Low rank approximations of matrices and tensors

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Overview

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- Statement of the problem

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- *CUR* approximation I – VII

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- Nonnegative *CUR*-approximation

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- Simulations

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- Simulations
- Conclusions

Statement of the problem

Data is presented in terms of a matrix

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Singular Value Decomposition (SVD)–$O(mnk)$–too expensive

CUR approximation-I

From $A \in \mathbb{R}^{m\times n}$ choose submatrices consisting of p-columns $C \in \mathbb{R}^{m\times p}$ and q rows $R \in \mathbb{R}^{q\times n}$

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & \dots & a_{2,n} \\ a_{3,1} & a_{3,2} & a_{3,3} & \dots & a_{3,n} \\ a_{4,1} & a_{4,2} & a_{4,3} & \dots & a_{4,n} \\ \vdots & \vdots & \vdots & \vdots & \\ a_{m-1,1} & a_{m-1,2} & a_{m-1,3} & \dots & a_{m-1,n} \\ a_{m,1} & a_{m,2} & a_{m,3} & \dots & a_{m,n} \end{bmatrix},$$

R - red - blue, C - red - magenta.

Approximate A using C, R.

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The set of read entries of A

$$\mathcal{S} := \langle m\rangle\times\langle n\rangle\backslash((\langle m\rangle\backslash I)\times(\langle n\rangle\backslash J)),\ \#\mathcal{S}=mp+qn-pq.$$

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Goal: approximate A by CUR for appropriately chosen C, R and $U \in \mathbb{R}^{p \times q}$.

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CUR approximation introduced by Goreinov, Tyrtyshnikov and Zmarashkin [7, 8].

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Then

$$||A - CA_{I,J}^{-1}R||_{\infty,e} \leq \frac{p+1}{\delta}\sigma_{p+1}(A).$$

CUR-approximations: IV

Object: Find a good algorithm by reading q rows and p columns of A at random and update the approximations.
View the rows and the columns read as the corresponding matrices

$$R \in \mathbb{R}^{q \times n}, C \in \mathbb{R}^{m \times p}.$$

Then a low rank approximation is of the form

$$A_k = CUR,$$

for a properly chosen $U \in \mathbb{R}^{p \times q}$.
The aim of this talk is to introduce an algorithm for finding an optimal U_{opt}, corresponding to $F := CU_{\mathrm{opt}}R$ and an optimal k-rank approximation B of F, if needed by updating the approximations.
Complexity $O(k^2 \max(m, n))$.

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Drineas, Kannan and Mahoney [1] different approach

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$$U_{\text{opt}} \in \arg\min_{U \in \mathbb{R}^{p\times q}} \sum_{(i,j)\in\mathcal{S}} (a_{i,j} - (CUR)_{i,j})^2,$$

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Average error

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Read additional rows and columns of A. Repeat the above process to obtain new k-rank approximation B_1 of A. Continue until either the error $\mathrm{Error}_{\mathrm{av}}(B_k)$ stabilizes, or we exceeded the allowed number of computational work.

CUR-approximations: VI

Given A the best choice of U is

$$U_b \in \arg \min_{U \in \mathbb{R}^{p \times q}} ||A - CUR||_F,$$

$$U_b = C^\dagger A R^\dagger,$$

$X^\dagger$ Moore-Penrose inverse of $X \in \mathbb{R}^{m \times n}$.
Complexity: $O(pqmn)$.

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SVD of CUR can be fast found by Golub-Reinsch *SVD* algorithm.
Complexity: $O(rpq \max(m, n))$.

CUR-approximations: VII

Least squares solution

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Example:

Cameraman: $n = m = 256$, $p = q = 80$.
Number of variables: $pq = 6400$.
Number of equations: $2 \times 256 \times 80 - 6400 = 34,560$.

Problems with executing least squares with Matlab:
very long time of execution time and poor precision.

Nonnegative CUR-approximation

$A \geq 0$: entries of A are nonnegative

$$U_{\text{opt}} \in \arg \min_{U \in \mathbb{R}^{p \times q}} \sum_{(i,j) \in \mathcal{S}} (a_{i,j} - (CUR)_{i,j})^2,$$

subject to constrains: $(CUR)_{i,j} \geq 0, \ (i,j) \in \mathcal{S}$.
Or

$$U_b \in \arg \min_{U \in \mathbb{R}^{p \times q}} ||A - CUR||_F,$$

subject to constrains: $(CUR) \geq 0$.

Minimization of strictly convex quadratic function in a convex polytope.

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back to Goreinov-Tyrtyshnykov.

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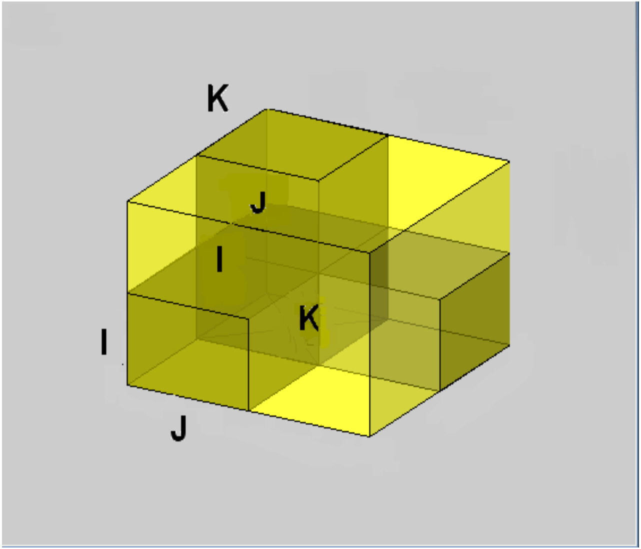
$$\tilde{U}_{\mathrm{opt}} := A_{I,J,r_p}^{\dagger}$$

A_{I,J,r_p} is the best rank r_p approximation of A_{IJ}.
A is approximated by $C\tilde{U}_{\mathrm{opt}}R$.

Extension to 3-tensors: I



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$R := \mathcal{A}_{\langle m \rangle, J, K} = [a_{i,j,k}]_{\langle m \rangle, J, K} \in \mathbb{R}^{m \times (\# J \cdot \# K)}$,
$C := \mathcal{A}_{I, \langle n \rangle, K} \in \mathbb{R}^{\langle n \rangle \times (\# I \cdot \# K)}$,
$D := \mathcal{A}_{I, J, \langle \ell \rangle} \in \mathbb{R}^{I \times (\# I \cdot \# J)}$

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$D := \mathcal{A}_{I,J,\langle \ell\rangle} \in \mathbb{R}^{I\times(\#I\cdot\#J)}$
Problem: Find 3-tensor $\mathcal{U} =\in \mathbb{R}^{(\#J\cdot\#K)\times(\#I\cdot\#K)\times(\#I\cdot\#J)}$
such that $\mathcal{A}$ is approximated by the Tucker tensor
$\mathcal{V} = \mathcal{U} \times_1 C \times_2 R \times_3 D$
where $\mathcal{U}$ is the least squares solution

$$\mathcal{U}_{\rm opt} \in \arg \min_{\mathcal{U}\in\mathbb{R}^{\rm three\ tensor}} \sum_{(i,j,k)\in\mathcal{S}} \left(a_{i,j,k} - (\mathcal{U} \times_1 C \times_2 R \times_3 D)_{i,j,k}\right)^2$$

$$\mathcal{S} = (\langle m\rangle \times J \times K) \cup (I \times \langle n\rangle \times K) \cup (I \times J \times \langle \ell\rangle)$$

Extension to 3-tensors: III

For $\#I = \#J = p, \ \#K = p^2, \ I \subset \langle m \rangle, J \subset \langle n \rangle, K \subset \langle \ell \rangle$
generically there is an exact solution to $\mathcal{U}_{\text{opt}} \in \mathbb{R}^{p^3 \times p^3 \times p^2}$
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View $\mathcal{A}$ as $A \in \mathbb{R}^{(mn) \times \ell}$ by identifying
$\langle m \rangle \times \langle n \rangle \equiv \langle mn \rangle,\ I_1 = I \times J,\ J_1 = K$ and apply CUR again.

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generically there is an exact solution to $\mathcal{U}_{\mathrm{opt}} \in \mathbb{R}^{p^3 \times p^3 \times p^2}$
obtained by unfolding in third direction

View $\mathcal{A}$ as $A \in \mathbb{R}^{(mn) \times \ell}$ by identifying
$\langle m \rangle \times \langle n \rangle \equiv \langle mn \rangle,\ I_1 = I \times J,\ J_1 = K$ and apply CUR again.

More generally, given $\#I = p,\ \#J = q,\ \#K = r$.
For $L = I \times J$ approximate $\mathcal{A}$ by $\mathcal{A}_{\langle m \rangle, \langle n \rangle, K} E^{\dagger}_{L,K} \mathcal{A}_{I,J,\langle \ell \rangle}$

Extension to 3-tensors: III

For $\#I = \#J = p,\ \#K = p^2,\ I \subset \langle m\rangle, J \subset \langle n\rangle, K \subset \langle \ell\rangle$
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More generally, given $\#I = p,\ \#J = q,\ \#K = r$.
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Then for each $k \in K$ approximate each matrix $\mathcal{A}_{\langle m\rangle,\langle n\rangle,\{k\}}$ by
$\mathcal{A}_{\langle m\rangle,J,\{k\}} E^{\dagger}_{I,J,\{k\}} \mathcal{A}_{I,\langle n\rangle,\{k\}}$

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Symmetric situation for 4-tensors $\mathcal{A} \in \mathbb{R}^{m \times n \times l \times q}$.

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$\mathcal{V}$ the Tucker approximation

$\mathcal{V} = \mathcal{U} \times_1 C \times_2 R \times_3 D \times_4 H$

$\mathcal{U} \in \mathbb{R}^{p^3 \times p^3 \times p^3 \times p^3}$ is the least squares solution

$$\mathcal{U}_{\text{opt}} \in \arg \min_{\mathcal{U} \in \mathbb{R}^{\text{four tensor}}} \sum_{(i,j,k,r) \in \mathcal{S}} \left(a_{i,j,k,r} - (\mathcal{U} \times_1 C \times_2 R \times_3 D \times_4 H)_{i,j,k,r}\right)^2$$

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At least $6 = \binom{4}{2}$ solutions in generic case:
Approximate $\mathcal{A}$ by $\mathcal{A}_{\langle m\rangle,\langle n\rangle,K,L} E^{\dagger}_{(I\times J),(K\times L)} \mathcal{A}_{I,J,\langle\ell\rangle,\langle q\rangle}$
More solutions?

Figure: Tire image compression (a) original, (b) SVD approximation, (c) CLS approximation, $t_{\text{max}} = 100$.

Figure 1 portrays the original image of the Tire picture from the Image Processing Toolbox of MATLAB, given by a matrix $A \in \mathbb{R}^{205 \times 232}$ of rank 205, the image compression given by the SVD (using the MATLAB function `svds`) of rank 30 and the image compression given by $B_b = CU_bR$.

Simulations: Tire II

The corresponding image compressions given by the approximations B_{opt_1}, B_{opt_2} and $\tilde{B}_{opt}$ are displayed respectively in Figure 2. Here, $t_{\max} = 100$ and $p = q = 30$. Note that the number of trials $t_{\max}$ is set to the large value of 100 for all simulations in order to be able to compare results for different (small and large) matrices.

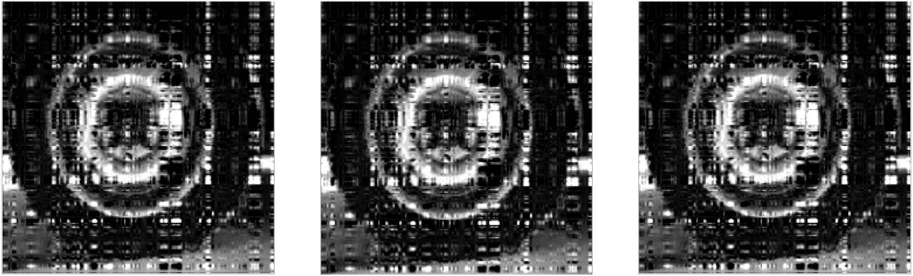

Figure: Tire image compression with (a) B_{opt_1}, (b) B_{opt_2}, (c) $\tilde{B}_{opt}$, $t_{\max} = 100$.

Simulations: Table 1

In Table 1 we present the S-average and total relative errors of the image data compression. Here, $B_b = CU_bR$, $B_{opt_2} = CU_{opt_2}R$ and $\tilde{B}_{opt} = C\tilde{U}_{opt}R$. Table 1 indicates that the less computationally costly FSVD with B_{opt_1}, B_{opt_2} and $\tilde{B}_{opt}$ obtains a smaller S-average error than the more expensive complete least squares solution CLS and the SVD. On the other hand, CLS and the SVD yield better results in terms of the total relative error. However, it should be noted that CLS is very costly and cannot be applied to very large matrices.

	rank	SAE	TRE
B_{svd}	30	0.0072	0.0851
B_b	30	0.0162	0.1920
B_{opt_1}	30	$1.6613 \cdot 10^{-26}$	0.8274
B_{opt_2}	30	$3.2886 \cdot 10^{-29}$	0.8274
$\tilde{B}_{opt}$	30	$1.9317 \cdot 10^{-29}$	0.8274

Table: Comparison of rank, S-average error and total relative error.

Simulations: Cameraman 1

Figure 3 shows the results for the compression of the data for the original image of a camera man from the Image Processing Toolbox of MATLAB. This data is a matrix $A \in \mathbb{R}^{256\times256}$ of rank 253 and the resulting image compression of rank 69 is derived using the SVD and the complete least square approximation CLS given by $B_b = CU_bR$.

Figure: Camera man image compression (a) original, (b) SVD approximation, (c) CLS approximation, $t_{\max} = 100$.

Simulations: Cameraman 2

Figure 4 is FSVD approximation $B_{opt_2} = CU_{opt_2}R$ and $\tilde{B}_{opt} = C\tilde{U}_{opt}R$. Here $t_{\max} = 100$ and $p = q = 80$. Table 2 gives S-average and total relative errors.

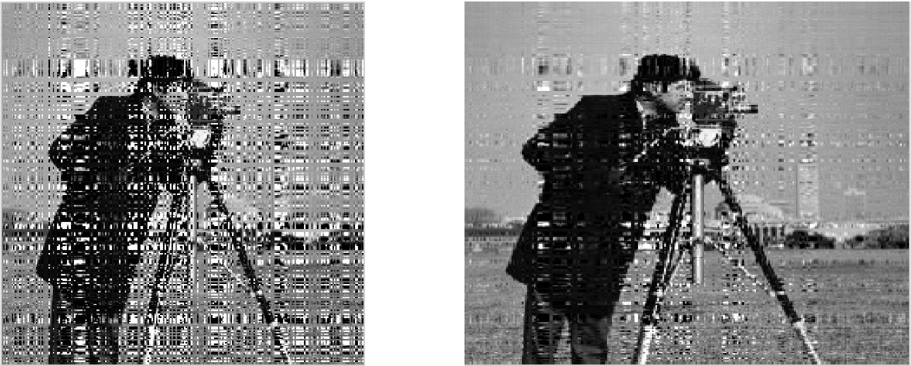

Figure: Camera man image compression. FSVD approximation with (a) $B_{opt_2} = CU_{opt_2}R$, (b) $\tilde{B}_{opt} = C\tilde{U}_{opt}R$. $t_{\max} = 100$.

Simulations: Table 2

	rank	SAE	TRE
B_{svd}	69	0.0020	0.0426
B_b	80	0.0049	0.0954
B_{opt_1}	—	—	—
B_{opt_2}	80	$3.7614 \cdot 10^{-27}$	1.5154
$\tilde{B}_{opt}$	69	$7.0114 \cdot 10^{-4}$	0.2175

Table: Comparison of rank, S-average error and total relative error.

Canal at night 1

Figure: Canal image (a) original, (b) SVD approximation, $t_{\max} = 100$.

Canal at night 2

Figure: Canal image compression (a) CLS approximation, (b) FSVD with $\tilde{B}_{opt}$, $t_{\max} = 100$.

Conclusions

Fast low rank approximation using CUR approximation of A of dimension $m \times n$, $C \in \mathbb{R}^{m \times p}$, $R \in \mathbb{R}^{q \times n}$ are submatrices of A $U \in \mathbb{R}^{p \times q}$ computable by least squares to fit best the entries of C and R.
Advantage: low complexity $O(pq \max(m, n))$.
Disadvantage: problems with computation time and approximation error
Drastic numerical improvement when using $\tilde{U}_{\mathrm{opt}}$.
Least squares can be straightforward generalized to tensors

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