to < n >^{\mathbb{Z}}, \quad \sigma((a_i)_{i \in \mathbb{Z}}) = (a_{i+1})_{i \in \mathbb{Z}}.$$

Note that the action of σ on a one sided shift results in a loss of the information (the first coordinate is dropped). On the two sided shift σ acts as invertible transformation $\sigma^{-1}((a_i)_{i \in \mathbb{Z}}) = (a_{i-1})_{i \in \mathbb{Z}}$. Let $k \in \mathbb{N}$. Then σ^k is the composition of σ k times, which act either on $< n >^{\mathbb{N}}$ or $< n >^{\mathbb{Z}}$. (σ^k shifts k-times to the left a given sequence.) σ^{-k} is the composition of σ^{-1} k times, which acts on $< n >^{\mathbb{Z}}$.

The origin of the action of σ on $< n >^{\mathbb{Z}}$ (or more precisely $\mathbb{Z}^2$ action) can be traced to Ising model in statistical mechanics [Kel, 1.2.2]. View $\mathbb{Z}$ as a discrete subset (lattice) of the real line $\mathbb{R}$. Place at each $i \in \mathbb{Z}$ a particle which has either spin down $a_i = 1$ or a spin up $a_i = 2$. Thus any $a \in < 2 >^{\mathbb{Z}}$ (called a state) corresponds to an arrangement of a countable number of particles on $\mathbb{Z}$ with the corresponding positions of spins. This state a depends on a choice of the "origin", which is located at the particle in the place 0. If we move the origin to the place $k \in \mathbb{Z}$ then the new state corresponds to the state $\sigma^k(a)$.

It is convenient to introduce metrics on $< n >^{\mathbb{N}}$ and $< n >^{\mathbb{Z}}$. Let $d_h :< n > \times < n > \to \mathbb{R}_+$ be the Hamming metric on $< n >$:

$$d_h(i,i) = 0, \quad i \in < n >,$$
$$d_h(i,j) = 1, \quad i \neq j \in < n > .$$

Then

$$d(a,b) = \sum_{i \in \mathbb{N}} \frac{d_h(a_i, b_i)}{2^i}, \quad a = (a_i)_{i\in\mathbb{N}}, b = (b_i)_{i\in\mathbb{N}} \in < n >^{\mathbb{N}},$$
$$d(a,b) = \sum_{i \in \mathbb{Z}} \frac{d_h(a_i, b_i)}{2^{|i|}}, \quad a = (a_i)_{i\in\mathbb{Z}}, b = (b_i)_{i\in\mathbb{Z}} \in < n >^{\mathbb{Z}} .$$

It is straightforward to show that $< n >^{\mathbb{N}}$ and $< n >^{\mathbb{Z}}$ are compact spaces. That is a sequence $a^1, a^2, ...$ in these spaces converges if and only if it converges coordinatewise. Clearly the shift is a continuous (Lipschitz) transformation

$$d(\sigma(a), \sigma(b)) \leq 2d(a,b).$$

DEFINITION 2.1. *A subset $\mathcal{S}$ of $< n >^{\mathbb{N}}$ or $< n >^{\mathbb{Z}}$ is called a subshift if*
(a) $\mathcal{S}$ is closed set with respect to the metric d;
(b) $\sigma(\mathcal{S}) = \mathcal{S}$. For $\mathcal{S} \subset < n >^{\mathbb{Z}}$ the condition (b) means that $a \in \mathcal{S} \Rightarrow \sigma^k(a) \in \mathcal{S}\ \forall k \in \mathbb{Z}$. Thus $\mathcal{S}$ is the set of all allowable states and the allowable states do not dependent on the choice of the origin. $\mathcal{S} = \emptyset$ is a subshift. For $k \geq m$ let $\pi_m((a_i)_1^k) = (a_i)_1^m$. Let $\pi_m :< n >^{\mathbb{N}}\ (< n >^{\mathbb{Z}}) \to < n >^{<m>}$ be the projection on the coordinates $1, ..., m$: $\pi_m((a_i)) = (a_i)_{i=1}^m$.

LEMMA 2.2. *Let $\mathcal{S} \subset < n >^{\mathbb{N}}\ (< n >^{\mathbb{Z}})$ be a subshift. Let $\tilde{\delta}_m := \log \#\pi_m(\mathcal{S})$ for $m \in \mathbb{N}$, where $\log 0 := -\infty$. Then the sequence $\tilde{\delta}_m$ is subadditive:*

$$\tilde{\delta}_p + \tilde{\delta}_q \geq \tilde{\delta}_{p+q} \qquad \forall\, p, q \in \mathbb{N}.$$

$(-\infty - \infty = -\infty.)$ *$\frac{\tilde{\delta}_m}{m}$ is the m-th density of $\mathcal{S}$ ($\frac{-\infty}{m} = -\infty$). The sequence $\frac{\tilde{\delta}_m}{m}$, $m = 1, ...,$ converges to h - the entropy of $\mathcal{S}$. Furthermore*

$$h \leq \frac{\tilde{\delta}_m}{m} \qquad \forall\, m \in \mathbb{N}. \tag{2.2}$$

Proof. It is enough to consider a subshift of one sided shift. If $\mathcal{S} = \emptyset$ the lemma is obvious. Assume that $\mathcal{S} \neq \emptyset$. Let $a = (a_i)_{i \in \mathbb{N}} \in \mathcal{S}$. Then $(a_i)_1^{p+q} \in \pi_{p+q}(\mathcal{S})$, $(a_i)_1^p \in \pi_p(\mathcal{S})$. As $\sigma^p(a) = (a_{p+i})_{i \in \mathbb{N}} \in \mathcal{S}$ it follows that $(a_i)_{i=p+1}^{p+q} \in \pi_q(\mathcal{S})$. Hence $\#\pi_{p+q}(\mathcal{S}) \leq \#\pi_p(\mathcal{S})\#\pi_q(\mathcal{S})$ and the sequence $\tilde{\delta}_m,\ m \in \mathbb{N}$ is a nonnegative subadditive sequence. In particular $\frac{\tilde{\delta}_m}{m},\ m \in \mathbb{N}$ is a convergent sequence, e.g. [Wal]. The subadditivity of $\tilde{\delta}_m,\ m \in \mathbb{N}$ yields that $\frac{\tilde{\delta}_{pm}}{pm} \leq \frac{\tilde{\delta}_m}{m}$. Let $p \to \infty$ to deduce (2.2). $\square$

DEFINITION 2.3. *A subshift $\mathcal{S} \subset <n>^{\mathbb{N}}$ $(<n>^{\mathbb{Z}})$ is called a subshift of finite type (SOFT) if it is the maximal subshift for the following condition:*
(a) There exists $r \in \mathbb{N}$ and a subset P (of allowable configurations) of $<n>^{<r>}$ such that $\pi_r(\mathcal{S}) \subset P$.

Equivalently, $\mathcal{S}$ is a SOFT if there exists a "window" of length r with an allowable set of configurations $P \subset <n>^{<r>}$ such that $a \in \mathcal{S}$ if and only if any consecutive string of r letters in a belongs to the allowable configuration P.

EXAMPLE 2.4. *Let $\Gamma \subset <n> \times <n>$. Identify Γ with a digraph on n vertices, where the directed edge (i,j) (from i to j) is in the graph if and only if $(i,j) \in \Gamma$. For $m \in \mathbb{N}$ denote by Γ^m and Γ^m_{per} the set of all possible walks on Γ of length m and the set of all periodic walks on Γ of period m:*

$$\Gamma^m := \{(a_i)_1^{m+1} \in <n>^{<m+1>}: \quad (a_i, a_{i+1}) \in \Gamma \text{ for } i = 1, ..., m.\},$$
$$\Gamma^m_{per} := \{(a_i)_1^{m+1} \in \Gamma^m: \quad a_1 = a_{m+1}\},$$

where $\Gamma^0 := <n>$. Let

$$\Gamma^{\mathbb{N}} := \{(a_i)_{i \in \mathbb{N}} \in <n>^{\mathbb{N}}: \quad (a_i, a_{i+1}) \in \Gamma, \text{ for } i \in \mathbb{N}\},$$
$$\Gamma^{\mathbb{Z}} := \{(a_i)_{i \in \mathbb{Z}} \in <n>^{\mathbb{Z}}: \quad (a_i, a_{i+1}) \in \Gamma, \text{ for } i \in \mathbb{Z}\},$$

be the sets of infinite and biinfinite walks on Γ respectively. Then $\Gamma^{\mathbb{N}} \subset <n>^{\mathbb{N}}$ and $\Gamma^{\mathbb{Z}} \subset <n>^{\mathbb{Z}}$ are SOFT induced by Γ.

For $\Gamma \subset <n> \times <n>$ denote by $A(\Gamma) = (a_{ij})_1^n$ the $0-1$ matrix induced by Γ. That is

$$a_{ij} = 1 \iff (i,j) \in \Gamma, \quad a_{ij} = 0 \iff (i,j) \notin \Gamma.$$

Vice versa any $0-1$ $n \times n$ matric $A = (a_{ij})_1^n$ induces a unique $\Gamma \subset <n> \times <n>$. Sometimes the SOFT $\Gamma^{\mathbb{N}}$ and $\Gamma^{\mathbb{Z}}$ are called *matrix* SOFT.

For any square matrix $A = (a_{ij})_1^n$ we denote by $\rho(A)$ the spectral radius of A (the maximum value of the absolute values of the eigenvalues of A.) The Perron-Frobenius theorem claims that if all the entries of A are nonnnegative then $\rho(A)$ is an eigenvalue of A (see Appendix). If in addition A is a $0-1$ matrix then $\rho(A)$ is an algebraic integer, i.e. $\rho(A)$ is a root of a normalized polynomial (the coefficient of the highest power is 1) with integer coefficients. For $\Gamma \subset <n> \times <n>$ we let $\rho(\Gamma) := \rho(A(\Gamma))$ to be the spectral radius of Γ. Then following lemma is straightforward (see Appendix):

LEMMA 2.5. *Let $\Gamma \subset < n > \times < n >$. Then either $\rho(\Gamma) = 0$ or $\rho(\Gamma) \geq 1$. Furthermore*

$$\rho(\Gamma) = 0 \iff \Gamma^n = \emptyset \iff \Gamma \text{ does not have cycles } \iff A(\Gamma)^n = 0.$$

In particular the SOFT induced by Γ is empty if and only if $\rho(\Gamma) = 0$.

Let $\mathcal{S} \subset < n >^{\mathbb{N}}$ $(< n >^{\mathbb{Z}})$ be a SOFT given by $P \subset < n >^{<r>}$. For $m \in \mathbb{N}$ let

$$\begin{aligned}
W^m(P) &:= < n >^{<m>} \quad \text{for } r > m,\\
W^m(P) &:= \{(a_i)_1^m \in < n >^{<m>} : (a_i)_{i=j}^{r+j-1} \in P,\\
&\qquad \text{for } j = 1, ..., m - r + 1\} \quad \text{for } r \leq m,\\
W^m_{per}(P) &:= \{(a_i)_1^{m+1} \in W^{m+1}(P) : a_i a_{i+1} ... a_{i+r-1} \in P,\\
&\qquad \text{for } i = 1, ..., m, \text{ where } a_i = a_j \text{ if } i - j \text{ is divisible by } m\},
\end{aligned}$$

be all P-allowable words of length m and all P-allowable periodic words of period m. Note that any word in $W^m_{per}(P)$ extends to a unique (bi)infinite m-periodic state in SOFT $\mathcal{S}$ induced by P. Let

$$\delta_m := \log \# W^m(P), \quad \delta_{m,per} = \log \# W^m_{per}(P), \quad m \in \mathbb{N}.$$

Then $\frac{\delta_m}{m}$ and $\frac{\delta_{m,per}}{m}$ is called the m-th capacity and the m-th periodic capacity respectively. Combine the results of Lemma 2.2 with the arguments of its proof to deduce:

LEMMA 2.6. *Let $\mathcal{S} \subset < n >^{\mathbb{N}}$ $(< n >^{\mathbb{Z}})$ be a SOFT given by the allowable configurations $P \subset < n >^r$. Then*

$$\delta_{m,per} \leq \tilde{\delta}_m \leq \delta_m, \quad m \in \mathbb{N}.$$

The sequence δ_m, $m \in \mathbb{N}$ is subadditive. Hence the sequence $\frac{\delta_m}{m}$, $m \in \mathbb{N}$ converges to the capacity h_{com}. Let $h_{per} := \limsup_{m \to \infty} \frac{\delta_{m,per}}{m}$ be the periodic capacity of $\mathcal{S}$. Then

$$h_{per} \leq h \leq h_{com} \leq \frac{\delta_m}{m},$$

for any $m \in \mathbb{N}$.

The following lemma shows that any SOFT can be (efficiently) coded as a matrix SOFT:

LEMMA 2.7. *Let $\mathcal{S} \subset < n >^{\mathbb{N}}$ $(< n >^{\mathbb{Z}})$ be a SOFT induced by the set of allowable configurations $P \subset < n >^{<r>}$, where $r \geq 2$. Let $N = \#\pi_{r-1}(P)$. Then there exists $\Gamma \subset < N > \times < N >$ such that there is one to one correspondence between any word $a \in W^m(P)$ and $b \in \Gamma^{m-r+1}$ for every $m \geq r$. Furthermore any $a \in W^m_{per}(P)$ corresponds to a unique $b \in \Gamma^m_{per}$ and vice versa for any $m \in \mathbb{N}$.*

Proof. Label configurations $(a_i)_1^{r-1}$ which can be extended to allowable configurations $(a_i)_1^r \in P$ by $1,...,N$. Let $u,v \in < N >$ correspond to $(a_i)_1^{r-1},(b_i)_1^{r-1} \in \pi_{r-1}P$ respectively. Then $(u,v) \in \Gamma$ if and only if $b_i = a_{i+1}$ for $i = 1,...,r-2$ (for $r = 2$ this condition is void) and $(a_i)_1^r \in P$, where $a_r = b_{r-1}$. It is straightforward to see that $(a_i)_1^m \in W^m(P)$ for $m \geq r$ if and only if $(u_j)_1^{m-r+2} \in \Gamma^{m-r+1}$ where u_j corresponds to $(a_i)_{i=j}^{j+r-2}$ for $j = 1,...,m-r+2$.

Let $a \in W^m_{per}(P)$. Extend a to the m-periodic state $\tilde{a} = (a_i)_{i\in\mathbb{N}} \in \mathcal{S}$. Then $\tilde{a}$ induces $(u_j)_1^{m+1} \in \Gamma^m_{per}$ where u_j corresponds to $(a_i)_{i=j}^{r+j-2}$. Similarly any $(u_j)_1^{m+1} \in \Gamma^m_{per}$ induces a unique $a \in W^m_{per}(P)$. □

THEOREM 2.8. *Let $\mathcal{S} \subset < n >^{\mathbb{N}}$ $(< n >^{\mathbb{Z}})$ be a SOFT induced by the set of allowable configurations $P \subset < n >^{<r>}$. Let $N = \#\pi_{r-1}P$ and $\Gamma \subset < N > \times < N >$ be the graph induced by P as in the proof of Lemma 2.7. Then*

$$h_{per} = h = h_{com} = \log\rho(\Gamma). \tag{2.3}$$

Proof. In view of Lemma 2.7 it is enough to prove the theorem in the case $P = \Gamma \subset < n > \times < n >$. Let $\mathbf{1} := (1,...,1)$ a vector whose all coordinates are equal to 1. It is straightforward to show (see Appendix)

$$\#\Gamma^m = \mathbf{1}A(\Gamma)^m\mathbf{1}^{\mathrm{T}}, \quad \#\Gamma^m_{per} = \operatorname{tr} A(\Gamma)^m, \tag{2.4}$$

where $\operatorname{tr} B$ is the trace of the square matrix B. It is known (see Appendix)

$$\begin{aligned} &\lim_{m\to\infty} \frac{1}{m}\log\mathbf{1}A^{m-1}\mathbf{1}^{\mathrm{T}} = \log\rho(A), \\ &\limsup_{m\to\infty}(\operatorname{tr} A^m)^{\frac{1}{m}} = \rho(A), \\ &A = (a_{ij})_1^n, \text{ and } a_{ij} \geq 0 \text{ for } ,i,j = 1,...,n. \end{aligned} \tag{2.5}$$

Hence $h_{com} = h_{per} = \log\rho(A(\Gamma))$. Combine these results with the last inequality of Lemma 2.6 to deduce (2.3). □

COROLLARY 2.9. *Let $\mathcal{S} \subset < n >^{\mathbb{N}}$ $(< n >^{\mathbb{Z}})$ be a SOFT. Then either $\mathcal{S} = \emptyset$ or $\mathcal{S}$ contains a periodic state.*

3. Multi-dimensional capacity. Let $2 \leq d \in \mathbb{N}$. Denote

$$\begin{aligned} &\mathbf{e}_j = (\delta_{1j},...,\delta_{dj}),\ j = 1,...,d, \quad \text{(the standard basis in } \mathbb{R}^d) \\ &\mathbf{m} = (m_1,...,m_d) \in \mathbb{Z}^d \text{ and } |\mathbf{m}| = |m_1|+...+|m_d|,\ |\mathbf{m}|_{pr} = |m_1|\ldots|m_d|, \\ &\mathbf{m} \leq \mathbf{n} \iff m_i \leq n_i, \quad i = 1,...,d, \\ &< \mathbf{m} > = < m_1 > \times ... \times < m_d > \text{ for } \mathbf{m} \in \mathbb{N}^d. \end{aligned}$$

View a map a: $< \mathbf{m} > \to < n >$ as $(a_{\mathbf{i}})_{\mathbf{i}\in<\mathbf{m}>}$. (Sometimes we denote this map by $(a_{\mathbf{i}})_{\mathbf{i}=\mathbf{1}}^{\mathbf{m}}$.) Then $< n >^{<\mathbf{m}>}$ is the set of all such maps a. Furthermore $< n >^{\mathbb{N}^d}$ and $< n >^{\mathbb{Z}^d}$ is the set of all maps from $\mathbb{N}^d$ and $\mathbb{Z}^d$

to $< n >$ respectively. That is $< n >^{\mathbb{N}^d}$ and $< n >^{\mathbb{Z}^d}$ are one sided and two sided d-shifts consisting of the sequences $(a_{\mathbf{i}})_{\mathbf{i}\in\mathbb{N}^d}$ and $(a_{\mathbf{i}})_{i\in\mathbb{Z}^d}$ respectively. A shift $\sigma_j : \mathbb{N}^d \to \mathbb{N}^d$ or $\sigma_j : \mathbb{Z}^d \to \mathbb{Z}^d$ is defined as

$$\sigma_j((a_{\mathbf{i}})) = (a_{\mathbf{i}+\mathbf{e}_j}), \quad j = 1, ..., d.$$

Note that $\sigma_1, ..., \sigma_d$ are commuting transformations on $\mathbb{N}^d$ and $\mathbb{Z}^d$ respectively. On $\mathbb{Z}^d$ each σ_j is an invertible transformation.

Define metrics on $\mathbb{N}^d$ and $\mathbb{Z}^d$

$$d(a, b) = \sum_{\mathbf{i}\in\mathbb{N}^d} \frac{d_h(a_{\mathbf{i}}, b_{\mathbf{i}})}{2^{|\mathbf{i}|}}, \quad a = (a_{\mathbf{i}})_{\mathbf{i}\in\mathbb{N}^d}, b = (b_{\mathbf{i}})_{\mathbf{i}\in\mathbb{N}^d} \in < n >^{\mathbb{N}^d},$$

$$d(a, b) = \sum_{\mathbf{i}\in\mathbb{Z}^d} \frac{d_h(a_{\mathbf{i}}, b_{\mathbf{i}})}{2^{|\mathbf{i}|}}, \quad a = (a_{\mathbf{i}})_{\mathbf{i}\in\mathbb{Z}^d}, b = (b_{\mathbf{i}})_{\mathbf{i}\in\mathbb{Z}^d} \in < n >^{\mathbb{Z}^d}.$$

Then $< n >^{\mathbb{N}^d}$ and $< n >^{\mathbb{Z}^d}$ are compact spaces. A sequence $a^1, a^2, ...$ in these spaces converges if and only if it converges coordinatewise. Clearly each σ_j is a continuous (Lipschitz) transformation. For $\mathbf{k} = (k_1, ..., k_d) \in \mathbb{Z}^d_+$ let $\sigma^{\mathbf{k}} = \sigma_1^{k_1}...\sigma_d^{k_d}$ be the composition of $\sigma_1^{k_1}, ..., \sigma^{k_d}$. Then $\sigma^{\mathbf{k}}$ is well defined on $\mathbb{N}^d$ and $\mathbb{Z}^d$. For $\mathbf{k} \in \mathbb{Z}^d$ the map $\sigma^{\mathbf{k}}$ is well defined on $\mathbb{Z}^d$.

DEFINITION 3.1. *A subset $\mathcal{S}$ of $< n >^{\mathbb{N}^d}$ or $< n >^{\mathbb{Z}^d}$ is called a subshift if*
(a) $\mathcal{S}$ is closed set with respect to the metric d;
(b) $\sigma_i(\mathcal{S}) = \mathcal{S}$ for I=1,...,d. For $\mathcal{S} \subset < n >^{\mathbb{Z}^d}$ $\mathcal{S}$ is the set of all allowable states which do not dependent on the choice of the origin. For $\mathbf{k} \geq \mathbf{m} \in \mathbb{N}^d$ let $\pi_{\mathbf{m}}((a_{\mathbf{i}})_{\mathbf{i}\in\mathbf{k}}) = (a_{\mathbf{i}})_{\mathbf{i}\in<\mathbf{m}>}$. Let $\pi_{\mathbf{m}} :< n >^{\mathbb{N}^d} \ (< n >^{\mathbb{Z}^d}) \to < n >^{<\mathbf{m}>}$ be the projection on the coordinates $\mathbf{1}$ to $\mathbf{m}$: $\pi_{\mathbf{m}}((a_{\mathbf{i}})) = (a_{\mathbf{i}})_{\mathbf{i}\in<\mathbf{m}>}$.

LEMMA 3.2. *Let $\mathcal{S} \subset < n >^{\mathbb{N}^d} \ (< n >^{\mathbb{Z}^d})$ be a subshift. Let $\tilde{\delta}_{\mathbf{m}} := \log \#\pi_{\mathbf{m}}(\mathcal{S})$ for $\mathbf{m} \in \mathbb{N}^d$. Then the sequence $\tilde{\delta}_{\mathbf{m}}$ is subadditive in each coordinate of $\mathbf{m}$:*

$$\tilde{\delta}_{\mathbf{p}} + \tilde{\delta}_{\mathbf{p}+q\mathbf{e}_i} \geq \tilde{\delta}_{\mathbf{p}+(p_i+q)\mathbf{e}_i} \ \ \forall \ \mathbf{p} = (p_1, ...p_d) \in \mathbb{N}^d, q \in \mathbb{Z}_+, \ i = 1, ..., d. \tag{3.1}$$

$\frac{\tilde{\delta}_{\mathbf{m}}}{|\mathbf{m}|_{pr}}$ is the $\mathbf{m}$-th density of $\mathcal{S}$. The sequence $\frac{\tilde{\delta}_{\mathbf{m}}}{|\mathbf{m}|_{pr}}$ converges to h - the entropy of $\mathcal{S}$ as each $m_i \to \infty$. Furthermore

$$h \leq \frac{\tilde{\delta}_{\mathbf{m}}}{|\mathbf{m}|_{pr}} \quad \forall m \in \mathbb{N}. \tag{3.2}$$

Proof. Let $\mathbf{m} = (m_1, ..., m_d) \in \mathbb{N}^d$ and assume that $\mathbf{m} \neq \mathbf{1}$, i.e. $m_i > 1$ for some $i \in < d >$. View $< \mathbf{m} >$ as a box in $\mathbb{R}^d$ of dimensions $m_1, ..., m_d$. Divide the box $< \mathbf{m} >$ to two boxes by the hyperplane $x_i = \ell$, where $\ell \in \mathbb{N}$ and $\ell < m_i$. Then the dimension of the smaller box is $\mathbf{p} = (p_1, ..., p_d)$, where $p_j = m_j$ for all $j \neq i$, and the dimension of the bigger box is

$\mathbf{p}+q\mathbf{e}_i$, where $m_i = 2p_i + q$ for some $q \ge 0$. As in the proof of Lemma 2.2 $\#\pi_{\mathbf{m}}(\mathcal{S}) \le \#\pi_{\mathbf{p}}(\mathcal{S})\#\pi_{\mathbf{p}+q\mathbf{e}_i}(\mathcal{S})$ and (3.1) follows.

For a nonempty strict subset τ of $< d >$ and $\mathbf{m} = (m_1, ..., m_d) \in \mathbb{N}^d$ denote

$$\begin{aligned}\tau^c &:= \ <d> \backslash \tau = \{q_1, q_2, ..., q_p\}, \quad 1 \le q_1 < ... < q_p \le d, \quad p = d - \#\tau,\\ \mathbf{m}^\tau &= (m_{q_1}, m_{q_2}, ..., m_{q_p}),\\ \mathbf{m}^{\{i\}} &= (m_1, ..., m_{i-1}, m_{i+1}, ..., m_d) \in \mathbb{N}^{d-1}, \ \mathbf{m} = (\mathbf{m}^{\{i\}}, m_i), \ \text{for } i \in <d>,\\ \Gamma^\tau &= (\Gamma_{q_1}, ..., \Gamma_{q_p}).\end{aligned}$$

Observe next that $\alpha_{\mathbf{m}^{\{i\}}, m_i} := \frac{\tilde\delta_{\mathbf{m}}}{m_i}$ is a subadditive sequence in each coordinate of $\mathbf{m}^{\{i\}} \in \mathbb{N}^{d-1}$ for a fixed value of $m_i \in \mathbb{N}$. Fix $\mathbf{m}^{\{i\}}$. Then $\tilde\delta_{\mathbf{m}}$ is a subadditive sequence in $m_i \in \mathbb{N}$. Hence $\lim_{m_i \to \infty} \alpha_{\mathbf{m}^{\{i\}}, m_i} = \tilde\alpha_{\mathbf{m}^{\{i\}}}$ exists and $\tilde\alpha_{\mathbf{m}^{\{i\}}} \le \alpha_{\mathbf{m}^{\{i\}}, m_i}$ for any $m_i \in \mathbb{N}$. Clearly, the sequence $\tilde\alpha_{\mathbf{m}^{\{i\}}}$, $\mathbf{m}^{\{i\}} \in \mathbb{N}^{d-1}$ is a subadditive sequence in each coordinate of $\mathbf{m}^{\{i\}}$. Continue this (contraction) process to deduce the lemma. □

For $\mathcal{S} =< n >^{\mathbb{N}^d}$ $(< n >^{\mathbb{Z}^d})$ $\tilde\delta_{\mathbf{m}} = |\mathbf{m}|_{pr} \log n$, which is an additive sequence in each coordinate of $\mathbf{m} \in \mathbb{N}^d$, and $h = \log n$.

DEFINITION 3.3. *A subshift $\mathcal{S} \subset< n >^{\mathbb{N}^d}$ $(< n >^{\mathbb{Z}^d})$ is called a subshift of finite type (SOFT) if it is maximal subshift for the following condition:*
(a) There exists $\mathbf{r} \in \mathbb{N}^d$ and a subset P (of allowable configurations) of $< n >^{<\mathbf{r}>}$ such that $\pi_{\mathbf{r}}(\mathcal{S}) \subset P$.

Equivalently, $\mathcal{S}$ is a SOFT if there exists a "window" of dimension $\mathbf{r}$ with an allowable set of configurations $P \subset< n >^{<\mathbf{r}>}$ such that $a \in \mathcal{S}$ if and only if any consecutive "box" of letters of dimension $\mathbf{r}$ in a belongs to the allowable configuration P.

EXAMPLE 3.4. *Let $\Gamma_1, ..., \Gamma_d \subset< n > \times < n >$ and denote $\Gamma = (\Gamma_1, ..., \Gamma_d)$. For $\mathbf{m} \in \mathbb{Z}_+^d$ $(\mathbb{N}^d)$ let $\Gamma^{\mathbf{m}}$ $(\Gamma^{\mathbf{m}}_{per}) \subset< n >^{\mathbf{m}+\mathbf{1}}$ be the set of all sequences $(a_{\mathbf{i}}) \in< n >^{\mathbf{m}+\mathbf{1}}$ such that the following condition holds:*
(a) Fix $k \in< d >$ and let $\mathbf{m} = (m_1, ..., m_d) = (\mathbf{m}^{\{k\}}, m_k)$ Fix $\mathbf{p}^{\{k\}} \in< \mathbf{m}^{\{k\}} + \mathbf{1}^{\{k\}} >\subset \mathbb{N}^{d-1}$ and let $\mathbf{p} = (\mathbf{p}^{\{k\}}, p_k) \in \mathbb{N}^d$. Then the sequence $(a_{(\mathbf{p}^{\{k\}}, p_k)})_{p_k=1}^{p_k=m_k+1}$ belongs to $\Gamma_k^{m_k}$ $(\Gamma_{per}^{m_k})$ (for each $k \in< d >$.)
Let $\Gamma^{\mathbb{N}^d}$ $(\Gamma^{\mathbb{Z}^d})$ be the set of all sequences $(a_{\mathbf{i}})_{\mathbf{i}\in\mathbb{N}^d}$ $((a_{\mathbf{i}})_{\mathbf{i}\in\mathbb{Z}^d})$ such that for each $k \in< d >$ and each $\mathbf{p}^{\{k\}} \in \mathbb{N}^{d-1}$ $(\mathbf{p}^{\{k\}} \in \mathbb{Z}^{d-1})$ the sequence $(a_{(\mathbf{p}^{\{k\}}, p_k)})_{p_k\in\mathbb{N}}$ $((a_{(\mathbf{p}^{\{k\}}, p_k)})_{p_k\in\mathbb{Z}})$ belongs to $\Gamma_k^{\mathbb{N}}$ $(\Gamma_k^{\mathbb{Z}})$. Then $\Gamma^{\mathbb{N}^d} \subset< n >^{\mathbb{N}}$ and $\Gamma^{\mathbb{Z}^d} \subset< n >^{\mathbb{Z}}$ are SOFT induced by Γ.

The above SOFT is called matrix SOFT. Let $\mathcal{S} \subset< n >^{\mathbb{N}^d}$ $(< n >^{\mathbb{Z}^d})$ be a SOFT given by $P \subset< n >^{<\mathbf{r}>}$. For $\mathbf{m} \in \mathbb{N}^d$ let

$$\begin{aligned}W^{\mathbf{m}}(P) &:= < n >^{<\mathbf{m}>} \quad \text{for } \mathbf{r} \not\le \mathbf{m},\\ W^{\mathbf{m}}(P) &:= \{(a_{\mathbf{i}})_{\mathbf{1}}^{\mathbf{m}} \in< n >^{<\mathbf{m}>}: (a_{\mathbf{i}})_{\mathbf{i}=\mathbf{j}}^{\mathbf{r}+\mathbf{j}-\mathbf{1}} \in P\\ &\qquad \text{for } \mathbf{1} \le \mathbf{j} \le \mathbf{m} - \mathbf{r} + \mathbf{1}\} \quad \text{for } \mathbf{r} \le \mathbf{m},\end{aligned}$$

$$W^{\mathbf{m}}_{per}(P) := \{(a_{\mathbf{i}})_{\mathbf{1}}^{\mathbf{m+1}} \in W^{\mathbf{m+1}}(P) : (a_{\mathbf{i}})_{\mathbf{i=j}}^{\mathbf{i=j+r-1}} \in P,$$
$$\text{for } \mathbf{j} \in < \mathbf{m} > \text{ where } a_{\mathbf{i}} = a_{\mathbf{j}} \text{ if } m_k | i_k - j_k \text{ for each } k \in < d >\},$$

be all P-allowable words of dimension $\mathbf{m}$ and all P-allowable periodic words of period $\mathbf{m}$. Note that any word in $W^{\mathbf{m}}_{per}(P)$ extends to a unique (bi)infinite $\mathbf{m}$-periodic state in SOFT $\mathcal{S}$ induced by P. Let

$$\delta_{\mathbf{m}} := \log \# W^{\mathbf{m}}(P), \quad \delta_{\mathbf{m},per} = \log \# W^{\mathbf{m}}_{per}(P), \quad \mathbf{m} \in \mathbb{N}^d.$$

Then $\frac{\delta_{\mathbf{m}}}{|\mathbf{m}|_{pr}}$ and $\frac{\delta_{\mathbf{m},per}}{|\mathbf{m}|_{pr}}$ is called the $\mathbf{m}$-th capacity and the $\mathbf{m}$-th periodic capacity respectively. Combine the results of Lemma 3.2 with the arguments of its proof to deduce:

LEMMA 3.5. *Let $\mathcal{S} \subset < n >^{\mathbb{N}^d}$ $(< n >^{\mathbb{Z}^d})$ be a SOFT given by the allowable configurations $P \subset < n >^{\mathbf{r}}$. Then*

$$\delta_{\mathbf{m},per} \le \tilde{\delta}_{\mathbf{m}} \le \delta_{\mathbf{m}}, \quad \mathbf{m} \in \mathbb{N}^d.$$

The sequence $\delta_{\mathbf{m}}$, $\mathbf{m} \in \mathbb{N}^d$ is subadditive in each coordinate. Hence the sequence $\frac{\delta_{\mathbf{m}}}{|\mathbf{m}|_{pr}}$, $\mathbf{m} \in \mathbb{N}^d$ converges to the capacity h_{com}. Let $h_{per} := \limsup_{\mathbf{m}\to\infty} \frac{\delta_{\mathbf{m},per}}{|m|_{pr}}$ be the periodic capacity of $\mathcal{S}$. Then

$$h_{per} \le h \le h_{com} \le \frac{\delta_{\mathbf{m}}}{|\mathbf{m}|_{pr}},$$

for any $\mathbf{m} \in \mathbb{N}^d$.

A simple argument yields [Fr2, Thm 1.3]:

LEMMA 3.6. *Let $\mathcal{S} \subset < n >^{\mathbb{N}^d}$ $(< n >^{\mathbb{Z}^d})$ be a SOFT given by $P \subset < n >^{\mathbf{r}}$. Then $\mathcal{S} = \emptyset \iff W^{\mathbf{m}}(P) = \emptyset$ for some $\mathbf{m} \in \mathbb{N}^d$.*

DEFINITION 3.7. *Let $\mathcal{S} \subset < n >^{\mathbb{N}^d}$ $(< n >^{\mathbb{Z}^d})$ be a SOFT given by $P \subset < n >^{\mathbf{r}}$. Then $\mathcal{S}$ is called a decidable SOFT if one of the following conditions hold:*
(a) There exists $\mathbf{m} \in \mathbb{N}^d$ such that $W^{\mathbf{m}}(P) = \emptyset$.
(b) There exists $\mathbf{m} \in \mathbb{N}^d$ such that $W^{\mathbf{m}}_{per}(P) \neq \emptyset$ (hence $\mathcal{S} \neq \emptyset$).

Corollary 2.9 implies that for $d = 1$ any SOFT is decidable. For $d = 2$, and hence for any $d \ge 2$, there are SOFT which are not decidable [Ber] and [Rob]. That is there exist nonempty multi-dimensional SOFT which do not contain a periodic state. This is a first example which shows that the theory of multi-dimensional SOFT is much more complicated then one dimensional SOFT.

THEOREM 3.8. *[Fr2, Thm 2.5] Let $\mathcal{S} \subset < n >^{\mathbb{N}^d}$ $(< n >^{\mathbb{Z}^d})$ be SOFT. Then $h = h_{com}$.*

In this paper we refer to $h = h_{com}$ as the *multi-dimensional capacity* (mdc) of the given SOFT. In mathematics h is called the *entropy* of $\mathcal{S}$. In statistical mechanics e^h is called the entropy of the given SOFT.

Note that if $\mathcal{S}$ is undecidable SOFT then

$$h_{per} = -\infty < 0 \le h = h_{com}.$$

(Compare that with (2.3).)

LEMMA 3.9. *Let $\mathcal{S} \subset <n>^{\mathbb{N}^d}$ $(<n>^{\mathbb{Z}^d})$ be a SOFT induced by the set of allowable configurations $P \subset <n>^{<\mathbf{r}>}$, where $\mathbf{r} \ge \mathbf{21}$. Let $N = \#P$. Then there exists $\Gamma = (\Gamma_1, ..., \Gamma_d)$ $\Gamma_i \subset <N> \times <N>$, $i = 1, ..., d$ such that there is one to one correspondence between any word $a \in W^{\mathbf{m}}(P)$ and $b \in \Gamma^{\mathbf{m}-\mathbf{r}+\mathbf{1}}$ for every $\mathbf{m} \ge \mathbf{r}+\mathbf{1}$. Moreover there is one to one correspondence between $\mathcal{S}$ and $\Gamma^{\mathbb{N}^d}$ $(\Gamma^{\mathbb{Z}^d})$. Furthermore any $a \in W^{\mathbf{m}}_{per}(P)$ corresponds to a unique $b \in \Gamma^{\mathbf{m}}_{per}$ and vice versa for any $\mathbf{m} \in \mathbb{N}$.*

Proof. Label allowable configurations $(a_{\mathbf{i}})_{\mathbf{1}}^{\mathbf{r}} \in P$ by $1, ..., N$. Let $u, v \in <N>$ correspond to $(a_{\mathbf{i}})_{\mathbf{1}}^{\mathbf{r}}, (b_i)_{\mathbf{1}}^{\mathbf{r}} \in P$. Then for each $k \in <d>$ the edge $(u, v) \in \Gamma_k$ if and only $b_{\mathbf{i}} = a_{\mathbf{i}+\mathbf{e}_k}$ for $\mathbf{i} \in <\mathbf{r}-\mathbf{e}_k>$. It is straightforward to see that $(a_{\mathbf{i}})_{\mathbf{1}}^{\mathbf{m}} \in W^{\mathbf{m}}(P)$ for $\mathbf{m} \ge \mathbf{r}+\mathbf{1}$ if and only if $(u_{\mathbf{j}})_{\mathbf{1}}^{\mathbf{m}-\mathbf{r}+\mathbf{1}} \in \Gamma^{\mathbf{m}-\mathbf{r}}$ where $u_{\mathbf{j}}$ corresponds to $(a_{\mathbf{i}})_{\mathbf{i}=\mathbf{j}}^{\mathbf{j}+\mathbf{r}-\mathbf{1}}$ for $\mathbf{j} \in <\mathbf{m}-\mathbf{r}>$. Clearly the any $(a_{\mathbf{i}})_{\mathbf{i}\in\mathbb{N}^d}$ $((a_{\mathbf{i}})_{\mathbf{i}\in\mathbb{Z}^d}) \in \mathcal{S}$ gives rise to a unique $(u_{\mathbf{j}})_{\mathbf{j}\in\mathbb{N}^d} \in \Gamma^{\mathbb{N}^d}$ $((u_{\mathbf{j}})_{\mathbf{j}\in\mathbb{Z}^d}) \in \Gamma^{\mathbb{Z}^d}$ and vise versa.

Let $a \in W^{\mathbf{m}}_{per}(P)$. Extend a to the $\mathbf{m}$-periodic state $\tilde{a} = (a_{\mathbf{i}})_{\mathbf{i}\in\mathbb{N}^d} \in \mathcal{S}$. Then $\tilde{a}$ induces $(u_{\mathbf{j}})_{\mathbf{1}}^{\mathbf{m}+\mathbf{1}} \in \Gamma^{\mathbf{m}}_{per}$ where $u_{\mathbf{j}}$ corresponds to $(a_{\mathbf{i}})_{\mathbf{i}=\mathbf{j}}^{\mathbf{r}+\mathbf{j}-\mathbf{1}}$. Similarly any $(u_{\mathbf{j}})_{\mathbf{1}}^{\mathbf{m}+\mathbf{1}} \in \Gamma^{\mathbf{m}}_{per}$ induces a unique $a \in W^{\mathbf{m}}_{per}(P)$. □

Note that the coding of a multi-dimensional SOFT as a matrix SOFT given in Lemma 3.9 is less efficient that the coding of 1-dimensional SOFT given by Lemma 2.7.

4. Estimates of mdc in terms of spectral radii *I*. From now and until the end of the paper we assume that $\mathcal{S} \subset <n>^{\mathbb{N}^d}$ $(<n>^{\mathbb{Z}^d})$ is a matrix SOFT type given by $\Gamma = (\Gamma_1, ..., \Gamma_d)$ with $d > 1$ unless stated otherwise.

DEFINITION 4.1. *For $k \in <d>$ and $\mathbf{m}^{\{k\}} \in \mathbb{N}^{d-1}$ let $\Gamma(k, \mathbf{m}^{\{k\}}) \subset (\Gamma^{\{k\}})^{\mathbf{m}^{\{k\}}-\mathbf{1}^{\{k\}}} \times (\Gamma^{\{k\}})^{\mathbf{m}^{\{k\}}-\mathbf{1}^{\{k\}}}$ be the graph on the vertices $(\Gamma^{\{k\}})^{\mathbf{m}^{\{k\}}-\mathbf{1}^{\{k\}}}$ such that the edge*

$$((b_{\mathbf{i}^{\{k\}}})_{\mathbf{i}^{\{k\}}\in<\mathbf{m}^{\{k\}}>}, (c_{\mathbf{i}^{\{k\}}})_{\mathbf{i}^{\{k\}}\in<\mathbf{m}^{\{k\}}>}) \in \Gamma(k, \mathbf{m}^{\{k\}})$$

if and only $(b_{\mathbf{i}^{\{k\}}}, c_{\mathbf{i}^{\{k\}}}) \in \Gamma_k$ for each $\mathbf{i}^{\{k\}} \in <\mathbf{m}^{\{k\}}>$. Let $\Gamma_{per}(k, \mathbf{m}^{\{k\}}) \subset (\Gamma^{\{k\}})^{\mathbf{m}^{\{k\}}}_{per} \times (\Gamma^{\{k\}})^{\mathbf{m}^{\{k\}}}_{per}$ be the graph on the vertices $(\Gamma^{\{k\}})^{\mathbf{m}^{\{k\}}}_{per}$ such that the edge

$$((b_{\mathbf{i}^{\{k\}}})_{\mathbf{i}^{\{k\}}\in<\mathbf{m}^{\{k\}}+\mathbf{1}^{\{k\}}>}, (c_{\mathbf{i}^{\{k\}}})_{\mathbf{i}^{\{k\}}\in<\mathbf{m}^{\{k\}}+\mathbf{1}^{\{k\}}>}) \in \Gamma_{per}(k, \mathbf{m}^{\{k\}})$$

if and only $(b_{\mathbf{i}^{\{k\}}}, c_{\mathbf{i}^{\{k\}}}) \in \Gamma_k$ for each $\mathbf{i}^{\{k\}} \in <\mathbf{m}^{\{k\}}+\mathbf{1}^{\{k\}}>$. Denote by $\rho(\Gamma(k, \mathbf{m}^{\{k\}}))$ and $\rho(\Gamma_{per}(k, \mathbf{m}^{\{k\}}))$ the spectral radii of $\Gamma(k, \mathbf{m}^{\{k\}})$ and $\Gamma_{per}(k, \mathbf{m}^{\{k\}})$ respectively.

That is $\Gamma(k,\mathbf{m}^{\{k\}})^{\mathbb{N}}$ $(\Gamma(k,\mathbf{m}^{\{k\}})^{\mathbb{Z}})$ is the 1-dimensional SOFT generated by the "strip" in direction k with the basis of dimension $\mathbf{m}^{\{k\}}$, which is an allowable configuration with respect to Γ. $\Gamma_{per}(k,\mathbf{m}^{\{k\}})^{\mathbb{N}}$ $(\Gamma_{per}(k,\mathbf{m}^{\{k\}})^{\mathbb{Z}})$ is the 1-dimensional SOFT generated by the "strip" in direction k with the $\mathbf{m}^{\{k\}}$-periodic basis of dimension $\mathbf{m}^{\{k\}}+\mathbf{1}^{\{k\}}$, which is an allowable configuration with respect to Γ. Note that we can view each $\mathbf{m}^{\{k\}}$ periodic configuration $(b_{\mathbf{i}^{\{k\}}})_{\mathbf{i}^{\{k\}}\in<\mathbf{m}^{\{k\}}+\mathbf{1}^{\{k\}}>}\in(\Gamma_{per}^{\{k\}})^{\mathbf{m}^{\{k\}}}$ as a configuration $(b_{\mathbf{i}^{\{k\}}})_{\mathbf{i}^{\{k\}}\in<\mathbf{m}^{\{k\}}>}\in(\Gamma^{\{k\}})^{\mathbf{m}^{\{k\}}-\mathbf{1}^{\{k\}}}$ which has a periodic extension to a configuration $(b_{\mathbf{i}^{\{k\}}})_{\mathbf{i}^{\{k\}}\in<\mathbf{m}^{\{k\}}+\mathbf{1}^{\{k\}}>}\in(\Gamma^{\mathbf{k}})^{\mathbf{m}^{\{k\}}}$. Thus (by abuse of notation) we view $(\Gamma^{\{k\}})_{per}^{\mathbf{m}^{\{k\}}}$ and $\Gamma_{per}(k,\mathbf{m}^{\{k\}})$ as subsets of $(\Gamma^{\{k\}})^{\mathbf{m}^{\{k\}}-\mathbf{1}^{\{k\}}}$ and $\Gamma(k,\mathbf{m}^{\{k\}})$ respectively. In statistical mechanics the matrix $A(\Gamma(k,\mathbf{m}^{\{k\}}))$ is called the *transfer* matrix.

LEMMA 4.2. *Let $\mathcal{S}\subset<n>^{\mathbb{N}^d}$ $(<n>^{\mathbb{Z}^d})$ be a SOFT given by $\Gamma=(\Gamma_1,...,\Gamma_k)$. Let $\delta_{\mathbf{m}}$, $\delta_{\mathbf{m},per}$, $\mathbf{m}\in\mathbb{N}^d$ be defined as in Lemma 3.5. Then*

$$\begin{aligned}\lim_{m_k\to\infty}\frac{\delta_{(\mathbf{m}^{\{k\}},m_k)}}{m_k}&=\log\rho(\Gamma(k,\mathbf{m}^{\{k\}})),\\ \limsup_{m_k\to\infty}\frac{\delta_{(\mathbf{m}^{\{k\}},m_k),per}}{m_k}&=\log\rho(\Gamma_{per}(k,\mathbf{m}^{\{k\}})).\end{aligned}\tag{4.1}$$

The sequence $\log\rho(\Gamma(k,\mathbf{m}^{\{k\}}))$ is subadditve in $\mathbf{m}^{\{k\}}\in\mathbb{N}^{d-1}$ for each $k\in<d>$ and $h\le\frac{\log\rho(\Gamma(k,\mathbf{m}^{\{k\}}))}{|\mathbf{m}^{\{k\}}|_{pr}}$ for each $\mathbf{m}^{\{k\}}\in\mathbb{N}^{d-1}$. Furthermore

$$\begin{aligned}\lim_{\mathbf{m}^{\{k\}}\to\infty}\frac{\log\rho(\Gamma(k,\mathbf{m}^{\{k\}}))}{|\mathbf{m}^{\{k\}}|_{pr}}&=h,\\ \limsup_{\mathbf{m}^{\{k\}}\to\infty}\frac{\log\rho(\Gamma_{per}(k,\mathbf{m}^{\{k\}}))}{|\mathbf{m}^{\{k\}}|_{pr}}&=h_{per}.\end{aligned}\tag{4.2}$$

Proof. Since $\delta_{\mathbf{m}}$ is a subadditive we deduce (as in the proof of Lemma 3.2) that the limit of the left-hand side in the first equation of (4.1) exists. Theorem 2.8 yields that this limit is equal to $\log\rho(\Gamma(k,\mathbf{m}^{\{k\}}))$. Since $\delta_{\mathbf{m}}$ is a subadditive sequence it follows that $\log\rho(\Gamma(k,\mathbf{m}^{\{k\}}))$ is a subadditive sequence. Hence the limit of in the left-hand side of the first equation of (4.2) exists. Use the definition of h_{com} and Theorem 3.8 to deduce the first equality of (4.2). The subadditivity of $\log\rho(\Gamma(k,\mathbf{m}^{\{k\}}))$ yields the inequality $h\le\frac{\log\rho(\Gamma(k,\mathbf{m}^{\{k\}}))}{|\mathbf{m}^{\{k\}}|_{pr}}$.

Use Theorem 2.8 to deduce that $\log\rho(\Gamma_{per}(k,\mathbf{m}^{\{k\}}))$ is the density of periodic paths on the graph $\Gamma_{per}(k,\mathbf{m}^{\{k\}})$. Hence the second equalities of (4.1-4.2) follow. $\square$

(4.1-4.2) can be viewed as contraction of the first inequality given in Lemma 3.5 in the direction k. It is not a coincidence that we have only upper bounds for the mdc as in Lemmas 3.5 and 4.2 but we do not have lower bounds. The undecidability of a general multi-dimensional SOFT implies that we can not have lower bounds for general multi-dimensional SOFT. In order to have lower bounds for h one has to assume some conditions on the SOFT.

Recall that a graph $\Delta \subset <n> \times <n>$ is called *symmetric* if $(u,v) \in \Delta \iff (v,u) \in \Delta$. That is any walk on Δ is *reversible.* Many SOFT $\Gamma = (\Gamma_1, ..., \Gamma_d)$ in statistical mechanics are given by the nearest neighbor graph $\Delta = \Gamma_1 = ... = \Gamma_k$ which is symmetric. We first consider the case $d = 2$. In that case for $k \in <2>$ $\mathbf{m}^{\{k\}} \in \mathbb{N}$.

THEOREM 4.3. *Let $d = 2$ and assume that $\mathcal{S} \subset <n>^{\mathbb{N}^2}$ $(<n>^{\mathbb{Z}^2})$ is a SOFT given by $\Gamma = (\Gamma_1, \Gamma_2)$. Suppose that Γ_2 is symmetric. Then*

$$\frac{\log \rho(\Gamma(1, p+2q+1) - \log \rho(\Gamma(1, 2q+1))}{p} \leq h \leq \frac{\log \rho(\Gamma_{per}(1, 2m))}{2m}, \tag{4.3}$$

for any $m, p \in \mathbb{N}$ and $q \in \mathbb{Z}_+$. In particular, $\mathcal{S} \neq \emptyset$ if and only if the graph $\Gamma(1,2)$ contains a cycle. Finally

$$h_{per} = h. \tag{4.4}$$

Proof. We first prove the upper bound of (4.3). The symmetricity of Γ_2 yields the symmetricity of each $\Gamma(2,i)$ and $\Gamma_{per}(2,i)$ for any $i \in \mathbb{N}$. Hence the eigenvalues of the symmetric matrix $A(\Gamma(2,i))$ are real. As $A(\Gamma(2,i))$is nonnegative the Perron-Frobenius theorem yields $\rho(\Gamma(2,i))$ is the maximal eigenvalue of $A(\Gamma(2,i))$. Assume that $\ell = 1 + 2m,\ m \in \mathbb{N}$. Then all the eigenvalues of $A(\Gamma(2,i))^{2m}$ are nonnegative. Thus $\rho(\Gamma(2,i))^{2m} \leq \operatorname{tr} A(\Gamma(2,i))^{2m}$. Recall that $\operatorname{tr} A(\Gamma(2,i))^{2m}$ is the number of periodic paths on the graph $\Gamma(2,i)$ of length $\ell = 1 + 2m$. Observe next that any periodic path on $\Gamma(2,i)$ of length ℓ is a path on $\Gamma_{per}(1,2m)$ of length i. Thus

$$\frac{\log \rho(\Gamma(2,i))}{i} \leq \frac{1}{2m} \frac{\log \#\Gamma_{per}(1,2m)^{i-1}}{i}.$$

Let $i \to \infty$. The first equality of (4.2) implies that the left-hand side of the above inequality tends to h. Theorem 2.8 implies that the right-hand side of the above inequality tends to $\frac{1}{2m} \log \rho(\Gamma_{per}(1,2m))$. This proves the upper bound of (4.3).

We now prove the lower bound of (4.3). Let $i \in \mathbb{N},\ N = \#\Gamma_1^{i-1}$ and $\mathbf{x} \in \mathbb{R}^N$ be a nonzero row vector. Let $p \in \mathbb{N}$. As $\rho(\Gamma(2,i))^p$ is the maximal eigenvalue of the symmetric matrix $A(\Gamma(2,i))^p$ the maximal characterization $\rho(\Gamma(2,i))^p$ yields (see Appendix)

$$\rho(\Gamma(2,i))^p \geq \frac{\mathbf{x} A(\Gamma(2,i))^p \mathbf{x}^{\mathrm{T}}}{\mathbf{x}\mathbf{x}^{\mathrm{T}}}.$$

For $q \in \mathbb{Z}_+$ let $\mathbf{x} = \mathbf{1} A(\Gamma(2,i))^q$. Then

$$\rho(\Gamma(2,i))^p \geq \frac{\mathbf{1} A(\Gamma(2,i))^{p+2q} \mathbf{1}^{\mathrm{T}}}{\mathbf{1} A(\Gamma(2,i))^{2q} \mathbf{1}^{\mathrm{T}}}. \tag{4.5}$$

Observe next

$$\#\Gamma^{(i-1,\ell-1)} = \#\Gamma(1,\ell)^{i-1} = \#\Gamma(2,i)^{\ell-1} = \mathbf{1}A(\Gamma(2,i))^{\ell-1}\mathbf{1}^{\mathrm{T}}. \tag{4.6}$$

Hence (4.5) is equivalent to

$$\frac{\log \rho(\Gamma(2,i))}{i} \geq \frac{1}{p}\frac{\log \#\Gamma(1,p+2q+1)^{i-1} - \log \#\Gamma(1,2q+1)^{i-1}}{i}.$$

Let $i \to \infty$ and use (4.2) and Theorem 2.8 to deduce the left-hand side of (4.3).

Consider the graph $\Gamma(1,2)$. Let $M = \#\Gamma_2$. Suppose first that $\Gamma(1,2)$ does not have a cycle. Then $\Gamma(1,2)^M = \emptyset$. Hence $\mathcal{S} = \emptyset$. Assume now that $\Gamma(1,2)$ has a cycle of length L. This cycle is $(u_1,v_1),...,(u_L,v_L)$ where $(u_j,v_j) \in \Gamma_2$. Here $j = 1,...,L$ and $u_L = u_1,\ v_L = v_1$. We view this cycle as a configuration in $\Gamma^{(L-1,1)}$ where u_j and v_j are in the position $(1,j)$ and $(2,j)$ respectively Since Γ_2 is symmetric we deduce that $(u_j,v_j,u_j) \in \Gamma_2^2$. Then $(u_1,v_1,u_1),...,(u_L,v_L,u_L)$ is a cycle in $\Gamma(1,3)$. This cycle is a configuration in $\Gamma^{(L-1,2)}$ which is $(L-1,2)$ periodic. Hence this periodic configuration extends to a periodic state in $\mathcal{S}$, i.e. $\mathcal{S} \neq \emptyset$.

We now prove the equality $h_{per} = h$. The second part of (4.2) implies that for any SOFT given by $\Gamma = (\Gamma_1,\Gamma_2)$ we have the equality

$$\limsup_{i\to\infty} \frac{\delta_{(i,j),per}}{i} = \log \rho(\Gamma_{per}(1,j)).$$

The upper bound of (4.3) yields

$$\limsup_{i\to\infty} \frac{\delta_{(i,2m),per}}{2mi} \geq h.$$

Let $m \to \infty$ to deduce that $h_{per} \geq h$. Use the obvious inequality $h_{per} \leq h$ (Lemma 3.5) to deduce $h_{per} = h$. ☐

The lower bound of (4.3) for $q = 0$ is given in [Fr2]. The equality $h_{per} = h$ is due to the author [Fr2, Thm. 3.1]. Use the upper bound in Lemma 4.2 to deduce the inequality [Fr2, Cor.3.4]

$$\frac{\log \rho(\Gamma(1,p+1))}{p} - \frac{\log(\Gamma(1,1))}{p} \leq h \leq \frac{\log \rho(\Gamma(1,p+1))}{p+1}, \text{ for any } p \in \mathbb{N}. \tag{4.7}$$

(Actually, the upper bound in the above inequalities is valid for $p = 0$.) Taking in account that $\log \rho(\Gamma(1,p))$ is a subadditive sequence in $p \in \mathbb{N}$ (Lemma 4.2) and the equality $\Gamma(1,1) = \Gamma_1$ we obtain that the difference between the upper and the lower bound in (4.7) satisfies

$$\begin{aligned}&\frac{\log \rho(\Gamma(1,p+1))}{p+1} - \left(\frac{\log \rho(\Gamma(1,p+1))}{p} - \frac{\log \rho(\Gamma(1,1))}{p}\right)\\ &= -\frac{\log \rho(\Gamma(1,p+1))}{p(p+1)} + \frac{\log \rho(\Gamma_1)}{p+1} \leq \frac{\log \rho(\Gamma_1) - h}{p} \leq \frac{\log \rho(\Gamma_1)}{p}.\end{aligned} \tag{4.8}$$

Thus, if $\mathcal{S} \neq \emptyset$ then $\rho(\Gamma(1,2)) \geq 1$, $\rho(\Gamma_1) \geq 1$ and we deduce that the right-hand side of (4.8) converges (slowly) to 0 as $p \to \infty$.

COROLLARY 4.4. *Let $\mathcal{S} \subset < n >^{\mathbb{N}^2}$ $(< n >^{\mathbb{Z}^2})$ be a SOFT given by $\Gamma = (\Gamma_1, \Gamma_2)$. Assume that either Γ_1 or Γ_2 is symmetric then the mdc of $\mathcal{S}$ is computable.* Recall that $\Gamma_{per}(1,\ell)$ can be viewed as a subgraph of $\Gamma(1,\ell)$. Hence

$$\rho(\Gamma_{per}(1,\ell)) \leq \rho(\Gamma(1,\ell)) \quad \text{for any } \ell \in \mathbb{N}. \tag{4.9}$$

Hence for $p = 2m - 1$ the upper bound in (4.3) is better then the upper bound in (4.7). For $\Gamma_1 = \Gamma_2 = \Delta$, where Δ is a special symmetric graph, the upper bound in (4.3) is due to Calkin-Wilf [CaW]. It turns out that the best lower bounds in (4.3) are obtained when $p = 1$ and q is increasing. (This is the "opposite" to the lower bound in the inequality (4.7).) See [CaW, WeB, NaZ] and §6.

5. Estimates of mdc in terms of spectral radii *II*. Most of the results of this section address the case $d \geq 3$. We always have an upper bound $h \leq \frac{\log \rho(\Gamma(k,\mathbf{m}^{\{k\}}))}{|\mathbf{m}^{\{k\}}|_{pr}}$ given in Lemma 4.2. To have improve this upper bound as in (4.3) we need to assume a symmetricity condition on some of the graphs Γ_i, $i \in < d >$.

DEFINITION 5.1. *Let $\Gamma_1, ..., \Gamma_d \subset < n > \times < n >$ and let $\Gamma = (\Gamma_1, ..., \Gamma_d)$. Let $i \in < d >$ and $\tau \subset < d > \backslash \{i\}$ be a nonempty subset and let $\mathbf{m} = (m_1, ..., m_d) \in \mathbb{N}^d$. Then $\Gamma^{\mathbf{m}}_{\tau,per}, \Gamma_{\tau,per}(i, \mathbf{m}^{\{i\}})$ are the following subsets of $\Gamma^{\mathbf{m}}, \Gamma(i, \mathbf{m}^{\{i\}})$ respectively:*
(a) $w = (w_{\mathbf{j}})_{\mathbf{j} \in < \mathbf{m}+\mathbf{1} >} \in \Gamma^{\mathbf{m}}_{\tau,per}$ if $w \in \Gamma^{\mathbf{m}}$ and the configuration w is periodic in the directions given by τ. That is let $\mathbf{j} = (j_1, ..., j_d)$, $\mathbf{t} = (t_1, ..., t_d) \in < \mathbf{m} + \mathbf{1} >$. Then $w_{\mathbf{j}} = w_{\mathbf{t}}$ if $m_q | j_q - t_q$ for each $q \in \tau$.
(b) $(u, v) \in \Gamma_{\tau,per}(i, \mathbf{m}^{\{i\}})$ if $(u, v) \in \Gamma(i, \mathbf{m}^{\{i\}})$ and the configurations

$$u = (u_{\mathbf{j}^{\{i\}}})_{\mathbf{j}^{\{i\}} \in < \mathbf{m}^{\{i\}} >}, v = (v_{\mathbf{j}^{\{i\}}})_{\mathbf{j}^{\{i\}} \in < \mathbf{m}^{\{i\}} >} \in (\Gamma^{\{i\}})^{\mathbf{m}^{\{i\}} - \mathbf{1}^{\{i\}}}$$

can be extended to $\Gamma^{\{i\}}$ allowable configurations, which are periodic in the directions given by τ. That is, let

$$\mathbf{p}^{\{i\}} = (\mathbf{m} + \sum_{q \in \tau} \mathbf{e}_q)^{\{i\}}.$$

Let $\hat{u} = (u_{\mathbf{j}^{\{i\}}})_{\mathbf{j}^{\{i\}} \in < \mathbf{p}^{\{i\}} >}$, $\hat{v} = (v_{\mathbf{j}^{\{i\}}})_{\mathbf{j}^{\{i\}} \in < \mathbf{p}^{\{i\}} >}$ where $u_{\mathbf{j}^{\{i\}}} = u_{\mathbf{t}^{\{i\}}}$, $v_{\mathbf{j}^{\{i\}}} = v_{\mathbf{t}^{\{i\}}}$ for $\mathbf{j}^{\{i\}}, \mathbf{t}^{\{i\}} \in < \mathbf{p}^{\{i\}} >$ if $m_q | j_q - t_q$ for each $q \in \tau$. Then $\hat{u}, \hat{v} \in (\Gamma^{\{i\}})^{< \mathbf{p}^{\{i\}} - \mathbf{1}^{\{i\}} >}$.

THEOREM 5.2. *Let $d \geq 2$ and assume that $\mathcal{S} \subset < n >^{\mathbb{N}^d}$ $(< n >^{\mathbb{Z}^d})$ be a SOFT given by $\Gamma = (\Gamma_1, ..., \Gamma_d)$. Let τ be a strict nonempty subset of $< d >$. Assume that Γ_j is symmetric for each $j \in \tau$. Let $i \in < d > \backslash \tau$, $\mathbf{m} = (m_1, ..., m_d) \in \mathbb{N}^d$. Assume that for each $j \in \tau$ m_j is even. Then*

$$h \leq \frac{\log \rho(\Gamma_{\tau,per}(i, \mathbf{m}^{\{i\}}))}{|\mathbf{m}^{\{i\}}|_{pr}}. \tag{5.1}$$

Proof. We prove the theorem by induction on d. For $d = 2$ (5.1) is equivalent to the upper bound of (4.3). Let $d \geq 3$ and assume that the theorem holds for $d-1$. Let $p \in \tau$. Then $\Gamma(p, \mathbf{m}^{\{p\}})$ is symmetric. Recall that

$$
\begin{aligned}
h &\leq \frac{\log \rho(\Gamma(p, \mathbf{m}^{\{p\}}))}{|\mathbf{m}^{\{p\}}|_{pr}} \\
&= \frac{\log \rho(\Gamma(p, \mathbf{m}^{\{p\}}))^{m_p}}{|\mathbf{m}|_{pr}} = \frac{\log \rho(A(\Gamma(p, \mathbf{m}^{\{p\}}))^{m_p})}{|\mathbf{m}|_{pr}}.
\end{aligned} \tag{5.2}
$$

Assume that m_p is even. As in the proof of Theorem 4.3 it follows that

$$
\begin{aligned}
\rho(\Gamma(p, \mathbf{m}^{\{p\}}))^{m_p} &\leq \operatorname{tr} A(\Gamma(p, \mathbf{m}^{\{p\}}))^{m_p}, \\
\operatorname{tr} A(\Gamma(p, \mathbf{m}^{\{p\}}))^{m_p} &= \#\Gamma_{\{p\},per}^{\mathbf{m}+\mathbf{e}_p-\mathbf{1}}.
\end{aligned} \tag{5.3}
$$

Assume first that $\tau = \{p\}$. Fix $\mathbf{m}^{\{i\}}$ and let m_i vary in $\mathbb{N}$. Then $\Gamma_{\{p\},per}^{\mathbf{m}+\mathbf{e}_p-\mathbf{1}}$ is a path of length m_i on the graph $\Gamma_{\{p\},per}(i, \mathbf{m}^{\{i\}})$ of length m_i. Hence

$$
\lim_{m_i \to \infty} \frac{\log \#\Gamma_{\{p\},per}^{\mathbf{m}+\mathbf{e}_p-\mathbf{1}}}{m_i} = \log \rho(\Gamma_{\{p\},per}(i, \mathbf{m}^{\{i\}})).
$$

Combine the above inequalities to deduce (5.1) for $\tau =< p >$.

Assume now that $p \in \tau$ and $\tau \backslash \{p\} \neq \emptyset$. Fix an even m_p and consider $\Gamma_{\{p\},per}^{\mathbf{m}+\mathbf{e}_p-\mathbf{1}}$ for all values of $\mathbf{m}^{\{p\}} \in \mathbb{N}^{d-1}$. These configurations gives rise to the following SOFT $\hat{\mathcal{S}} \subset < N >^{d-1}$ $(< N >^{\mathbb{Z}^{d-1}})$ given by $\hat{\Gamma} = (\hat{\Gamma}_1, ..., \hat{\Gamma}_{p-1}, \hat{\Gamma}_{p+1}, ..., \hat{\Gamma}_d)$:

(a) $N = \#(\Gamma_p)_{per}^{m_p}$. That is $q \in < N >$ is represented by a periodic path $u = (u_j)_1^{m_p+1}$ of length m_p in Γ_p.

(b) For $\ell \in < d > \backslash \{p\}$ $\hat{\Gamma}_\ell \subset < N > \times < N >$ is the following subgraph. $(u, v) \in \hat{\Gamma}_\ell,\ u = (u_j)_1^{m_p+1}, v = (v_j)_1^{m_p+1} \in (\Gamma_p)_{per}^{m_p}$ if and only if $(u_j, v_j) \in \Gamma_\ell$ for $j = 1, ..., m_p + 1$.

Let $\hat{h}$ be the entropy of $\hat{\mathcal{S}}$:

$$
\hat{h} = \lim_{\mathbf{m}^{\{p\}} \to \infty} \frac{\log \#\Gamma_{\{p\},per}^{\mathbf{m}+\mathbf{e}_p-\mathbf{1}}}{|\mathbf{m}^{\{p\}}|_{pr}} = \lim_{m_j \to \infty, j \in <d> \backslash \{i,p\}} \frac{\log \rho(\Gamma_{\{p\},per}(i, \mathbf{m}^{\{i\}}))}{\prod_{j \in <d> \backslash \{i,p\}} m_j}.
$$

Use (5.2-5.3) to deduce $h \leq \frac{\hat{h}}{m_p}$. Since Γ_q is symmetric for each $q \in \tau$ it follows that $\hat{\Gamma}_q$ is symmetric for each $q \in \tau' := \tau \backslash \{p\}$. Assume that each m_q is even for $q \in \tau'$. Then the induction hypothesis yields

$$
\hat{h} \leq \frac{\log \rho(\hat{\Gamma}_{\tau',per}(i, \mathbf{m}^{\{i,p\}}))}{|\mathbf{m}^{\{i,p\}}|_{pr}}.
$$

Hence

$$h \le \frac{\hat{h}}{m_p} \le \frac{\log \rho(\hat{\Gamma}_{\tau',per}(i, \mathbf{m}^{\{i,p\}}))}{|\mathbf{m}^{\{i\}}|_{pr}}.$$

Observe finally that $\Gamma_{\tau',per}(i, \mathbf{m}^{\{i,p\}})$ is isomorphic to $\Gamma_{\tau,per}(i, \mathbf{m}^{\{i\}})$. Then the above inequality implies (5.1). □

The arguments of the proof of (4.4) yield:

COROLLARY 5.3. *[Fr2, Thm 3.1] Let* $d \ge 3$ *and* $i \in < d >$. *Let* $\mathcal{S} \subset < n >^{\mathbb{N}^d}$ $(< n >^{\mathbb{Z}^d})$ *be a SOFT given by* $\Gamma = (\Gamma_1, ..., \Gamma_d)$. *Assume that* Γ_j *is symmetric for each* $j \in < d > \backslash\{i\}$. *Then* $h_{per} = h$.

To give lower bounds for the entropy of SOFT $\mathcal{S} \subset < n >^{\mathbb{N}^d}$ $(< n >^{\mathbb{Z}^d})$ for $d \ge 3$ as in Theorem 4.3 we need to define $\tilde{\mathcal{S}} \subset < n >^{\mathbb{N}^{d-1}}$ $(< n >^{\mathbb{Z}^{d-1}})$ similar to $\hat{\mathcal{S}}$ defined in the proof of Theorem 5.2.

DEFINITION 5.4. *Let* $d \ge 3$ *and* $\Gamma_1, ..., \Gamma_d \subset < n > \times < n >$. *Fix* $i \in < d >$ *and* $q \in \mathbb{N}$. *Let* $\tilde{\mathcal{S}} \subset < N >^{\mathbb{N}^{d-1}}$ $(< N >^{\mathbb{Z}^{d-1}})$ *given by* $\Gamma^{\{i\},q} = (\Gamma_1^{\{i\},q}, ..., \Gamma_{i-1}^{\{i\},q}, \Gamma_{i+1}^{\{i\},q}, ..., \Gamma_d^{\{i\},q})$ *which are defined as follows:*
(a) $N = \#\Gamma_i^{q-1}$. *That is* $p \in < N >$ *is represented by a path* $u = (u_j)_1^q$ *of length* $q-1$ *in* Γ_i.
(b) For $\ell \in < d > \backslash\{i\}$ $\Gamma_\ell^{\{i\},q} \subset < N > \times < N >$ *is the following subgraph.* $(u, v) \in \Gamma_\ell^{\{i\},q}$, $u = (u_j)_1^q, v = (v_j)_1^q \in \Gamma_i^{q-1}$ *if and only if* $(u_j, v_j) \in \Gamma_\ell$ *for* $j = 1, ..., q$.

Let $h^{\{i\},q}$ *be the entropy of* $\tilde{\mathcal{S}}$. *For* $q = 1$ *let* $h^{\{i\}} = h^{\{i\},1}$. *(Note that* $\Gamma^{\{i\},1} = \Gamma^{\{i\}}$.*)*

THEOREM 5.5. *Let* $d \ge 3$ *and assume that* $\mathcal{S} \subset < n >^{\mathbb{N}^d}$ $(< n >^{\mathbb{Z}^d})$ *be a SOFT given by* $\Gamma = (\Gamma_1, ..., \Gamma_d)$. *Let* $i \in < d >$ *and* $\mathbf{m} = (m_1, ..., m_d) \in \mathbb{N}^d$. *Then for each* $j \in < d > \backslash\{i\}$ *the limit* $\lim_{m_i \to \infty} \frac{\log \rho(\Gamma(j, \mathbf{m}^{\{j\}})}{m_i}$ *exists an is denoted by* $\log \rho_{\{i\}}(j, \mathbf{m}^{(i,j\}})$. $(\rho_{\{i\}}(j, \mathbf{m}^{(i,j\}}) \ge 0$, *and* $\rho_{\{i\}}(j, \mathbf{m}^{(i,j\}}) = 0 \Rightarrow \mathcal{S} = \emptyset$.*) Assume that* Γ_j *is symmetric for* $j \in < d > \backslash\{i\}$. *Then for any* $p \in \mathbb{N}$ *and* $q \in \mathbb{Z}_+$

$$\begin{aligned} \log \rho(\Gamma(i, (\mathbf{m}^{\{i,j\}}, p + 2q + 1))) \\ \le \log \rho(\Gamma(i, (\mathbf{m}^{\{i,j\}}, 2q + 1)) + p \log \rho_{\{i\}}(j, \mathbf{m}^{\{i,j\}}). \end{aligned} \tag{5.4}$$

In particular

$$\frac{h^{\{j\},p+2q+1} - h^{\{j\},2q+1}}{p} \le h \le \frac{h^{\{j\},p+2q+1}}{p + 2q + 1}. \tag{5.5}$$

Suppose furthermore that Γ_j *is symmetric for each* $j \in < d > \backslash\{i\}$. *Then* h *is computable.*

Proof. Recall that $\log \rho(\Gamma(j, \mathbf{m}^{\{j\}}))$ is subadditive sequence in each coordinate of $\mathbf{m}^{\{j\}}$. Hence $\lim_{m_i \to \infty} \frac{\log \rho(\Gamma(j, \mathbf{m}^{\{j\}})}{m_i} = \log \rho_{\{i\}}(j, \mathbf{m}^{(i,j\}})$. If

$\mathcal{S} \neq \emptyset$ then $\log \rho(\Gamma(j, \mathbf{m}^{\{j\}}) \geq 0$ and $\rho_{\{i\}}(j, \mathbf{m}^{(i,j\}}) \geq 1$. If $\rho(\Gamma(j, \mathbf{m}^{\{j\}}) = 0$ for some $\mathbf{m}^{\{j\}}$ we deduce that $\mathcal{S} = \emptyset$. Thus $\rho_{\{i\}}(j, \mathbf{m}^{(i,j\}}) = 0 \Rightarrow \mathcal{S} = \emptyset$.

Assume that Γ_j is symmetric. Then the graph $\Gamma(j, \mathbf{m}^{\{j\}})$ is symmetric. Hence $\rho(\Gamma(j, \mathbf{m}^{\{j\}})^p$ is the maximal eigenvalue of $A(\Gamma(j, \mathbf{m}^{\{j\}}))^p$. The maximal characterization of $\rho(\Gamma(j, \mathbf{m}^{\{j\}}))^p$ yields

$$\rho(\Gamma(j, \mathbf{m}^{\{j\}}))^p \geq \frac{\mathbf{x}A(\Gamma(j, \mathbf{m}^{\{j\}}))^p \mathbf{x}^{\mathrm{T}}}{\mathbf{x}\mathbf{x}^{\mathrm{T}}}$$

for any $\mathbf{x} \neq 0$. Let $\mathbf{x} = \mathbf{1}A(\Gamma(j, \mathbf{m}^{\{j\}})^q$. Recall that

$$\mathbf{1}A(\Gamma(j, (\mathbf{m}^{\{j\}}))^{\ell}\mathbf{1}^{\mathrm{T}} = \#\Gamma^{\mathbf{r}},$$
$$\mathbf{r} = (m_1 - 1, ..., m_{j-1} - 1, \ell, m_{j+1} - 1, ..., m_d - 1),$$

For any $\ell \in \mathbb{Z}_+$. Hence the above inequality is equivalent to

$$p\frac{\log \rho(\Gamma(j, \mathbf{m}^{\{j\}}))}{m_i} + \log \frac{\#\Gamma^{(\mathbf{m}^{\{j\}}, 2q+1) - \mathbf{1}}}{m_i} \geq \log \frac{\#\Gamma^{(\mathbf{m}^{\{j\}}, p+2q+1) - \mathbf{1}}}{m_i},$$

where $(\mathbf{m}^{\{j\}}, \ell) := (m_1, ..., m_{j-1}, \ell, m_{j+1}, ..., m_d)$. Let $m_i \to \infty$ in the above inequality to obtain (5.4). Divide (5.4) by $|\mathbf{m}^{\{i,j\}}|_{pr}$ and let $m_q \to \infty$ for each $q \in< d > \backslash\{i, j\}$ to obtain $h^{\{j\}, p+2q+1} \leq h^{\{j\}, 2q+1} + ph$. This proves establishes the left-hand side of (5.5). Recall that

$$h \leq \frac{\log \rho(\Gamma(j, (\mathbf{m}^{\{i,j\}}, p + 2q + 1))}{|\mathbf{m}^{\{i,j\}}|_{pr}(p + 2q + 1)}.$$

Let $m_q \to \infty$ for each $q \in< d > \backslash\{i, j\}$ to deduce the right-hand side of (5.5). Observe that for $q = 0$ in (5.5) the difference between lower and the upper bound is bounded above by $\frac{h^{\{j\}}}{p}$. (Note that if $h^{\{j\}} = -\infty$ then $\mathcal{S} = \emptyset$.)

We now show by induction on d that if $d-1$ graphs out of the d graphs $\Gamma_1, ..., \Gamma_d$ are symmetric then h is computable. For $d = 2$ the computability of h follows from Theorem 4.3. Assume that $d \geq 3$ and suppose that h is computable for $d-1$ dimensional SOFT if $d-2$ graphs of $d-1$ given graphs are symmetric. Assume that each Γ_j is symmetric for $j \in< d > \backslash\{i\}$. The arguments of the proof of Theorem 4.3 yield that $\mathcal{S} \neq \emptyset$ if and only if $\Gamma(i, (2, ..., 2))$ has a cycle. Assume that $\Gamma(i, (2, ..., 2))$ has a cycle. Consider the lower and the upper estimates given in (5.5) for $q = 0$. For each $p \in \mathbb{N}$ the $d-1$ subshift given by $\Gamma^{\{j\},p} = (\Gamma_1^{\{j\},p}, ..., \Gamma_{j-1}^{\{j\},p}, \Gamma_{j+1}^{\{j\},p}, ..., \Gamma_d^{\{j\},p})$ has at least $d - 2$ symmetric graphs. Hence the entropy of each $h^{\{j\},p}$ is computable by the induction hypothesis. The difference between lower and upper bounds in (5.5) is bounded above by $\frac{h^{\{i\}}}{p}$. By the induction hypothesis $h^{\{i\}}$ is computable. Hence h is computable. □

Similar lower bounds for h are given in [Fr2, §3].

6. Examples. EXAMPLE 6.1. *The residual entropy of square ice can be described as the entropy of the following $\mathbb{Z}^2$-SOFT: The coloring of the $\mathbb{Z}^2$ lattice in three colors, where no two identical colors are adjacent [Lie, p'169]. The admissible colorings correspond to a SOFT $\mathcal{S} \subset < n >^{\mathbb{Z}^2}$, where $n = 3$ and $\Gamma_1 = \Gamma_2 = \Delta$ is a complete graph on 3 vertices without self loops:*

$$\Delta = < 3 > \times < 3 > \backslash\{(1,1),(2,2),(3,3)\} \tag{6.1}$$

In a brilliant paper Lieb [Lie] computed the periodic entropy $\mathcal{S}$.

$$h_{per} = \frac{3}{2}\log\frac{4}{3} = 0.43152... \tag{6.2}$$

It was shown in [BKW] that for this $\mathcal{S}$ $h_{per} = h_{com}$. Note that Δ is a symmetric graph. Hence the equality $h_{per} = h (= h_{com})$ follows from Theorem 4.3. There are just a few cases of SOFT $\mathcal{S}$ in statistical mechanics in which the exact value of the entropy of $\mathcal{S}$ is known. Note that h is not a logarithm of an algebraic integer, as in the case of the entropy of one dimensional SOFT (Theorem 2.8). (h is a logarithm of $u = (\frac{4}{3})^{\frac{3}{2}}$, which is an algebraic number: $27u^2 = 64$.)

We computed several spectral radii which give upper and lower bounds of h using Lemma 4.2 and (4.3).

$$\rho(\Delta) = \rho(\Gamma(1,1)) = \rho(\Gamma_{per}(1,3)) = 2, \qquad \rho(\Gamma(1,2)) = \rho(\Gamma_{per}(1,2)) = 3,$$
$$\rho(\Gamma(1,3)) = 4.561552813, \qquad \rho(\Gamma_{per}(1,4)) = 6.372281326.$$

Note that $\#\Delta^{k-1} = 3\cdot 2^{k-1}$ for any $k \in \mathbb{N}$. That is $\Gamma(1,k)$ has $3\cdot 2^{k-1}$ vertices. A straightforward argument shows

$$\#\Delta^k_{per} = \#\Delta^{k-1} - \#\Delta^{k-1}_{per} \ \text{ for } k \geq 2, \quad \#\Delta_{per} = \emptyset.$$

Hence $\Gamma_{per}(1,k)$ is a graph with the number of vertices v_k, which we denote by (k, v_k):

$$(1,0), \quad (2,6), \quad (3,6), \quad (4,18), \quad (5,30), \quad (6,66).$$

The lower bound of (4.3) for $p = 1,\ q = 0$ yields

$$h \geq \log\frac{3}{2} = 0.405465108.$$

The estimate $h \approx \log\frac{3}{2}$ goes back to Pauling [Pau]. For $p = 2,\ q = 0$ the lower bound of (4.3) yields

$$h \geq 0.4122579570.$$

For $p = 0, 1, 2$ the upper bounds of h in (4.7) are

$$0.693147181 > 0.549306145 > 0.505887698 \geq h.$$

The upper bounds of h in (4.3) for $m = 1, 2$ are

$$0.549306145 > 0.462989385 \geq h.$$

Note that

$$\frac{\log \rho(\Gamma_{per}(1,3))}{3} = \frac{\log 2}{3} = .0231049060 < h.$$

That is $\frac{\log \rho(\Gamma_{per}(1,k))}{k}$ does not have to be a an upper bound for h for an odd $k > 1$ (contrary to the upper bound in (4.3)).

Physics literature has many asymptotic expansions which estimate the values of the SOFT related to some models in statistical mechanics. (As we pointed out before physicists refer to e^h as the entropy of $\mathcal{S}$. We adjusted the results of Lieb to our notation.) We already pointed out the Pauling estimate $h \approx \log \frac{3}{2}$ for the residual entropy of square ice. A remarkable estimate using the asymptotic expansions is due to Nagle [Na1] $h = 0.432 \pm 0.001$.

EXAMPLE 6.2. $(0,1)$ *run length limited channel is* $\mathbb{Z}^d$*-SOFT can be described by the graphs* $\Gamma_1 = ... = \Gamma_d = \Delta \subset < 2 > \times < 2 >$*, where* $\Delta = \{(1,2),(2,1),(2,2)\}$*. (In our notation* $2 \equiv 0$*.) That is each point of the lattice is filled out by* 0 *and* 1 *and no two* 1*'s are neighbors. Let* h_d *be the entropy of this* $\mathbb{Z}^d$*-SOFT.* In information theory the logarithms are on base 2. Let $\overline{h}_d$ be the entropy with respect to the basis 2. Note

$$\overline{h}_d = \frac{h_d}{\log 2}.$$

Theorem 2.8 yields

$$\overline{h}_1 = \log_2 \rho(\Delta) = \log_2 \frac{1+\sqrt{5}}{2} = 0.694241914.$$

The exact value of $\overline{h}_2$ is not known. In fact the aim of the paper [CaW] was to find good estimates of $\overline{h}_2$. As we pointed above the inequality (4.3) was proven in [CaW] for this particular case. The bounds in [CaW] were calculated with greater precision in [WeB]. In [FoJ] $\overline{h}_2$ was estimated up to 8 digits. In [NaZ] $\overline{h}_2$ is estimated up to 9 digits:

$$0.587891161775 \leq \overline{h}_2 \leq 0.587891161868.$$

In [NaZ] $\overline{h}_3$ is estimated to two digits using the inequalities given in the previous section:

$$0.5225017411838 \leq \overline{h}_3 \leq 0.526880847825.$$

The improvement of these estimates by the methods of §5 needs better computer ability than available now. Indeed, the upper bound given for $\overline{h}_3$

involves the computation of the spectral radius with more that 40 million elements.

The above two examples follow the following pattern:

DEFINITION 6.3. *Let $\Delta \subset < n > \times < n >$ be a given graph. For $d \in \mathbb{N}$ denote by $\mathcal{S}_d(\Delta) \subset < n >^{\mathbb{Z}^d}$ $(< n >^{\mathbb{N}^d})$ the SOFT given by $\Gamma_1 = ... = \Gamma_d = \Delta$. Let $h_d(\Delta)$ be the entropy of $\mathcal{S}_d(\Delta)$.* Many $\mathbb{Z}^d$-SOFT in statistical mechanics are of the form $\mathcal{S}_d(\Delta)$ where Δ is a symmetric graph.

PROPOSITION 6.4. *Let $\Delta \subset < n > \times < n >$ and consider $\mathcal{S}_d(\Delta)$ for $d \in \mathbb{N}$ as defined above. Then the entropy sequence $\{h_d(\Delta)\}$ is a decreasing sequence which converges to $h_\infty(\Delta)$.*

Proof. Let $\Gamma(d) = (\underbrace{\Delta, ..., \Delta}_{d})$. Let $\mathbf{m} = (m_1, ..., m_{d-1}, 1)$ Then $\Gamma(d)^{\mathbf{m}-\mathbf{1}}$ is isomorphic to $\Gamma(d-1)^{\mathbf{m}^{\{d\}}-\mathbf{1}^{\{d\}}}$. Hence

$$\lim_{m_j \to \infty,\, j \in <d> \setminus \{d\}} \frac{\log \# \Gamma(d)^{\mathbf{m}-\mathbf{1}}}{|\mathbf{m}|_{pr}} = h_{d-1}(\Delta).$$

Use the last inequality of Lemma 3.5 to deduce that $\frac{\log \# \Gamma(d)^{\mathbf{m}-\mathbf{1}}}{|\mathbf{m}|_{pr}} \geq h_d(\Delta)$. Hence $h_{d-1}(\Delta) \geq h_d(\Delta)$. □

It is of interest to find $h_\infty(\Delta)$ for the two graphs Δ discussed in the above two examples. Our last example shows that there are $\mathbb{Z}^d$-SOFT in statistical mechanics which are not of the form $\mathcal{S}_d(\Delta)$ and they do not have an obvious symmetricity.

EXAMPLE 6.5. *Let $d \in \mathbb{N}$. A state θ is partitioning $\mathbb{Z}^d$ to dimers (dominoes) which is obtained by joining some adjacent lattice points together to form a dimer, such that every element in $\mathbb{Z}^d$ is covered by one dimer exactly. (Equivalently, a state θ is a 1-factor of the graph on $\mathbb{Z}^d$ where the vertices $u, v \in \mathbb{Z}^d$ are joint by an undirected edge (u, v) if $d_h(u, v) = 1$.)*

PROPOSITION 6.6. *Let $d \in \mathbb{N}$. Then the set of dimer partitions of $\mathbb{Z}^d$ is a $\mathbb{Z}^d$-SOFT $\mathcal{S}_d \subset < 2d >^{\mathbb{Z}^d}$ given by $\Gamma(d) = (\Gamma_1, ..., \Gamma_d)$, where each $\Gamma_i \subset < 2d > \times < 2d >$ is given as follows:*
(a) $(i, i+d) \in \Gamma_i$ and $(i, j) \notin \Gamma_i$ for $j \neq i+d$.
(b) For $k \neq i$ $(k, i+d) \notin \Gamma_i$.
(c) For $k \neq i$, $j \neq i+d$ $(k, j) \in \Gamma_i$.

Proof. A dimer in direction of $\mathbf{e}_i$ is viewed as an edge $(i, i+d)$, where i is the "left" part of the dimer and $i+d$ is the "right" part of the dimer. That is if the two positions of the dimer in direction $\mathbf{e}_i$ is $\mathbf{p} \in \mathbb{Z}^d$ and $\mathbf{p} + \mathbf{e}_i$ then the position $\mathbf{p}$ is colored by i and the position $\mathbf{p} + \mathbf{e}_i$ is colored by $i+d$. Thus any partition θ of $\mathbb{Z}^d$ to dimers induces a unique coloring $\mathbb{Z}^d$ to $2d$ colors. Vice versa any $\phi \in \Gamma^{\mathbb{Z}^d}$ induces a unique dimer partition of $\mathbb{Z}^d$. □

Note that Γ_i is not symmetric. However, the matrix $A(\Gamma_i)$ is a diagonable rank two matrix with the two nonzero eigenvalues d–$1 \pm \sqrt{(d-1)^2+1}$.

PROPOSITION 6.7. *Let $d \in \mathbb{N}$ and let h_d be the entropy of the d-dimensional dimer SOFT given by $\mathcal{S}_d$. Then h_d is an increasing sequence.*

Proof. Consider all dimer partitions of $\mathbb{Z}^d$ in the directions $\mathbf{e}_1, \ldots, \mathbf{e}_{d-1}$. These dimer partitions give rise to a SOFT $\tilde{\mathcal{S}}_d \subset \mathcal{S}_d$. Hence the entropy of $\tilde{\mathcal{S}}_d$ is not more then h_d. Clearly, the entropy of $\tilde{\mathcal{S}}_d$ is h_{d-1}, i.e. $h_{d-1} \le h_d$. □

Note that $\mathcal{S}_1$ has only two states. Hence $h_1 = 0$. The dimer problem, i.e. finding the value h_d for $d > 1$ can be traced to 1930's. Fowler and Rushbrooke estimate $h_2 \approx 0.29$ and $h_3 \approx 0.43$. Furthermore they prove that $h_d \le \frac{1}{2} \log d$ [FoR]. Fisher, Kasteleyn and Tempreley (each working independently of the other two) found [Fis] and [Kas]

$$h_2 = \frac{1}{\pi} \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)^2} = 0.29156090... \tag{6.3}$$

Until today no exact formula is known for h_3. Hammersley [Ha2] was the first one to relate the computation of any h_d to permanents. (A permanent of a square matrix $A = (a_{ij})_1^n$, denoted by perm A, is the sum of all products $(n!)$ appearing in determinant of A but all the signs are chosen to be +.) Let $\mathbf{m} = (m_1, ..., m_d) \in \mathbb{N}^d$ and view $< \mathbf{m} >$ as a box in $\mathbb{Z}^d$ of volume $|\mathbf{m}|_{pr}$. Suppose that $|\mathbf{m}|_{pr}$ is even. Let $P(\mathbf{m})$ be the number of partitions on all points in $< \mathbf{m} >$ to dimers. (That is consider the adjacency graph on vertices in $< \mathbf{m} >$ and let $P(\mathbf{m})$ be the number of 1-factors of the graph.) It was shown by Hammersley [Ha1] that

$$\limsup_{\mathbf{m} \to \infty} \frac{\log P(\mathbf{m})}{|\mathbf{m}|_{pr}} = h_d.$$

(In our notation $\#\Gamma(d)^{\mathbf{m}-\mathbf{1}} \ge P(\mathbf{m})$, and if $|\mathbf{m}|_{pr}$ is even and each m_i is big then $P(\mathbf{m})$ is close to $\#\Gamma(d)^{\mathbf{m}-\mathbf{1}}$.) Since $|\mathbf{m}|_{pr}$ is even the adjacency graph on $< \mathbf{m} >$ is bipartite. This bipartite graph can be presented by $0-1$ matrix $A(\mathbf{m})$ of dimension $\frac{|\mathbf{m}|_{pr}}{2}$. Then $P(\mathbf{m}) = \text{perm A}$. Let B be a matrix which obtained from $A(\mathbf{m})$ by changing some entries of $A(\mathbf{m})$ equal to 1 to -1. Then $|\text{det B}| \le \text{perA}$, where det B is the determinant of B. Following the ideas of Fisher [Fis] Hammersley chooses a special matrix B, whose determinant for $d = 2$ gives perm A. Replacing perm A by |det B| Hammersley obtains the lower bound $h_3 \ge 0.418347$. Next Hammersley relates h_d to the famous van der Waerden permanent conjecture as follows. Let $\mathbf{m} = (m_1, ..., m_d) \in \mathbb{N}^d$ where each m_i ie even. Then $\Gamma(d)^{\mathbf{m}}_{per}$ is the set of all partitions of the torus $\mathbf{T}^{\mathbf{m}} := \mathbb{Z}^d/(m_1\mathbb{Z} \times ... \times m_d\mathbb{Z})$ to dimers. The number of these partitions is equal to perm $\mathrm{A}_{\mathrm{per}}(\mathbf{m})$, where $A_{per}(\mathbf{m})$ is the adjacency matrix of the bipartite graph corresponding the torus $\mathbf{T}^{\mathbf{m}}$. (The dimension of $A_{per}(\mathbf{m})$ is $\frac{|\mathbf{m}|_{pr}}{2}$). Note that each row and column of $A_{per}(\mathbf{m})$ has $2d$ ones. Hence $\frac{1}{2d} A_{per}(\mathbf{m})$ is a doubly stochastic matrix. The van der

Waerden permanent conjecture proved by Egorichev [Ego] and Falikman [Fan] implies that

$$\text{perm C} \geq \frac{\text{k!}}{\text{k}^{\text{k}}} > \text{e}^{-\text{k}}, \quad \text{for any k} \times \text{k doubly stochastic C.} \tag{6.4}$$

This weaker form of the van der Waerden conjecture perm C $\geq \mathrm{e}^{-\mathrm{k}}$ was proved by the author in [Fr1]. Hence

$$\frac{\log \text{perm } \mathrm{A}_{\text{per}}(\mathbf{m})}{|\mathbf{m}|_{\text{pr}}} \geq \frac{\frac{|\mathbf{m}|_{pr}}{2}(\log(2d)-1)}{|\mathbf{m}|_{pr}} = \frac{1}{2}\log(2d) - \frac{1}{2}.$$

The definition of h_d and [Ha1] yield that

$$h_d = lim_{\mathbf{m}\to\infty} \frac{\log \text{perm } \mathrm{A}_{\text{per}}(\mathbf{m})}{|\mathbf{m}|_{\text{pr}}}.$$

Hence

$$h_d \geq \frac{1}{2}\log(2d) - \frac{1}{2}. \tag{6.5}$$

Note that for $d = 3$ this bound gives $h_3 \geq 0.3958797354$ which is worth then Hammersley's lower bound for h_3. However for $d \geq 4$ the lower estimate (6.5) is better the lower estimates in [Ha2]. To obtain improve the upper bound $h_d \leq \frac{1}{2}\log d$ given in [FoR] one needs an upper estimate the permanent of $0-1$ matrix $A_{per}(\mathbf{m})$. An upper bound for a permanent of $0-1$ matrix was conjectured by Minc and proved by Bregman [Bre]. From this bound it follows

$$\text{perm } \mathrm{A}_{\text{per}}(\mathbf{m}) \leq ((2\mathrm{d})!)^{\frac{|\mathbf{m}|_{\text{pr}}}{4\mathrm{d}}}.$$

Hence

$$h_d \leq \frac{1}{4d}\log(2d)!. \tag{6.6}$$

(See [Min] for details.) Use Stirling formula for $(2d)!$ for large d to deduce

COROLLARY 6.8. *For $d >> 1$*

$$h_d = \frac{1}{2}\log(2d) - \frac{1}{2} + o(1) \ \left(= \frac{1}{2}\log(2d) - \frac{1}{2} + O\left(\frac{\log d}{d}\right)\right).$$

Recently there were significant improvements of the above results. First, a lower bound on permanents of $0-1$ matrices containing in each row and column k ones due to Schrijver [Sch] implies

$$\text{perm } \mathrm{A}_{\text{per}}(\mathbf{m}) \geq \left(\frac{(2\mathrm{d}-1)^{2\mathrm{d}-1}}{(2\mathrm{d})^{2\mathrm{d}-2}}\right)^{\frac{|\mathbf{m}|_{\text{pr}}}{2}}. \tag{6.7}$$

Hence

$$h_d \geq \frac{1}{2}((2d-1)\log(2d-1) - (2d-2)\log(2d)). \tag{6.8}$$

In particular

$$h_3 > 0.440075842.$$

Ciucu showed [Ciu] (without using permanents) that

$$h_3 \leq 0.463107.$$

Finally we mention two heuristic results. The first one is an old result of Nagle [Na2] $h_3 = 0.44645 \pm 0.00005$ which uses asymptotic expansions. The second recent estimate $h_3 = 0.4466 \pm 0.0006$ is due to Beichl-Sullivan [BeS] is based on special Monte Carlo methods to estimate permanents .

7. Subadditive functions on $\mathbb{Z}_+^d$-SOFT. In what follows we assume that $\mathcal{S} \subset < n >^{\mathbb{N}^d}$ is a nonempty SOFT given by $\Gamma = (\Gamma_1, ..., \Gamma_d)$. For each $\mathbf{m} \in \mathbb{N}^d$ let

$$\begin{aligned} C((a_{\mathbf{i}})_{\mathbf{i} \in <\mathbf{m}>}) &:= \{(b_{\mathbf{i}})_{\mathbf{i} \in \mathbb{N}^d} \in \mathcal{S} : \quad b_{\mathbf{i}} = a_{\mathbf{i}} \text{ for } \mathbf{i} \in < \mathbf{m} >\}, \\ \mathcal{C}(\mathbf{m}) &:= \{(a_{\mathbf{i}})_{\mathbf{i} \in <\mathbf{m}>} \in < n >^{<\mathbf{m}>} : \quad \emptyset \neq C((a_{\mathbf{i}})_{\mathbf{i} \in <\mathbf{m}>})\}. \end{aligned} \tag{7.1}$$

$C((a_{\mathbf{i}})_{\mathbf{i} \in <\mathbf{m}>})$ is called a cylinder of $\mathcal{S}$ corresponding to $(a_{\mathbf{i}})_{\mathbf{i} \in <\mathbf{m}>}$. $\mathcal{C}(\mathbf{m})$ is the set of all nonempty cylinders of dimension $\mathbf{m}$ in $\mathcal{S}$. Note that $\mathcal{C}(\mathbf{m}) \subset \Gamma^{\mathbf{m}-\mathbf{1}}$.

Let Π be the set of probability measures on the sigma-algebra $\mathcal{B} \subset 2^{\mathcal{S}}$ generated by all cylinders in $\mathcal{S}$. $\mu \in \Pi$ is called σ-invariant if $\mu(T) = \mu(\sigma_i^{-1}(T))$ for every $T \in \mathcal{B}$ and $i \in < d >$. $\mu \in \Pi$ is called ergodic if μ is σ-invariant and for each $T \in \mathcal{B}$ such that $\sigma_i^{-1}(T) = T$ for all $i \in < d >$ one has $\mu(T) = 0, 1$. Let $\Pi_i \supset \Pi_e$ be the set of σ-invariant probability measures and ergodic measures respectively. For $\mu \in \Pi_i$ let $h(\mu)$ be the entropy of $\mu \in \Pi_i$. Consult with [Wal] for a good reference on ergodic theory for $\mathbb{Z}_+$ actions and with [Kre] for $\mathbb{Z}_+^d$ actions.

Denote by $C(\mathcal{S})$ the Banach space of continuous functions $f : \mathcal{S} \to \mathbb{R}$ with the maximal norm $||f||$. In what follows we assume that $\{\psi_{\mathbf{m}}\}_{\mathbf{m} \in \mathbb{N}^d}$ is a family of continuous functions on $\mathcal{S}$ satisfying the condition

$$||\psi_{\mathbf{m}}|| \leq K|\mathbf{m}|_{pr}, \quad \text{for all} \quad \mathbf{m} \in \mathbb{N}^d. \tag{7.2}$$

Then for each $\mu \in \Pi$ the following sequence

$$\alpha_{\mathbf{m}}(\mu) := \int \frac{\psi_{\mathbf{m}}(x)}{|\mathbf{m}|_{pr}}, \quad \mathbf{m} \in \mathbb{N}^d, \tag{7.3}$$

is bounded by K. A family $\psi_{\mathbf{m}}$ is called σ-subadditive if

$$\begin{aligned} &\psi_{(m_1,\ldots,m_{j-1},m'_j+m''_j,m_{j+1},\ldots,m_d)}(x) \\ (7.4)\quad &\leq \psi_{(m_1,\ldots,m_{j-1},m'_j,m_{j+1},\ldots,m_d)}(x)+\psi_{(m_1,\ldots,m_{j-1},m''_j,m_{j+1},\ldots,m_d)}(\sigma_j^{m'_j}x), \\ &\text{for all } m_1,\ldots,m_{j-1},m'_j,m''_j,m_{j+1},\ldots,m_d\in\mathbb{N},\ j\in<d>,\ x\in\mathcal{S}. \end{aligned}$$

Assume that $\psi_{\mathbf{m}}$ is σ-subadditive and (7.2) holds. It is straightforward to show that if $\mu\in\Pi_i$ then the sequence $|\mathbf{m}|_{pr}\alpha_{\mathbf{m}}(\mu),\ \mathbf{m}\in\mathbb{N}^d$ is subadditive. (See the proof of Proposition 7.1.) Hence

$$\lim_{\mathbf{m}\to\infty}\alpha_{\mathbf{m}}(\mu)=\alpha(\mu). \tag{7.5}$$

A celebrated Kingman subergodic theorem [Kin] claims that under the above conditions

$$\begin{aligned} &\lim_{\mathbf{m}\to\infty}\frac{\psi_{\mathbf{m}}(x)}{|\mathbf{m}|_{pr}} \overset{\mu\text{ a.e.}}{=} \theta(x), \qquad \text{for any } \mu\in\Pi_i, \\ &\text{and}\qquad \theta(x) \overset{\mu\text{ a.e.}}{=} \alpha(\mu), \qquad \text{for } \mu\in\Pi_e. \end{aligned} \tag{7.6}$$

(Originally Kingman proved his result for $d=1$. The extension to $d>1$ is straightforward as pointed out in the proof of Proposition 7.1.) As in [Fr3], to each $(a_{\mathbf{i}})_{\mathbf{i}\in<m>}\in\mathcal{C}(\mathbf{m})$ we associate a positive real number $\phi_{\mathbf{m}}((a_{\mathbf{i}})_{\mathbf{i}\in<\mathbf{m}>})$. This number can be viewed as the "volume" of the configuration $(a_{\mathbf{i}})_{\mathbf{i}\in<\mathbf{m}>}$. We assume that the family ϕ satisfies the following σ-subadditivity conditions

$$\begin{aligned} &\phi_{(m_1,\ldots,m_{j-1},m'_j+m''_j,m_{j+1},\ldots,m_d)}((a_{(i_1,\ldots,i_d)})_{i_1=\ldots=i_d=1}^{m_1,\ldots,m_{j-1},m'_j+m''_j,m_{j+1},\ldots,m_d}) \\ (7.7)\ &\leq \phi_{(m_1,\ldots,m_{j-1},m'_j,m_{j+1},\ldots,m_d)}((a_{(i_1,\ldots,i_d)})_{i_1=\ldots=i_d=1}^{m_1,\ldots,m_{j-1},m'_j,m_{j+1},\ldots,m_d}) \\ &\quad+\phi_{(m_1,\ldots,m_{j-1},m''_j,m_{j+1},\ldots,m_d)}((a_{(i_1,\ldots,i_d)})_{i_1=\ldots=i_{j-1}=i_j-m'_j=i_{j+1}=\ldots=i_d=1}^{m_1,\ldots,m_{j-1},m'_j+m''_j,m_{j+1},\ldots,m_d}), \end{aligned}$$

for all $m'_j,m''_j\in\mathbb{N}$ and $j\in<d>$. Note that if the left-hand side of the above inequality is defined then the right-hand side is also defined. One way to obtain a family of σ-subadditive functions is as follows. Let $q\in C(\mathcal{S})$. Define

$$S_{\mathbf{m}}(q)(x):=\sum_{\mathbf{i}\in<m>}q(\sigma^{\mathbf{i}-\mathbf{1}}(x)),\quad x\in\mathcal{S}. \tag{7.8}$$

Then

$$\phi_{\mathbf{m}}((a_{\mathbf{i}})_{\mathbf{i}\in<\mathbf{m}>})=\max_{x\in C((a_{\mathbf{i}})_{\mathbf{i}\in<\mathbf{m}>})}S_{\mathbf{m}}(q)(x), \tag{7.9}$$

satisfies (7.7). If q is a positive function on $\mathcal{S}$ then each $\phi_{\mathbf{m}}((a_{\mathbf{i}})_{\mathbf{i}\in<\mathbf{m}>})$ defined by (7.8) is positive. Another natural families of ϕ satisfying (7.7) for $\mathbb{Z}_+$-SOFT are discussed in [Fr3]. Let $\{\tilde{\phi}_{\mathbf{m}}\}$ be defined as follows:

$$\tilde{\phi}_{\mathbf{m}}((a_{\mathbf{i}})_{\mathbf{i}\in<\mathbf{m}>}):=\max_{\mathbf{k}\in<\mathbf{m}>}\phi_{\mathbf{k}}((a_{\mathbf{i}})_{\mathbf{i}\in<\mathbf{k}>}),\qquad (a_{\mathbf{i}})_{\mathbf{i}\in<\mathbf{m}>}\in\mathcal{C}(\mathbf{m}). \tag{7.10}$$

Clearly

$$\tilde{\phi}_{\mathbf{k}}((a_{\mathbf{i}})_{\mathbf{i}\in<\mathbf{k}>}) \le \tilde{\phi}_{\mathbf{m}}((a_{\mathbf{i}})_{\mathbf{i}\in<\mathbf{m}>}) \quad \text{for} \quad \mathbf{k} \in <\mathbf{m}> . \tag{7.11}$$

Let $\psi_{\mathbf{m}}, \tilde{\psi}_{\mathbf{m}} \in C(\mathcal{S})$ be the unique continuous functions which are constant on each $C((a_{\mathbf{i}})_{\mathbf{i}\in<\mathbf{m}>})$ with the value $\phi_{\mathbf{m}}((a_{\mathbf{i}})_{\mathbf{i}\in<\mathbf{m}>}), \tilde{\phi}_{\mathbf{m}}((a_{\mathbf{i}})_{\mathbf{i}\in<\mathbf{m}>})$ respectively.

PROPOSITION 7.1. *Let $\{\phi_{\mathbf{m}}\}_{\mathbf{m}\in\mathbb{N}^d}$ be a family of positive σ-subadditive functions. Then the following holds:*
(a) *The family $\{\tilde{\phi}_{\mathbf{m}}\}_{\mathbf{m}\in\mathbb{N}^d}$ is a family of positive σ-subadditive functions.*
(b) *The families $\{\psi_{\mathbf{m}}\}_{\mathbf{m}\in\mathbb{N}^d}$ and $\{\tilde{\psi}_{\mathbf{m}}\}_{\mathbf{m}\in\mathbb{N}^d}$ satisfy the condition (7.2).*
(c) *The function $d : \mathcal{S} \times \mathcal{S} \to \mathbb{R}_+$ given by*

$$\begin{aligned}
d(x,x) &= 0,\\
d(x,y) &= 1 \text{ if } x \text{ and } y \text{ are not in the same cylinder of dimension } \mathbf{1},\\
d(x,y) &= e^{-\tilde{\phi}_{(k,\dots,k)}((a_{\mathbf{i}})_{\mathbf{i}\in<(k,\dots,k)>})} \text{ if } x,y \in C((a_{\mathbf{i}})_{\mathbf{i}\in<(k,\dots,k)>}),\\
&\quad \text{and } x \text{ and } y \text{ are not in the same cylinder of dimension}\\
&\quad (k+1,\dots,k+1), \text{ for some } k \in \mathbb{N},
\end{aligned} \tag{7.12}$$

is a distance on $\mathcal{S}$.
(d) *For all $x \in \mathcal{S}$*

$$\limsup_{\mathbf{m}\to\infty} \frac{\psi_{\mathbf{m}}(x)}{|\mathbf{m}|_{pr}} = \limsup_{\mathbf{m}\to\infty} \frac{\tilde{\psi}_{\mathbf{m}}(x)}{|\mathbf{m}|_{pr}}, \tag{7.13}$$

(e) *Let $\mu \in \Pi_i$. Then the limits below exist μ a.e. and*

$$\lim_{\mathbf{m}\to\infty} \frac{\psi_{\mathbf{m}}(x)}{|\mathbf{m}|_{pr}} = \lim_{\mathbf{m}\to\infty} \frac{\tilde{\psi}_{\mathbf{m}}(x)}{|\mathbf{m}|_{pr}}. \tag{7.14}$$

The sequences $\{\beta_{\mathbf{m}}\}_{\mathbf{m}\in\mathbb{N}^d}$ and $\{\tilde{\beta}_{\mathbf{m}}\}_{\mathbf{m}\in\mathbb{N}^d}$ given by

$$0 < \beta_{\mathbf{m}}(\mu) := \int \psi_{\mathbf{m}} d\mu \le \tilde{\beta}_{\mathbf{m}}(\mu) := \int \tilde{\psi}_{\mathbf{m}} d\mu, \quad \mathbf{m} \in \mathbb{N}^d \tag{7.15}$$

are subadditive sequences. The sequences $\{\alpha_{\mathbf{m}}\}_{\mathbf{m}\in\mathbb{N}^d}$ and $\{\tilde{\alpha}_{\mathbf{m}}\}_{\mathbf{m}\in\mathbb{N}^d}$ given as in (7.3) converge to the same limit $\alpha(\mu) = \tilde{\alpha}(\mu)$. Furthermore

$$\alpha(\mu) \le \alpha_{\mathbf{m}}(\mu) \le \tilde{\alpha}_{\mathbf{m}}(\mu), \quad \mathbf{m} \in \mathbb{N}^d. \tag{7.16}$$

(f) *Let $\mu \in \Pi_e$. Then μ a.e.*

$$\lim_{\mathbf{m}\to\infty} \frac{\psi_{\mathbf{m}}(x)}{|\mathbf{m}|_{pr}} = \lim_{\mathbf{m}\to\infty} \frac{\tilde{\psi}_{\mathbf{m}}(x)}{|\mathbf{m}|_{pr}} = \alpha(\mu). \tag{7.17}$$

Proof. (a) Let $\mathbf{m},\ \mathbf{m}',\ \mathbf{m}''$ be defined as follows

$$\begin{aligned} \mathbf{m} &= (m_1, ..., m_{j-1}, m_j' + m_j'', m_{j+1}, ..., m_d), \\ \mathbf{m}' &= (m_1, ..., m_{j-1}, m_j', m_{j+1}, ..., m_d), \\ \mathbf{m}'' &= (m_1, ..., m_{j-1}, m_j'', m_{j+1}, ..., m_d) \in \mathbb{N}^d. \end{aligned} \tag{7.18}$$

Assume that

$$\tilde\phi_{\mathbf{m}}((a_{\mathbf{i}})_{\mathbf{i}\in<\mathbf{m}>}) = \phi_{\mathbf{k}}((a_{\mathbf{i}})_{\mathbf{i}\in<\mathbf{k}>}), \quad \mathbf{k} \le \mathbf{m}.$$

Suppose first that $k_j \le m_j'$. Then (7.10) and (7.11) yield the inequalities

$$\tilde\phi_{\mathbf{m}}((a_{\mathbf{i}})_{\mathbf{i}\in<\mathbf{m}>}) = \phi_{\mathbf{k}}((a_{\mathbf{i}})_{\mathbf{i}\in<\mathbf{k}>}) \le \tilde\phi_{\mathbf{m}'}((a_{\mathbf{i}})_{\mathbf{i}\in<\mathbf{m}'>}) \le \tilde\phi_{\mathbf{m}}((a_{\mathbf{i}})_{\mathbf{i}\in<\mathbf{m}>}).$$

(Hence equalities hold in the above inequalities.) As each $\tilde\phi_{\mathbf{j}}$ is positive we deduce (7.7) holds for $\tilde\phi_{\mathbf{m}}((a_{\mathbf{i}})_{\mathbf{i}\in<\mathbf{m}>})$. Suppose that $k_j > m_j'$. Let $k_j' = m_j', k_j'' = k_j - m_j' \le m_j''$. Assume that $\mathbf{k}',\ \mathbf{k}''$ are defined as $\mathbf{m}',\ \mathbf{m}''$ respectively. Then the σ-additivity of ϕ and (7.11) yield

$$\begin{aligned} \tilde\phi_{\mathbf{m}}((a_{\mathbf{i}})_{\mathbf{i}\in<\mathbf{m}>}) &= \phi_{\mathbf{k}}((a_{\mathbf{i}})_{\mathbf{i}\in<\mathbf{k}>}) \\ &\le \phi_{\mathbf{k}'}((a_{\mathbf{i}})_{\mathbf{i}\in<\mathbf{k}'}) + \phi_{\mathbf{k}''}((a_{\mathbf{i}})^{\mathbf{k}}_{(1,...,1,k_j'+1,1,...,1)}) \\ &\le \tilde\phi_{\mathbf{m}'}((a_{\mathbf{i}})_{\mathbf{i}\in<\mathbf{m}'>}) + \tilde\phi_{\mathbf{m}''}((a_{\mathbf{i}})^{\mathbf{m}}_{(1,...,1,m_j'+1,1,...,1)}). \end{aligned}$$

(b) Let

$$K := \max_{(a_{\mathbf{1}})\in\mathcal{C}(\mathbf{1})} \phi_{\mathbf{1}}((a_{\mathbf{1}})).$$

Then the positivity of ϕ and σ-subadditivity imply

$$0 < \phi_{\mathbf{m}}((a_{\mathbf{i}})_{\mathbf{i}\in<\mathbf{m}>}) \le |\mathbf{m}|_{pr}K \Rightarrow 0 < \tilde\phi_{\mathbf{m}}((a_{\mathbf{i}})_{\mathbf{i}\in<\mathbf{m}>}) \le |\mathbf{m}|_{pr}K.$$

(c) We claim that

$$d(x,y) \le \max(d(x,z), d(z,y)). \tag{7.19}$$

Note that $d(x,y) \le 1$ and equality holds iff x and y are not contained in any joint cylinder. Hence (7.19) holds if either x,z or z,y are not contained in any joint cylinder. Suppose that $C((a_{\mathbf{i}})_{\mathbf{i}\in<(m,...,m)>})$, $C((a_{\mathbf{i}})_{\mathbf{i}\in<(j,...,j)>})$, $C((a_{\mathbf{i}})_{\mathbf{i}\in<(k,...,k)>})$ are the longest cylinders containing the pairs (x,y), (x,z), (z,y) respectively. Clearly, $m \ge \min(j,k)$. Use (7.12) and (7.11) to obtain (7.19).

(d) Let $\limsup_{\mathbf{m}\to\infty} \frac{\psi_{\mathbf{m}}(x)}{|\mathbf{m}|_{pr}} = a \ge 0$. Then for any $\epsilon > 0$ there exists $N(\epsilon) \in \mathbb{N}$ so that $\psi_{\mathbf{m}} \le |\mathbf{m}|_{pr}(a+\epsilon)$ for $\mathbf{m} > \mathbf{N}(\epsilon) := (N(\epsilon), ..., N(\epsilon))$. The definition of $\tilde\phi$ and (b) yield that

$$\tilde\psi_{\mathbf{m}} \le |\mathbf{m}|_{pr} \max(a+\epsilon, \frac{KN(\epsilon)}{\min_{1\le i\le d} m_i}) \quad \text{for} \quad \mathbf{m} > \mathbf{N}(\epsilon).$$

Hence $\limsup_{\mathbf{m}\to\infty} \frac{\tilde{\psi}_{\mathbf{m}}(x)}{|\mathbf{m}|_{pr}} \le a+\epsilon$ for any $\epsilon > 0$. Clearly

$$\limsup_{\mathbf{m}\to\infty} \frac{\psi_{\mathbf{m}}(x)}{|\mathbf{m}|_{pr}} \le \limsup_{\mathbf{m}\to\infty} \frac{\tilde{\psi}_{\mathbf{m}}(x)}{|\mathbf{m}|_{pr}}.$$

Hence (7.13) holds.
(e) Suppose first that $d = 1$. Kingman's subergodic theorem [Kin] yields that that μ a.e.

$$\lim_{m\to\infty} \frac{\psi_m(x)}{m} = \Psi(x) \ge 0, \qquad \lim_{m\to\infty} \frac{\tilde{\psi}_m(x)}{m} = \tilde{\Psi}(x) \ge 0.$$

Furthermore, Ψ and $\tilde{\Psi}(x)$ are σ invariant:

$$\Psi(\sigma(x)) = \Psi(x), \qquad \tilde{\Psi}(\sigma(x)) = \tilde{\Psi}(x).$$

Use (d) to deduce $\Psi(x) = \tilde{\Psi}(x)$ μ a.e.. Assume now that $d > 1$. We prove the existence of the limits in (7.14) by the repeated use of Kingman's subergodic theorem as follows. Fix $m_1, ..., m_{d-1}$ and consider the family $\{\psi_{(m_1,...,m_d)}\}_{m_d\in\mathbb{N}}$. As this family is σ_d-subadditive we deduce that

$$\lim_{m_d\to\infty} \frac{\psi_{\mathbf{m}}}{m_d} := \psi_{\mathbf{m}^{\{d\}}}(x) \ge 0, \qquad \mathbf{m}^{\{d\}} := (m_1, ..., m_{d-1})$$

μ-a.e. Note that $\psi_{\mathbf{m}^{\{d\}}}(x)$ is σ_d invariant. Observe next that for a fixed m_d the family $\{\frac{\psi_{\mathbf{m}}}{m_d}\}_{\mathbf{m}^{\{d\}}\in\mathbb{N}^{d-1}}$ is σ_j-subadditive for $j = 1, ..., d-1$. Hence $\{\psi_{\mathbf{m}^{\{d\}}}\}$ is σ_j-subadditive for $j = 1, ..., d-1$. Continuing this process we deduce $\lim_{\mathbf{m}\to\infty} \frac{\psi_{\mathbf{m}}(x)}{|\mathbf{m}|_{pr}} = \Psi(x)$ μ a.e.. Furthermore, $\Psi(x)$ is σ_j invariant for $j = 1, ..., d$. Similar claim holds the family $\tilde{\psi}$. (7.13) yields (7.14). As μ is σ_j invariant for $j = 1, ..., d$ the σ-subadditivity of ψ and $\tilde{\psi}$ imply the subadditivity of positive sequences $\beta_{\mathbf{m}}$ and $\tilde{\beta}_{\mathbf{m}}$ respectively. Clearly (7.15) holds. Hence the sequences $\alpha_{\mathbf{m}}$ and $\tilde{\alpha}_{\mathbf{m}}$ converge to $\alpha(\mu)$ and $\tilde{\alpha}(\mu)$ respectively. Clearly (7.16) holds. (7.14) implies the equality $\alpha(\mu) = \tilde{\alpha}(\mu)$.
(f) As μ is ergodic and $\Psi(x)$ is σ_j invariant for $j = 1, ..., d$ we deduce that $\Psi(x)$ is a constant function μ a.e, which is equal to $\alpha(\mu)$. □

As [Fr3], for a given family of positive σ-subadditive ϕ let

$$B(\phi, t) := \{(a_{\mathbf{i}})_{\mathbf{i}\in<\mathbf{m}>} : \quad C((a_{\mathbf{i}})_{\mathbf{i}\in<\mathbf{m}>}) \ne \emptyset, \ \ \phi_{\mathbf{m}}((a_{\mathbf{i}})_{\mathbf{i}\in<\mathbf{m}>}) \le t\}, \ \ t > 0. \tag{7.20}$$

Let $|B(\phi, t)|$ be the cardinality of $B(\phi, t)$. Then the topology induced by the metric d on $\mathcal{S}$ is given by the Tychonoff topology if

$$|B(\phi, t)| < \infty \quad \text{for all} \quad t > 0. \tag{7.21}$$

In what follows we assume that (7.21) holds. Let

$$\kappa(\phi) := \limsup_{t\to\infty} \frac{\log |B(\phi, t)|}{t}. \tag{7.22}$$

Then

$$(7.23) \qquad \begin{aligned} &B(\tilde{\phi}, t) \subset B(\phi, t) \quad \text{for all} \quad t > 0, \\ &\kappa(\tilde{\phi}) \le \kappa(\phi). \end{aligned}$$

8. Topological pressure. Assume that the family $\{\psi_{\mathbf{m}}\}_{\mathbf{m}\in\mathbb{N}^d} \subset C(\mathcal{S})$ satisfies the condition (7.2). Define the topological pressure $P(\psi)$ as follows:

$$(8.1) \qquad \begin{aligned} P_{\mathbf{m}} &:= \sum_{(a_{\mathbf{i}})_{\mathbf{i}\in<\mathbf{m}>}\in\mathcal{C}(\mathbf{m})} \max_{x\in C((a_{\mathbf{i}})_{\mathbf{i}\in<\mathbf{m}>})} e^{\psi_{\mathbf{m}}(x)}, \quad \mathbf{m}\in\mathbb{N}^d, \\ P(\psi) &:= \limsup_{\mathbf{m}\to\infty} \frac{\log P_{\mathbf{m}}}{|\mathbf{m}|_{pr}}. \end{aligned}$$

PROPOSITION 8.1. *Let $\mathcal{S}$ be a nonempty $\mathbb{Z}_+^d$-SOFT of $< n >^{\mathbb{N}^d}$. Assume that the family $\{\psi_{\mathbf{m}}\}_{\mathbf{m}\in\mathbb{N}^d} \subset C(\mathcal{S})$ satisfies the condition* (7.2). *Then*

$$-K \le P(\psi) \le K + \log n.$$

Proof. Since $\mathcal{S}$ is nonempty $P_{\mathbf{m}} \ge e^{-|\mathbf{m}|_{pr}K}$. Hence $P(\psi) \ge -K$. For $\mathcal{S} =< n >^{\mathbb{N}^d}$ $|\mathcal{C}(\mathbf{m})| = n^{|\mathbf{m}|_{pr}}$. Hence $P_{\mathbf{m}} \le e^{|\mathbf{m}|_{pr}K} n^{|\mathbf{m}|_{pr}}$ and the proposition follows. □

Let $\mu \in \Pi_i$. Then the entropy of μ is given by

$$(8.2)\ h(\mu) = \lim_{\mathbf{m}\to\infty} -\frac{1}{|\mathbf{m}|_{pr}} \sum_{(a_{\mathbf{i}})_{\mathbf{i}\in<\mathbf{m}>}\in\mathcal{C}(\mathbf{m})} \mu(C((a_{\mathbf{i}})_{\mathbf{i}\in<\mathbf{m}>})) \log \mu(C((a_{\mathbf{i}})_{\mathbf{i}\in<\mathbf{m}>})).$$

It is known that $h(\mu)$ is upper semicontinuous. For a $q \in C(\mathcal{S})$ let the family $\{S_{\mathbf{m}}(q)\}$ be defined by (7.8). Then the standard topological pressure $P(q)$ [Rue] is the topological pressure corresponding to the family $\{S_{\mathbf{m}}(q)\}$. Then $P(q)$ has the following maximal characterization [Rue] or [Mis]:

$$(8.3)\quad P(q) = \max_{\mu\in\Pi_i} h(\mu) + \int q d\mu = \max_{\mu\in\Pi_e} h(\mu) + \int q d\mu = h(\mu^*) + \int q d\mu^*$$

for some ergodic μ^*.

PROPOSITION 8.2. *Let the family $\{\psi_{\mathbf{m}}\}_{\mathbf{m}\in\mathbb{N}^d} \subset C(\mathcal{S})$ satisfy the condition*

$$(8.4) \qquad \lim_{\mathbf{m}\to\infty} \frac{\psi_{\mathbf{m}} - S_{\mathbf{m}}(q)}{|\mathbf{m}|_{pr}} = 0$$

in the maximal norm of $C(\mathcal{S})$ for some $q \in C(\mathcal{S})$. Then $P(\psi) = P(q)$.

Proof. Let $P_{\mathbf{m}}$, $P_{\mathbf{m}}(q)$ be defined by (8.1) for the families $\{\psi_{\mathbf{m}}\}$ and $\{S_{\mathbf{m}}\}(q)$ respectively. Set $||\frac{\psi_{\mathbf{m}} - S_{\mathbf{m}}(q)}{|\mathbf{m}|_{pr}}|| = \epsilon_{\mathbf{m}}$. Then

$$\frac{1}{|\mathbf{m}|_{pr}} \log P_{\mathbf{m}}(q) - \epsilon_{\mathbf{m}} \le \frac{1}{|\mathbf{m}|_{pr}} \log P_{\mathbf{m}} \le \frac{1}{|\mathbf{m}|_{pr}} \log P_{\mathbf{m}}(q) + \epsilon_{\mathbf{m}}.$$

As $\lim_{\mathbf{m}\to\infty} \epsilon_{\mathbf{m}} = 0$ we deduce $P(\psi) = P(q)$. $\square$

As in [Fr3, Lemma 3.10] it is possible to replace the condition (8.4), in the maximal norm of $C(\mathcal{S})$ for some $q \in C(\mathcal{S})$, by an intrinsic condition. Let

$$C_i(\mathcal{S}) := \{f \in C(\mathcal{S}) : \quad \int f d\mu = 0, \quad \text{for all} \quad \mu \in \Pi_i\}. \tag{8.5}$$

Then $C_i(\mathcal{S})$ is a closed subspace of $C(\mathcal{S})$. Let $Q(\mathcal{S}) := C(\mathcal{S})/C_i(\mathcal{S})$ be the quotient space. Then $Q(\mathcal{S})$ induces the following seminorm on $C(\mathcal{S})$:

$$|||f||| = \inf_{g\in C_i(\mathcal{S})} ||f-g|| = \inf_{g\in C_i(\mathcal{S})} \max_{x\in\mathcal{S}} |f(x)-g(x)|, \quad f \in C(\mathcal{S}). \tag{8.6}$$

Note that $|||\cdot|||$ is the induced norm on $Q(\mathcal{S})$. Furthermore

$$\begin{aligned} &f(x) - f(\sigma_i x) && \text{is the zero element in } Q(\mathcal{S}) \quad \text{for } f \in C(\mathcal{S}) \\ & && \text{and} \quad i = 1, ..., d, \\ &f(x) - \frac{S_{\mathbf{m}}(f)}{|\mathbf{m}|_{pr}} && \text{is the zero element in } Q(\mathcal{S}) \quad \text{for } f \in C(\mathcal{S}). \end{aligned} \tag{8.7}$$

That is, the above trivial cocycles equal to the zero element in $Q(\mathcal{S})$. Let

$$\mathcal{M} := \{\mu \in C(\mathcal{S})^* : \quad \mu = a_1\mu_1 - a_2\mu_2, \quad \mu_1, \mu_2 \in \Pi_i, \quad a_1, a_2 \in \mathbb{R}_+\}.$$

Then $\mathcal{M} = Q(\mathcal{S})^*$. (For a Banach space B, B^* is the Banach space of the bounded linear functionals on B.) Furthermore, Π_i can be viewed as a subset of the unit sphere of $Q(\mathcal{S})^*$ with respect to the dual norm to $|||\cdot|||$.

THEOREM 8.3. *Let $\{\psi_{\mathbf{m}}\}_{\mathbf{m}\in\mathbb{N}^d} \subset C(\mathcal{S})$ be a σ-subadditive family that satisfies the condition* (7.2). *Then*
(a) *The equalities* (7.3), (7.6) *and the first part of the inequality* (7.16) *hold.*
(b) *$P(\psi)$ is characterized by*

$$P(\psi) = \lim_{\mathbf{m}\to\infty} P(\frac{\psi_{\mathbf{m}}}{|\mathbf{m}|_{pr}}) = \lim_{\mathbf{m}\to\infty} \max_{\mu\in\Pi_e} h(\mu) + \int \frac{\psi_{\mathbf{m}}}{|\mathbf{m}|_{pr}} d\mu. \tag{8.8}$$

(c) *The following inequalities hold:*

$$P(\psi) \le P(\frac{\psi_{\mathbf{m}}}{|\mathbf{m}|_{pr}}) \qquad \text{for all} \quad \mathbf{m} \in \mathbb{N}^d, \tag{8.9}$$

$$P(\psi) \ge \sup_{\mu\in\Pi_e} h(\mu) + \alpha(\mu). \tag{8.10}$$

(d) *Assume that for any $\epsilon > 0$ there exists $N(\epsilon) \in \mathbb{N}$ such that*

$$\begin{aligned} &|\alpha_{\mathbf{m}}(\mu) - \alpha(\mu)| < \epsilon, \qquad \textit{for all} \quad \mu \in \Pi_i \\ &\text{and} \quad \mathbf{m} > \mathbf{N}(\epsilon) = (N(\epsilon), ..., N(\epsilon)). \end{aligned} \tag{8.11}$$

Then equality holds in (8.10).

(e) *Assume that*

$$\lim_{\mathbf{m}\to\infty} |||\frac{\psi_{\mathbf{m}}}{|\mathbf{m}|_{pr}} - q||| = 0. \tag{8.12}$$

Then $P(\psi) = P(q)$ *and*

$$P(\psi) = \max_{\mu\in\Pi_e} h(\mu) + \alpha(\mu) = h(\mu^*) + \alpha(\mu^*), \quad \text{for some} \quad \mu^* \in \Pi_e. \tag{8.13}$$

Proof. (a) The σ-subadditivity of $\{\psi_{\mathbf{m}}\}$ implies the subadditivity of the sequence $\{\alpha_{\mathbf{m}}\}$. Hence (7.3) and the first part of the inequality (7.16) hold. (7.6) follows from the Kingman subergodic theorem, as in the proof of part (f) of Proposition 7.1.
(b)+(c) The σ-subadditivity of $\{\psi_{\mathbf{m}}\}$ yields that the sequence $\{P_{\mathbf{m}}\}$ is log-subadditive, i.e. $\{\log P_{\mathbf{m}}\}$ is subadditive. Hence

$$\begin{aligned} &P(\psi) \le \frac{\log P_{\mathbf{m}}}{|\mathbf{m}|_{pr}}, \qquad \mathbf{m}\in\mathbb{N}^d, \\ &\lim_{\mathbf{m}\to\infty} \frac{\log P_{\mathbf{m}}}{|\mathbf{m}|_{pr}} = P(\psi). \end{aligned} \tag{8.14}$$

Furthermore

$$\psi_{\mathbf{m}}(x) \le S_{\mathbf{m}}(\psi_{\mathbf{1}})(x), \quad x\in\mathcal{S}.$$

Hence (8.9) holds for $\mathbf{m} = 1$.
Fix $\mathbf{k} = (k_1, ..., k_d) \in \mathbb{N}^d$. Let

$$\psi_{\mathbf{m},\mathbf{k}} := \psi_{\mathbf{m}\circ\mathbf{k}}, \quad \mathbf{m}\circ\mathbf{k} := (m_1k_1, ..., m_dk_d), \quad \mathbf{m}\in\mathbb{N}^d.$$

Let $\overline{\sigma}_j = \sigma_j^{k_j}$, $j = 1, ..., d$ and $\overline{\sigma} = (\overline{\sigma}_1, ..., \overline{\sigma}_d)$. Then the family $\{\psi_{\mathbf{m},\mathbf{k}}\}_{\mathbf{m}\in\mathbb{N}^d}$ is $\overline{\sigma}$-subadditive family. Let $P_{\mathbf{m},\mathbf{k}}$, $\mathbf{m}\in\mathbb{N}^d$ and $P_{\mathbf{k}}(\psi)$ be the quantities defined in (8.1) for the family $\{\psi_{\mathbf{m},\mathbf{k}}\}_{\mathbf{m}\in\mathbb{N}^d}$ and the commuting transformations $\overline{\sigma}$. Furthermore, let $P_{\mathbf{k}}(\psi_{\mathbf{m},\mathbf{k}})$ be the topological pressure for $\psi_{\mathbf{m},\mathbf{k}}$ with respect to $\overline{\sigma}$. Note that the Markov partition for $\overline{\sigma}$ are all nonempty cylinders of dimension $\mathbf{k}$. Hence $P_{\mathbf{k}} = P_{\mathbf{1},\mathbf{k}}$. Use the second part of (8.14) to show straightforward [Fr3, §2-3]

$$\begin{aligned} P_{\mathbf{k}}(\psi) &= |\mathbf{k}|_{pr} P(\psi), \\ P_{\mathbf{k}}(\psi_{\mathbf{1},k}) &= |\mathbf{k}|_{pr} P(\frac{\psi_{\mathbf{k}}}{|\mathbf{k}|_{pr}}). \end{aligned} \tag{8.15}$$

(The second equality follows from the maximum principle for the standard topological pressure.) Hence

$$P_{\mathbf{k}}(\psi) \le P_{\mathbf{k}}(\psi_{\mathbf{1},\mathbf{k}}) = |\mathbf{k}|_{pr} P(\frac{\psi_{\mathbf{k}}}{|\mathbf{k}|_{pr}})$$

and (8.9) follows. Thus

$$P(\psi) \le \liminf_{\mathbf{m}\to\infty} P(\frac{\psi_{\mathbf{m}}}{|\mathbf{m}|_{pr}}).$$

As $\{\log P_{\mathbf{k}}\}$ is a subadditive sequence, for each $\epsilon > 0$ there exists $N(\epsilon) \in \mathbb{N}$ such that

$$P(\psi) \ge \frac{\log P_{\mathbf{k}}}{|\mathbf{k}|_{pr}} - \epsilon, \qquad \mathbf{k} > \mathbf{N}(\epsilon) = (N(\epsilon), ..., N(\epsilon)).$$

Clearly $P_{\mathbf{k}}(\psi) \le \log P_{\mathbf{1},\mathbf{k}} = \log P_{\mathbf{k}}$. Hence

$$P(\psi) \ge P(\frac{\psi_{\mathbf{k}}}{|\mathbf{k}|_{pr}}) - \epsilon, \quad \mathbf{k} > \mathbf{N}(\epsilon).$$

Thus

$$P(\psi) \ge \limsup_{\mathbf{m}\to\infty} P(\frac{\psi_{\mathbf{m}}}{|\mathbf{m}|_{pr}}).$$

This establishes the first equality in (8.8). The second equality follows from the maximal characterization of the topological pressure. The first part of the inequality (7.16) and the maximal characterization of the topological pressure imply

$$P(\frac{\psi_{\mathbf{k}}}{|\mathbf{k}|_{pr}}) \ge \sup_{\mu\in\Pi_e} h(\mu) + \alpha(\mu). \tag{8.16}$$

The characterization (8.8) yields (8.10).
(d) The inequality (8.11) combined with (8.8) implies equality in (8.10).
(e) The assumption (8.12) yields that $\alpha(\mu) = \int q d\mu,\ \mu \in \Pi_i$. Hence $\alpha(\mu)$ is continuous on Π_i with respect to the weak* convergence. Since $h(\mu)$ is upper semicontinuous, it follows that $\sup_{\mu\in\Pi_e} h(\mu) + \alpha(\mu)$ is achieved for some $\mu^* \in \Pi_e$. Furthermore (8.11) holds. Use (d) to obtain (8.13). □

9. Hausdorff dimension. For $x \in \mathcal{S}$ let $\mathcal{P}_{\mathbf{m}}(x)$ be the unique cylinder $C((a_{\mathbf{i}})_{\mathbf{i}\in<\mathbf{m}>})$ which contains x. The Shannon-McMillan-Breiman theorem for $\mathbb{Z}_+^d$-SOFT [Kre] states:

THEOREM 9.1. *Let $\mathcal{S}$ be a $\mathbb{Z}_+^d$-SOFT. Let $\mu \in \Pi_i$. Then μ a.e.*

$$\lim_{\mathbf{m}\to\infty} -\frac{1}{|\mathbf{m}_{pr}|} \log \mu(\mathcal{P}_{\mathbf{m}}(x)) = h(\mu, x) \ge 0, \tag{9.1}$$

where $h(\mu, x)$ σ-invariant L^1 function satisfying

$$\int h(\mu, x) d\mu = h(\mu). \tag{9.2}$$

In particular, if μ is ergodic then $h(\mu, x) = h(\mu)$ μ a.e.. Let $d : \mathcal{S}\times\mathcal{S} \to \mathbb{R}_+$ be given by (7.10) and (7.12) respectively. (Note that the definition of d in [Fr3, (0.3)] should be corrected to the definition (7.12).) Let $\delta(\phi)$ be the Hausdorff dimension of $\mathcal{S}$ for the metric d. (See for example [Pes] for a definition of Hausdorff dimension $\dim_H X$ of a subset X of a metric space.) For $\mu \in \Pi$ let

$$\delta(\phi, \mu) = \dim_H \mu := \inf_{X\in\mathcal{B},\ \mu(X)=1} \mathrm{dim_H X},$$

be the μ-Hausdorff dimension of $\mathcal{S}$.

THEOREM 9.2. *Let $\{\phi_{\mathbf{m}}\}$ be a family of positive functions satisfying* (7.7) *and* (7.21). *Let $\{\tilde{\phi}_{\mathbf{m}}\}$ and $d : \mathcal{S} \times \mathcal{S} \to \mathbb{R}_+$ be given by* (7.10) *and* (7.12) *respectively. Then for any ergodic measure $\mu \in \Pi_e$ which satisfies $h(\mu) + \alpha(\mu) > 0$, the μ-Hausdorff dimension of $\mathcal{S}$ with respect to the metric d is equal to $\frac{h(\mu)}{\alpha(\mu)}$.*

Proof. For $r > 0$ let

$$B(y, r) := \{x \in \mathcal{S} : \quad d(y, x) \le r\}.$$

Assume that $Y \subset \mathcal{S}$ is a Borel set and $\mu(Y) > 0$. Suppose furthermore that for each $y \in Y$ the following inequality holds:

$$\underline{\delta} \le \liminf_{r\to0^+} \frac{\log\mu(B(y, r))}{\log r} \le \limsup_{r\to0^+} \frac{\log\mu(B(y, r))}{\log r} \le \overline{\delta}.$$

Then $\underline{\delta} \le \dim_H Y \le \overline{\delta}$ [You]. We claim that

$$\lim_{r\to0^+} \frac{\log\mu(B(y, r))}{\log r} = \frac{h(\mu)}{\alpha(\mu)}, \tag{9.3}$$

for μ-almost all $y \in \mathcal{S}$. Observe first that for $0 < r < 1$

$$B(y, r) = \mathcal{P}_{\mathbf{m}(k(r,y))}(y), \qquad \mathbf{m}(k) = (k, ..., k) \in \mathbb{N}^d,$$
$$\tilde{\psi}_{\mathbf{m}(k(r,y))}(y) \ge -\log r, \qquad \tilde{\psi}_{\mathbf{m}(k(r,y))-\mathbf{1}}(y) < -\log r.$$

Assume first that $\alpha(\mu) > 0$. Let $\{\tilde{\psi}_{\mathbf{m}}\}$ be defined as in §7 (before Proposition 7.1). Combine Shannon-McMillan-Breiman theorem with Proposition 7.1 to obtain

$$\lim_{\mathbf{m}\to\infty} \frac{\log\mu(\mathcal{P}_{\mathbf{m}}(y))}{\tilde{\psi}_{\mathbf{m}}(y)} \overset{\mu \text{ a.e.}}{=} -\frac{h(\mu)}{\alpha(\mu)}.$$

The assumption that $\alpha(\mu) > 0$ and the definition of the metric d implies

$$\mathcal{P}_{\mathbf{m}(k)}(y) = B(y, r_k(y)), \qquad r_k(y) \approx e^{-\alpha(\mu)k^d}$$

for μ-almost all y. Combine these results to deduce (9.3).

Suppose that $\alpha(\mu) = 0$ and $h(\mu) > 0$. It is left to show that $\delta(\phi, \mu) = \infty$. Fix $\epsilon > 0$ and for each $\mathbf{m} \in \mathbb{N}^d$ let

$$\phi_{\mathbf{m},\epsilon}((a_{\mathbf{i}})_{\mathbf{i}\in<\mathbf{m}>}) = \phi_{\mathbf{m}}((a_{\mathbf{i}})_{\mathbf{i}\in<\mathbf{m}>}) + |\mathbf{m}|_{pr}\epsilon,$$
$$\tilde{\phi}_{\mathbf{m},\epsilon}((a_{\mathbf{i}})_{\mathbf{i}\in<\mathbf{m}>}) = \tilde{\phi}_{\mathbf{m}}((a_{\mathbf{i}})_{\mathbf{i}\in<\mathbf{m}>}) + |\mathbf{m}|_{pr}\epsilon,$$
$$\psi_{\mathbf{m},\epsilon} = \psi_{\mathbf{m}} + |\mathbf{m}|_{pr}\epsilon, \quad \tilde{\psi}_{\mathbf{m},\epsilon} = \tilde{\psi}_{\mathbf{m}} + |\mathbf{m}|_{pr}\epsilon.$$

Let d_ϵ be the metric induced by the σ-additive family $\{\phi_{\mathbf{m},\epsilon}\}$. Clearly $d_\epsilon(x, y) \le d(x, y)$ for all $x, y \in \mathcal{S}$. Furthermore $\alpha_\epsilon(\mu) = \alpha(\mu) + \epsilon = \epsilon$. Hence

$$\delta(\phi, \mu) \ge \delta(\phi_\epsilon, \mu) = \frac{h(\mu)}{\epsilon}.$$

As ϵ was an arbitrary positive number we deduce that $\delta(\phi, \mu) = \infty$. □

Theorem 9.2 is a generalization of Young's formula of the μ-Hausdorff dimension of the unstable manifold for a diffeomorphism of a surface. The quantity $\alpha(\mu)$ can be viewed as the discrete Lyapunov exponent of the σ-subadditive family $\{\phi_{\mathbf{m}}\}$ [Fr3]. Use the definition of the μ-Hausdorff dimension and the part (e) of Proposition 7.1 to deduce computable lower bounds

$$\delta(\phi) \ge \delta(\phi, \mu) \ge \frac{h(\mu)}{\alpha_{\mathbf{m}}(\mu)}, \qquad \mu \in \Pi_e \text{ and } \mathbf{m} \in \mathbb{N}^d, \tag{9.4}$$

for $\delta(\phi)$ [Fr3].

PROPOSITION 9.3. *Let the assumptions of Theorem* 9.2 *hold. Let* $\kappa(\phi)$ *be defined by* (7.22). *Then* $\delta(\phi) \le \kappa(\tilde{\phi}) \le \kappa(\phi)$.

Proof. In view of the inequality (7.23) it is enough to show $\kappa(\tilde{\phi}) \ge \delta(\phi)$. Consider the "Poincaré" series [Fr3]

$$\sum_{\mathbf{m}\in\mathbb{N}^d} \sum_{(a_{\mathbf{i}})_{\mathbf{i}\in<\mathbf{m}>}\in\mathcal{C}(\mathbf{m})} e^{-s\tilde{\phi}((a_{\mathbf{i}})_{\mathbf{i}\in<\mathbf{m}>})}, \qquad s > 0. \tag{9.5}$$

It is well known that the above series converges for $s > \kappa(\tilde{\phi})$ and diverges for $s < \kappa(\tilde{\phi})$. Assume that $s > \kappa(\tilde{\phi})$. Let $u(s)$ be the value of the sum of the above series. Hence

$$\sum_{(a_{\mathbf{i}})_{\mathbf{i}\in<\mathbf{m}(k)>}\in\mathcal{C}(\mathbf{m}(k))} e^{-s\tilde{\phi}((a_{\mathbf{i}})_{\mathbf{i}\in<\mathbf{m}(k)>})} < u(s).$$

Fix $\epsilon \in (0, 1)$. Let $t = -\log \epsilon$. The assumption (7.21) implies the existence of $k >> 1$ such that each $(a_{\mathbf{i}})_{\mathbf{i}\in<\mathbf{m}(k)>} \in \mathcal{C}(\mathbf{m}(k))$ is not in $B(\tilde{\phi}, t)$. Hence

$$\cup_{(a_{\mathbf{i}})_{\mathbf{i}\in<\mathbf{m}(k)>}\in\mathcal{C}(\mathbf{m}(k))} C((a_{\mathbf{i}})_{\mathbf{i}\in<\mathbf{m}(k)>})$$

is a finite cover of $\mathcal{S}$ such that the diameter of $C((a_{\mathbf{i}})_{\mathbf{i}\in<\mathbf{m}(k)>})$ is $e^{-\tilde{\phi}((a_{\mathbf{i}})_{\mathbf{i}\in<\mathbf{m}(k)>})} \le \epsilon$. The definition of the s-Hausdorff measure of $\mathcal{S}$-$\mathcal{H}^s(\mathcal{S})$

and the above inequality yields that $\mathcal{H}^s(\mathcal{S}) < u(s)$. Hence $\delta(\phi) \le s$ and the proposition follows. $\square$

The above proposition is a generalization of [Fr3, Thm 1.14].

THEOREM 9.4. *Let $\{\phi_{\mathbf{m}}\}$ be a family of positive functions satisfying (7.7) and (7.21). Let $\{\tilde{\phi}_{\mathbf{m}}\}$ and $d : \mathcal{S} \times \mathcal{S} \to \mathbb{R}_+$ be given by (7.10) and (7.12) respectively. Let $\{\psi_{\mathbf{m}}\}$ and $\{\tilde{\psi}_{\mathbf{m}}\}$ be the induced σ-subadditive functions. For $t \ge 0$ let $P(-t\psi)$, $P(t\tilde{\psi})$ be the topological pressure associated with $\{-t\psi_{\mathbf{m}}\}$, $\{-t\tilde{\psi}_{\mathbf{m}}\}$ respectively. Then $P(-t\psi) \ge P(t\tilde{\psi})$. Suppose that $P(-\tau\tilde{\psi}) < 0$. Then $\delta(\phi) \le \tau$. Suppose that there exits a positive function $q \in C(\mathcal{S})$ such that (8.4) holds in the seminorm $|||\cdot|||$ ((8.12) holds). Then there exists an ergodic μ^* such that $\delta(\phi, \mu^*) = \hat{\delta}(\phi)$, where*

$$\hat{\delta}(\phi) := \sup_{\mu \in \Pi_e,\ h(\mu)+\alpha(\mu)>0} \delta(\phi, \mu). \tag{9.6}$$

Suppose furthermore that (8.4) holds in the maximal norm in $C(\mathcal{S})$. Then $\delta(\phi) = \hat{\delta}(\phi)$. Furthermore, $\delta(\phi)$ is the unique solution of the Bowen equation $P(-tq) = 0$. In particular, there exists an ergodic μ^* such that $\dim_H \mu^* = \dim_H \mathcal{S}$.

Proof. As $\psi_{\mathbf{m}} \le \tilde{\psi}_{\mathbf{m}}$ for all $\mathbf{m} \in \mathbb{N}^d$ we deduce that $P(-t\psi) \ge P(-t\tilde{\psi})$ for any $t \ge 0$. Suppose that $P(-\tau\tilde{\psi}) < 0$. The arguments of the proof of Proposition 9.3 yields that $\mathcal{H}^\tau(\mathcal{S}) = 0$. Hence $\delta(\phi) \le \tau$.

Suppose that (8.4) holds in the seminorm $|||\cdot|||$ in $Q(\mathcal{S})$. As $\min_{x \in \mathcal{S}} q(x) = a > 0$ and $\alpha(\mu) = \int q d\mu > a$, the upper semicontinuity of $h(\mu)$ implies the existence of an ergodic $\mu^* \in \Pi_e$ such that $\delta(\phi, \mu^*) = \hat{\delta}(\phi)$. Assume furthermore that (8.4) holds in the maximal norm in $C(\mathcal{S})$. Proposition 8.2 implies that $P(-t\psi) = P(-tq)$. $P(-tq)$ is a strictly decreasing function for $t \in [0, \infty)$, such that $P(0) \ge 0$ and $P(-\infty q) = -\infty$. Hence, there exists a unique $t_0 \in [0, \infty)$ such that $P(-t_0 q) = 0$. The first part of the theorem implies that $\delta(\phi) \le t_0$. The maximal characterization of $P(-t_0 q)$ and the equality $P(-t_0 q) = 0$ yield that $t_0 = \hat{\delta}(\phi)$. The inequality $\delta(\phi) \ge \hat{\delta}(\phi)$ yields the equalities $t_0 = \hat{\delta}(\phi) = \delta(\phi)$. In particular $\dim_H \mu^* = \dim_H \mathcal{S}$. $\square$

10. Appendix: Some results on matrices. In this sections we recall well known results in matrices which are used in this paper. Most of the results can be found in the classical book [Gan] or the modern book [HoJ]. Let $\mathbb{F} = \mathbb{R}, \mathbb{C}$. Let $||\cdot|| : \mathbb{F}^n \to \mathbb{R}_+$ be a norm. Let $||(x_1, ..., x_n)||_2 = (\sum_{i=1}^n |x_i|^2)^{\frac{1}{2}}$ be the Euclidean norm on $\mathbb{F}^n$. Then any norm $||\cdot||$ is equivalent to Euclidean norm. That is there exists $0 < K_1 \le K_2$ (depending on $||\cdot||$) such that

$$K_1 ||\mathbf{x}||_2 \le ||\mathbf{x}|| \le K_2 ||\mathbf{x}||_2, \qquad \text{for all } \mathbf{x} = (x_1, ..., x_n) \in \mathbb{F}^n.$$

(Hence any two norms are equivalent.) Let $\mathrm{M}_n(\mathbb{F})$ be the set of $n \times n$ matrices with entries in $\mathbb{F}$. Then any $A \in \mathrm{M}_n(\mathbb{F})$ can be viewed as a linear operator on $\mathbf{x} \mapsto \mathbf{x}A$. The operator norm $||\cdot|| : \mathrm{M}_n(\mathbb{F}) \to \mathbb{R}_+$ induced by $||\cdot||$ on $\mathbb{F}^n$ is given by:

$$||A|| = \sup_{\mathbf{x} \in \mathbb{F}^n \setminus \{0\}} \frac{||\mathbf{x}A||}{||\mathbf{x}||}.$$

That is $||A||$ is vector norm, $||I|| = 1$ (I the identity matrix) and $||AB|| \leq ||A|| \, ||B||$. Let A^{T}, A^* be the transpose and the conjugate transpose of $A \in \mathrm{M}_n(\mathbb{C})$. Then $||A||$ is equivalent to the Frobenius norm $(\operatorname{tr} AA^*)^{\frac{1}{2}}$. The spectrum of A-$\sigma(A)$ is the set of distinct eigenvalues of A. Note that even if $A \in \mathrm{M}_n(\mathbb{R})$ then $\sigma(A) \subset \mathbb{C}$. Let $\rho(A) = \max_{\lambda \in \sigma(A)} |\lambda|$ be the spectral radius of A. $A \in \mathrm{M}_n(\mathbb{R})$ is called nonnegative, denoted by $A \geq 0$, if all the entries of A are nonnegative.

PROPOSITION 10.1. *Let $A \in \mathrm{M}_n(\mathbb{C})$. Let let $||\cdot|| : \mathrm{M}_n(\mathbb{C}) \to \mathbb{R}_+$ be a vector norm on $\mathrm{M}_n(\mathbb{C})$. Then*

$$\rho(A) = \lim_{k \to \infty} ||A^k||^{\frac{1}{k}}. \tag{10.1}$$

In particular for $A \geq 0$

$$\rho(A) = \lim_{k \to \infty} (\mathbf{1}A^k\mathbf{1}^t)^{\frac{1}{k}}. \tag{10.2}$$

Proof. (10.1) is well known if a given matrix norm is an operator norm, e.g. [HoJ, Cor.5.6.14]. Since any two vector norms on $\mathrm{M}_n(\mathbb{C}) \approx C^{n^2}$ we deduce (10.1) for any vector norm on $\mathrm{M}_n(\mathbb{C})$.

Clearly

$$||A||_1 := \sum_{i,j=1}^{n} |a_{ij}|, \quad A = (a_{ij})_1^n \in \mathrm{M}_n(\mathbb{C})$$

is a vector norm. If $A \geq 0$ then $||A||_1 = \mathbf{1}A\mathbf{1}^{\mathrm{T}}$ and (10.2) follows from (10.1). □

Clearly, the first equality of (2.5) follows from (10.2). Let $A = (a_{ij})_1^n \geq 0$. Associate with A the incidence graph $\Gamma(A) \subset < n > \times < n >$:

$$(i,j) \in \Gamma(A) \iff a_{ij} > 0, \qquad \text{for all } \; i,j \in < n > .$$

Clearly, for $\Gamma \subset < n > \times < n >$ we have the equality $\Gamma(A(\Gamma)) = \Gamma$.

PROPOSITION 10.2. *Let $\Gamma \subset < n > \times < n >$. Then equalities (2.4) hold.*

Proof. Let $A = (a_{ij})_1^n \in \mathrm{M}_n(\mathbb{C})$. For $m \in \mathbb{N}$ let $A^m = (a_{ij}^{(m)})_1^n$. It is straightforward to see that

$$a_{ij}^{(m)} = \sum_{i_1,\ldots,i_{m-1} \in <n>} a_{ii_1} a_{i_1 i_2} \ldots a_{i_{m-1}j}, \quad i,j \in < n > .$$

Let $A = A(\Gamma)$. Then each $a_{ij} \in \{0,1\}$. Consider the product $a_{ii_1}a_{i_1i_2}\cdots a_{i_{m-1}j} \in \{0,1\}$. It is equal to 1 if and only if the directed path of length m: $i \to i_1 \to .. \to i_{m-1} \to j$ is in Γ^m. Hence $a_{ij}^{(m)}$ is the number of directed paths of length m from i to j in Γ. Thus $\mathbf{1}A(\Gamma)^m\mathbf{1}^{\mathrm{T}} = \#\Gamma^m$. Clearly $\operatorname{tr} A(\Gamma)^m = \sum_{i=1}^n a_{ii}^{(m)}$. Since $a_{ii}^{(m)}$ is the number of periodic paths of length m starting and ending at the vertex i we deduce $\operatorname{tr} A(\Gamma)^m = \#\Gamma_{per}^m$. □

We now recall some spectral properties of nonnegative square matrices, referred as Perron-Frobenius theorem [Gan] or [HoJ]. Let $A \geq 0$. Then $\rho(A) \in \sigma(A)$. If A is not nilpotent then all the eigenvalues of $\frac{1}{\rho(A)}A$ on the unit circle are roots of unity.

PROPOSITION 10.3. *Let $A \geq 0$. Then $\rho(A) = \limsup_{k\to\infty}(\operatorname{tr} A^k)^{\frac{1}{k}}$.*

Proof. Clearly $\operatorname{tr} A^k \leq \mathbf{1}A^k\mathbf{1}^{\mathrm{T}}$. Hence (10.2) yields the inequality $\rho(A) \geq \limsup_{k\to\infty}(\operatorname{tr} A^k)^{\frac{1}{k}}$. It is left to prove the proposition in case $\rho(A) > 0$. Assume that $\lambda_1, ..., \lambda_n$ are the eigenvalues of A counted with their multiplicities. Arrange these eigenvalues in the order

$$\rho(A) = \lambda_1 = |\lambda_2| = ... = |\lambda_p| > |\lambda_{p+1}| \geq ... \geq |\lambda_n|.$$

There exists $q \in \mathbb{N}$ such that $\lambda_i^q = \rho(A)^q,\ i = 1, ..., p$. Hence

$$\operatorname{tr} A^{mq} = p\,\rho(A)^{mq} + \sum_{i=p+1}^{n} \lambda_i^{mq}.$$

Thus $\lim_{m\to\infty}(\operatorname{tr} A^{mq})^{\frac{1}{mq}} = \rho(A)$ and the proposition follows. □

Let $A = (a_{ij})_1^n, B = (b_{ij})_1^n \in \mathrm{M}_n(\mathbb{R})$. Then $A \leq B \iff a_{ij} \leq b_{ij}$ for each $i, j \in < n >$. If $0 \leq A \leq B$ then $\rho(A) \leq \rho(B)$.

Proof of Lemma 2.5. Clearly $\rho(\Gamma) = 0$ if and only if $A(\Gamma)$ is nilpotent. This is equivalent to the statement that Γ does not have a path of length greater than n. The last statement is equivalent to the statement that Γ does not have cycles. Assume that Γ has a cycle. Let $\Gamma' \subset \Gamma$ consist of one cycle. As $A(\Gamma') \leq A(\Gamma)$ we obtain that $\rho(\Gamma') \leq \rho(\Gamma)$. It is straightforward to see that $\rho(\Gamma') = 1$. □

To estimate h of $\mathbb{Z}^d$-SOFT it is of importance to find effectively good and converging estimates of $\rho(\Gamma)$. This is usually done using Wielandt's inequalities (characterization) [Gan] or [HoJ, 8.1.26]. For $\mathbf{x} = (x_1, ..., x_n) \in \mathbb{R}^n$ we let $\mathbf{x} > 0 \iff x_1, ..., x_n > 0$. Then for any $\mathbf{x} > 0$ and any $A = (a_{ij})_1^n \geq 0$ we have the estimate

$$\min_{i\in<n>} \frac{\sum_{j=1}^n a_{ij}x_j}{x_i} \leq \rho(A) \leq \max_{i\in<n>} \frac{\sum_{j=1}^n a_{ij}x_j}{x_i}. \tag{10.3}$$

Recall that $A \geq 0$ irreducible if $\Gamma(A)$ is a connected graph. A is called primitive if the A is irreducible and the gcd of all cycles of $\Gamma(A)$ is 1. (Equivalently A^m has positive entries for some $m \in \mathbb{N}$.) If A is primitive than we repeat the estimates (10.3) for $x^{(m)} = \frac{\mathbf{x}^{(m-1)}A^{\mathrm{T}}}{a_{m-1}}$ where $a_{m-1} > 0$

is a normalizing constant starting with $x^{(0)} > 0$. Then the upper and lower bound converge in (10.3) converge fast to $\rho(A)$. If A is irreducible but not primitive than replace A by $B = B + I$ now B is primitive and $\rho(B) = \rho(A) + 1$. So any lower and upper bounds for $\rho(B)$ translate easily to lower and upper bounds for A.

Assume finally that $0 \le A = A^{\mathrm{T}}$, i.e. A is a symmetric and nonnegative (entrywise) matrix. It is known that all the eigenvalues of A are real. Hence

$$\operatorname{tr}(A^{2m}) = \sum_{i=1}^{n} \lambda_i^{2m} \ge \rho(A)^{2m}.$$

Then Rayleigh characterization yields that

$$\rho(A) = \max_{\mathbf{x}\in\mathbb{R}^n\setminus\{0\}} \frac{\mathbf{x}A\mathbf{x}^{\mathrm{T}}}{\mathbf{x}\mathbf{x}^{\mathrm{T}}}.$$

In that case one can improve slightly the lower bound of (10.3) for $\mathbf{x} > 0$:

$$\rho(A) \ge \frac{\sum_{i=1}^{n} (\sum_{j=1}^{n} a_{ij}x_j)x_i}{\sum_{i=1}^{n} x_i x_i} \ge \min_{i\in<n>} \frac{\sum_{j=1}^{n} a_{ij}x_j}{x_i}.$$

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