Results and problems for 3-tensors

Shmuel Friedland
Univ. Illinois at Chicago

Haifa Matrix Conference, 21 May, 2009

Overview

Overview

1. The rank of 3-tensors.

Overview

1. The rank of 3-tensors.
2. Best rank one approximation.

Overview

1. The rank of 3-tensors.
2. Best rank one approximation.
3. Perron-Frobenius theorem for *irreducible* nonnegative tensors.

Overview

1. The rank of 3-tensors.
2. Best rank one approximation.
3. Perron-Frobenius theorem for *irreducible* nonnegative tensors.
4. Analogs of SVD decomposition of 3-tensors.

Overview

1. The rank of 3-tensors.
2. Best rank one approximation.
3. Perron-Frobenius theorem for *irreducible* nonnegative tensors.
4. Analogs of SVD decomposition of 3-tensors.
5. CUR decompositions for tensors

Overview

1. The rank of 3-tensors.
2. Best rank one approximation.
3. Perron-Frobenius theorem for *irreducible* nonnegative tensors.
4. Analogs of SVD decomposition of 3-tensors.
5. CUR decompositions for tensors
6. Scaling of nonnegative tensors to balanced tensors.
   (The analog of scaling to doubly stochastic matrices.)

Rank of tensor

Rank of tensor

$\mathbb{F}^{m\times n\times l} := \{\mathcal{T} = [t_{i,j,k}]_{i=j=k}^{m,n,l}, t_{i,j,k} \in \mathbb{F}\}.$

Rank of tensor

$\mathbb{F}^{m\times n\times l} := \{\mathcal{T} = [t_{i,j,k}]_{i=j=k}^{m,n,l}, t_{i,j,k} \in \mathbb{F}\}$.

rank one tensor: $\mathbf{x}\otimes\mathbf{y}\otimes\mathbf{z} = [x_i y_j z_k] = (a\mathbf{x})\otimes(b\mathbf{y})((ab)^{-1}\mathbf{z})$

Rank of tensor

$\mathbb{F}^{m\times n\times l} := \{\mathcal{T} = [t_{i,j,k}]_{i=j=k}^{m,n,l}, t_{i,j,k} \in \mathbb{F}\}$.

rank one tensor: $\mathbf{x}\otimes\mathbf{y}\otimes\mathbf{z} = [x_i y_j z_k] = (a\mathbf{x})\otimes(b\mathbf{y})((ab)^{-1}\mathbf{z})$

rank $\mathcal{T}$ minimal r:
$\mathcal{T} = f_r(\mathbf{x}_1,\mathbf{y}_1,\mathbf{z}_1,\ldots,\mathbf{x}_r,\mathbf{y}_r,\mathbf{z}_r) := \sum_{i=1}^r \mathbf{x}_i\otimes\mathbf{y}_i\otimes\mathbf{z}_i$,
$\mathbf{x}_i\in\mathbb{F}^m, \mathbf{y}_i\in\mathbb{F}^n, \mathbf{z}_i\in\mathbb{F}^l$

Rank of tensor

$\mathbb{F}^{m\times n\times l} := \{\mathcal{T} = [t_{i,j,k}]_{i=j=k}^{m,n,l}, t_{i,j,k} \in \mathbb{F}\}$.

rank one tensor: $\mathbf{x}\otimes\mathbf{y}\otimes\mathbf{z} = [x_i y_j z_k] = (a\mathbf{x})\otimes(b\mathbf{y})((ab)^{-1}\mathbf{z})$

$\mathrm{rank}\ \mathcal{T}$ minimal r:
$\mathcal{T} = f_r(\mathbf{x}_1,\mathbf{y}_1,\mathbf{z}_1,\ldots,\mathbf{x}_r,\mathbf{y}_r,\mathbf{z}_r) := \sum_{i=1}^r \mathbf{x}_i\otimes\mathbf{y}_i\otimes\mathbf{z}_i$,
$\mathbf{x}_i \in \mathbb{F}^m, \mathbf{y}_i \in \mathbb{F}^n, \mathbf{z}_i \in \mathbb{F}^l$

THM Let $\mathcal{T} = [t_{i,j,k}] \in \mathbb{F}^{m\times n\times l}$. $T_k := [t_{i,j,k}]_{i,j=1}^{m,n}, k = 1,\ldots,l$. Then $\mathrm{rank}\ \mathcal{T}$ is the minimal dimension of subspace spanned by rank one matrices containing span $(T_1,\ldots,T_l)$.

Rank of tensor

$\mathbb{F}^{m\times n\times l} := \{\mathcal{T} = [t_{i,j,k}]_{i=j=k}^{m,n,l}, t_{i,j,k} \in \mathbb{F}\}$.

rank one tensor: $\mathbf{x}\otimes\mathbf{y}\otimes\mathbf{z} = [x_i y_j z_k] = (a\mathbf{x})\otimes(b\mathbf{y})((ab)^{-1}\mathbf{z})$

$\mathrm{rank}\,\mathcal{T}$ minimal r:
$\mathcal{T} = f_r(\mathbf{x}_1,\mathbf{y}_1,\mathbf{z}_1,\ldots,\mathbf{x}_r,\mathbf{y}_r,\mathbf{z}_r) := \sum_{i=1}^r \mathbf{x}_i\otimes\mathbf{y}_i\otimes\mathbf{z}_i$,
$\mathbf{x}_i \in \mathbb{F}^m, \mathbf{y}_i \in \mathbb{F}^n, \mathbf{z}_i \in \mathbb{F}^l$

THM Let $\mathcal{T} = [t_{i,j,k}] \in \mathbb{F}^{m\times n\times l}$. $T_k := [t_{i,j,k}]_{i,j=1}^{m,n}, k = 1,\ldots,l$. Then $\mathrm{rank}\,\mathcal{T}$ is the minimal dimension of subspace spanned by rank one matrices containing span $(T_1,\ldots,T_l)$.

Cor For $\mathcal{T} \in \mathbb{F}^{m\times n\times l}$ and $l \geq mn$ $\mathrm{rank}_{\mathbb{F}}\mathcal{T} \leq mn$ (sharp).

Rank of tensor

$\mathbb{F}^{m\times n\times l} := \{\mathcal{T} = [t_{i,j,k}]_{i=j=k}^{m,n,l}, t_{i,j,k} \in \mathbb{F}\}$.

rank one tensor: $\mathbf{x} \otimes \mathbf{y} \otimes \mathbf{z} = [x_i y_j z_k] = (a\mathbf{x}) \otimes (b\mathbf{y})((ab)^{-1}\mathbf{z})$

$\mathrm{rank}\ \mathcal{T}$ minimal r:
$\mathcal{T} = f_r(\mathbf{x}_1, \mathbf{y}_1, \mathbf{z}_1, \ldots, \mathbf{x}_r, \mathbf{y}_r, \mathbf{z}_r) := \sum_{i=1}^{r} \mathbf{x}_i \otimes \mathbf{y}_i \otimes \mathbf{z}_i$,
$\mathbf{x}_i \in \mathbb{F}^m, \mathbf{y}_i \in \mathbb{F}^n, \mathbf{z}_i \in \mathbb{F}^l$

THM Let $\mathcal{T} = [t_{i,j,k}] \in \mathbb{F}^{m\times n\times l}$. $T_k := [t_{i,j,k}]_{i,j=1}^{m,n}, k = 1, \ldots, l$. Then $\mathrm{rank}\ \mathcal{T}$ is the minimal dimension of subspace spanned by rank one matrices containing span $(T_1, \ldots, T_l)$.

Cor For $\mathcal{T} \in \mathbb{F}^{m\times n\times l}$ and $l \geq mn$ $\mathrm{rank}_{\mathbb{F}} \mathcal{T} \leq mn$ (sharp).

Normalization: $2 \leq m \leq n \leq l \leq mn$

Generic rank I

Generic rank I

generic rank: $\mathrm{grank}_{\mathbb{F}}(m, n, l)$ - the rank of a random tensor $\mathcal{T} \in \mathbb{F}^{m \times n \times l}$

Generic rank I

generic rank: $\mathrm{grank}_{\mathbb{F}}(m,n,l)$ - the rank of a random tensor $\mathcal{T} \in \mathbb{F}^{m\times n\times l}$

Thm: $\mathrm{grank}_{\mathbb{C}}(m,n,l) = \min(l, mn)$ for $(m-1)(n-1)+1 \leq l$.

Generic rank I

generic rank: $\mathrm{grank}_{\mathbb{F}}(m, n, l)$ - the rank of a random tensor $\mathcal{T} \in \mathbb{F}^{m\times n\times l}$

Thm: $\mathrm{grank}_{\mathbb{C}}(m, n, l) = \min(l, mn)$ for $(m-1)(n-1)+1 \le l$.

PRF: Generic subspace $L \subset \mathbb{C}^{m\times n}$ of dimension $l = (m-1)(n-1)+1$ intersects $\mathcal{R}(n, m, 1, \mathbb{C})$ at $\binom{m+n-2}{m-1}$, $(\ge l)$, s.t. any l are linearly independent

Generic rank I

generic rank: $\mathrm{grank}_{\mathbb{F}}(m,n,l)$ - the rank of a random tensor $\mathcal{T} \in \mathbb{F}^{m\times n\times l}$

Thm: $\mathrm{grank}_{\mathbb{C}}(m,n,l) = \min(l, mn)$ for $(m-1)(n-1)+1 \le l$.

PRF: Generic subspace $L \subset \mathbb{C}^{m\times n}$ of dimension $l = (m-1)(n-1)+1$ intersects $\mathcal{R}(n,m,1,\mathbb{C})$ at $\binom{m+n-2}{m-1}$, $(\ge l)$, s.t. any l are linearly independent

Any generic subspace $L \subset \mathbb{F}^{m\times n}$ of dimension $l \ge (m-1)(n-1)+1$ spanned by rank one matrices

Generic rank I

generic rank: $\mathrm{grank}_{\mathbb{F}}(m,n,l)$ - the rank of a random tensor $\mathcal{T} \in \mathbb{F}^{m\times n\times l}$

Thm: $\mathrm{grank}_{\mathbb{C}}(m,n,l) = \min(l, mn)$ for $(m-1)(n-1)+1 \le l$.

PRF: Generic subspace $L \subset \mathbb{C}^{m\times n}$ of dimension $l = (m-1)(n-1)+1$ intersects $\mathcal{R}(n,m,1,\mathbb{C})$ at $\binom{m+n-2}{m-1}$, $(\geq l)$, s.t. any l are linearly independent

Any generic subspace $L \subset \mathbb{F}^{m\times n}$ of dimension $l \geq (m-1)(n-1)+1$ spanned by rank one matrices

Dimension count for $\mathbb{F} = \mathbb{C}$ and $2 \le m \le n \le l \le (m-1)(n-1)$:

Generic rank I

generic rank: $\mathrm{grank}_{\mathbb{F}}(m, n, l)$ - the rank of a random tensor $\mathcal{T} \in \mathbb{F}^{m \times n \times l}$

Thm: $\mathrm{grank}_{\mathbb{C}}(m, n, l) = \min(l, mn)$ for $(m-1)(n-1)+1 \leq l$.

PRF: Generic subspace $L \subset \mathbb{C}^{m \times n}$ of dimension $l = (m-1)(n-1)+1$ intersects $\mathcal{R}(n, m, 1, \mathbb{C})$ at $\binom{m+n-2}{m-1}$, $(\geq l)$, s.t. any l are linearly independent

Any generic subspace $L \subset \mathbb{F}^{m \times n}$ of dimension $l \geq (m-1)(n-1)+1$ spanned by rank one matrices

Dimension count for $\mathbb{F} = \mathbb{C}$ and $2 \leq m \leq n \leq l \leq (m-1)(n-1)$:
$f_r : (\mathbb{C}^m \times \mathbb{C}^n \times \mathbb{C}^l)^r \to \mathbb{C}^{m \times n \times l}$

Generic rank I

generic rank: $\mathrm{grank}_{\mathbb{F}}(m, n, l)$ - the rank of a random tensor $\mathcal{T} \in \mathbb{F}^{m\times n\times l}$

Thm: $\mathrm{grank}_{\mathbb{C}}(m, n, l) = \min(l, mn)$ for $(m-1)(n-1)+1 \le l$.

PRF: Generic subspace $L \subset \mathbb{C}^{m\times n}$ of dimension $l = (m-1)(n-1)+1$ intersects $\mathcal{R}(n, m, 1, \mathbb{C})$ at $\binom{m+n-2}{m-1}$, $(\ge l)$, s.t. any l are linearly independent

Any generic subspace $L \subset \mathbb{F}^{m\times n}$ of dimension $l \ge (m-1)(n-1)+1$ spanned by rank one matrices

Dimension count for $\mathbb{F} = \mathbb{C}$ and $2 \le m \le n \le l \le (m-1)(n-1)$:
$f_r : (\mathbb{C}^m \times \mathbb{C}^n \times \mathbb{C}^l)^r \to \mathbb{C}^{m\times n\times l}$
$\mathrm{grank}_{\mathbb{C}}(m, n, l)(m+n+l-2) \ge mnl \Rightarrow \mathrm{grank}_{\mathbb{C}}(m, n, l) \ge \lceil \frac{mnl}{(m+n+l-2)} \rceil$

Generic rank I

generic rank: $\mathrm{grank}_{\mathbb{F}}(m,n,l)$ - the rank of a random tensor $\mathcal{T} \in \mathbb{F}^{m\times n\times l}$

Thm: $\mathrm{grank}_{\mathbb{C}}(m,n,l) = \min(l, mn)$ for $(m-1)(n-1)+1 \leq l$.

PRF: Generic subspace $L \subset \mathbb{C}^{m\times n}$ of dimension $l = (m-1)(n-1)+1$ intersects $\mathcal{R}(n,m,1,\mathbb{C})$ at $\binom{m+n-2}{m-1}$, $(\geq l)$, s.t. any l are linearly independent

Any generic subspace $L \subset \mathbb{F}^{m\times n}$ of dimension $l \geq (m-1)(n-1)+1$ spanned by rank one matrices

Dimension count for $\mathbb{F} = \mathbb{C}$ and $2 \leq m \leq n \leq l \leq (m-1)(n-1)$:
$f_r : (\mathbb{C}^m \times \mathbb{C}^n \times \mathbb{C}^l)^r \to \mathbb{C}^{m\times n\times l}$
$\mathrm{grank}_{\mathbb{C}}(m,n,l)(m+n+l-2) \geq mnl \Rightarrow \mathrm{grank}_{\mathbb{C}}(m,n,l) \geq \lceil \frac{mnl}{(m+n+l-2)} \rceil$
$\frac{mnl}{(m+n+l-2)} = l$ for $l = (m-1)(n-1)+1$

Generic rank I

generic rank: $\mathrm{grank}_{\mathbb{F}}(m,n,l)$ - the rank of a random tensor $\mathcal{T} \in \mathbb{F}^{m\times n\times l}$

Thm: $\mathrm{grank}_{\mathbb{C}}(m,n,l) = \min(l, mn)$ for $(m-1)(n-1)+1 \le l$.

PRF: Generic subspace $L \subset \mathbb{C}^{m\times n}$ of dimension $l = (m-1)(n-1)+1$ intersects $\mathcal{R}(n,m,1,\mathbb{C})$ at $\binom{m+n-2}{m-1}$, $(\ge l)$, s.t. any l are linearly independent

Any generic subspace $L \subset \mathbb{F}^{m\times n}$ of dimension $l \ge (m-1)(n-1)+1$ spanned by rank one matrices

Dimension count for $\mathbb{F} = \mathbb{C}$ and $2 \le m \le n \le l \le (m-1)(n-1)$:
$f_r : (\mathbb{C}^m \times \mathbb{C}^n \times \mathbb{C}^l)^r \to \mathbb{C}^{m\times n\times l}$
$\mathrm{grank}_{\mathbb{C}}(m,n,l)(m+n+l-2) \ge mnl \Rightarrow \mathrm{grank}_{\mathbb{C}}(m,n,l) \ge \lceil \frac{mnl}{(m+n+l-2)} \rceil$
$\frac{mnl}{(m+n+l-2)} = l$ for $l = (m-1)(n-1)+1$
COR $\mathrm{grank}_{\mathbb{C}}(m,n,(m-1)(n-1)) = (m-1)(n-1)+1$.

Generic rank II

Conjecture $\mathrm{grank}_{\mathbb{C}}(m, n, l) = \lceil \frac{mnl}{(m+n+l-2)} \rceil$
for $2 \le m \le n \le l < (m-1)(n-1)$ and $(3, n, l) \ne (3, 2p+1, 2p+1)$

Generic rank II

Conjecture $\mathrm{grank}_{\mathbb{C}}(m, n, l) = \lceil \frac{mnl}{(m+n+l-2)} \rceil$
for $2 \le m \le n \le l < (m-1)(n-1)$ and $(3, n, l) \ne (3, 2p+1, 2p+1)$

Fact: $\mathrm{grank}_{\mathbb{C}}(3, 2p+1, 2p+1) = \lceil \frac{3(2p+1)^2}{4p+3} \rceil + 1$

Generic rank II

Conjecture $\mathrm{grank}_{\mathbb{C}}(m,n,l) = \lceil \frac{mnl}{(m+n+l-2)} \rceil$
for $2 \le m \le n \le l < (m-1)(n-1)$ and $(3,n,l) \neq (3,2p+1,2p+1)$

Fact: $\mathrm{grank}_{\mathbb{C}}(3,2p+1,2p+1) = \lceil \frac{3(2p+1)^2}{4p+3} \rceil + 1$

Conjecture is known in some cases

Generic rank II

Conjecture $\mathrm{grank}_{\mathbb{C}}(m, n, l) = \lceil \frac{mnl}{(m+n+l-2)} \rceil$
for $2 \le m \le n \le l < (m-1)(n-1)$ and $(3, n, l) \ne (3, 2p+1, 2p+1)$

Fact: $\mathrm{grank}_{\mathbb{C}}(3, 2p+1, 2p+1) = \lceil \frac{3(2p+1)^2}{4p+3} \rceil + 1$

Conjecture is known in some cases
For symmetric 3-tensors: Alexander-Hirschowitz Theorem

Generic rank II

Conjecture $\mathrm{grank}_{\mathbb{C}}(m,n,l) = \lceil \frac{mnl}{(m+n+l-2)} \rceil$
for $2 \le m \le n \le l < (m-1)(n-1)$ and $(3,n,l) \neq (3,2p+1,2p+1)$

Fact: $\mathrm{grank}_{\mathbb{C}}(3,2p+1,2p+1) = \lceil \frac{3(2p+1)^2}{4p+3} \rceil + 1$

Conjecture is known in some cases
For symmetric 3-tensors: Alexander-Hirschowitz Theorem

Easy to compute $\mathrm{grank}_{\mathbb{C}}(m,n,l)$:
Pick at random $\mathbf{w}_r := (\mathbf{x}_1,\mathbf{y}_1,\mathbf{z}_1,\ldots,\mathbf{x}_r,\mathbf{y}_r,\mathbf{z}_r) \in (\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^l)^r$
The minimal $r \ge \lceil \frac{mnl}{(m+n+l-2)} \rceil$ s.t. $\mathrm{rank}\, J(f_r)(\mathbf{w}_r) = mnl$
is $\mathrm{grank}_{\mathbb{C}}(m,n,l)$

Generic rank II

Conjecture $\mathrm{grank}_{\mathbb{C}}(m,n,l) = \lceil \frac{mnl}{(m+n+l-2)} \rceil$
for $2 \le m \le n \le l < (m-1)(n-1)$ and $(3,n,l) \ne (3,2p+1,2p+1)$

Fact: $\mathrm{grank}_{\mathbb{C}}(3,2p+1,2p+1) = \lceil \frac{3(2p+1)^2}{4p+3} \rceil + 1$

Conjecture is known in some cases
For symmetric 3-tensors: Alexander-Hirschowitz Theorem

Easy to compute $\mathrm{grank}_{\mathbb{C}}(m,n,l)$:
Pick at random $\mathbf{w}_r := (\mathbf{x}_1,\mathbf{y}_1,\mathbf{z}_1,\ldots,\mathbf{x}_r,\mathbf{y}_r,\mathbf{z}_r) \in (\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^l)^r$
The minimal $r \ge \lceil \frac{mnl}{(m+n+l-2)} \rceil$ s.t. $\mathrm{rank}\, J(f_r)(\mathbf{w}_r) = mnl$
is $\mathrm{grank}_{\mathbb{C}}(m,n,l)$

Avoid round-off error:
$\mathbf{w}_r \in (\mathbb{Z}^m \times \mathbb{Z}^n \times \mathbb{Z}^l)^r$ find $\mathrm{rank}\, J(f_r)(\mathbf{w}_r)$ exact arithmetic

Generic rank II

Conjecture $\mathrm{grank}_{\mathbb{C}}(m,n,l) = \lceil \frac{mnl}{(m+n+l-2)} \rceil$
for $2 \le m \le n \le l < (m-1)(n-1)$ and $(3,n,l) \ne (3, 2p+1, 2p+1)$

Fact: $\mathrm{grank}_{\mathbb{C}}(3, 2p+1, 2p+1) = \lceil \frac{3(2p+1)^2}{4p+3} \rceil + 1$

Conjecture is known in some cases
For symmetric 3-tensors: Alexander-Hirschowitz Theorem

Easy to compute $\mathrm{grank}_{\mathbb{C}}(m,n,l)$:
Pick at random $\mathbf{w}_r := (\mathbf{x}_1, \mathbf{y}_1, \mathbf{z}_1, \ldots, \mathbf{x}_r, \mathbf{y}_r, \mathbf{z}_r) \in (\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^l)^r$
The minimal $r \ge \lceil \frac{mnl}{(m+n+l-2)} \rceil$ s.t. $\mathrm{rank}\, J(f_r)(\mathbf{w}_r) = mnl$
is $\mathrm{grank}_{\mathbb{C}}(m,n,l)$

Avoid round-off error:
$\mathbf{w}_r \in (\mathbb{Z}^m \times \mathbb{Z}^n \times \mathbb{Z}^l)^r$ find $\mathrm{rank}\, J(f_r)(\mathbf{w}_r)$ exact arithmetic
I checked the conjecture up to $m, n, l \le 14$

Generic rank III - the real case

For $mn \leq l$ $\mathrm{grank}_{\mathbb{R}}(m, n, l) = mn$.

Generic rank III - the real case

For $mn \le l$ $\mathrm{grank}_{\mathbb{R}}(m,n,l) = mn$.

For $2 \le m \le n \le l < mn - 1$, there exist $V_1, \ldots, V_{c(m,n,l)} \subset \mathbb{R}^{m \times n \times l}$ pairwise disjoint open connected semi-algebraic sets s.t.

$\mathrm{Closure}(\cup_{i=1}^{c(m,n,l)} V_i) = \mathbb{R}^{m \times n \times l}$

$\mathrm{rank}\ \mathcal{T} = \mathrm{grank}_{\mathbb{C}}(m,n,l)$ for each $\mathcal{T} \in V_1$

$\mathrm{rank}\ \mathcal{T} = \rho_i$ for each $\mathcal{T} \in V_i$

$\rho_i \ge \mathrm{grank}_{\mathbb{C}}(m,n,l)$ for $i = 2, \ldots, c(m,n,l)$

Generic rank III - the real case

For $mn \le l$ $\mathrm{grank}_{\mathbb{R}}(m,n,l) = mn$.

For $2 \le m \le n \le l < mn-1$, there exist $V_1, \ldots, V_{c(m,n,l)} \subset \mathbb{R}^{m \times n \times l}$ pairwise disjoint open connected semi-algebraic sets s.t.
Closure$(\cup_{i=1}^{c(m,n,l)} V_i) = \mathbb{R}^{m \times n \times l}$
$\mathrm{rank}\ \mathcal{T} = \mathrm{grank}_{\mathbb{C}}(m,n,l)$ for each $\mathcal{T} \in V_1$
$\mathrm{rank}\ \mathcal{T} = \rho_i$ for each $\mathcal{T} \in V_i$
$\rho_i \ge \mathrm{grank}_{\mathbb{C}}(m,n,l)$ for $i = 2, \ldots, c(m,n,l)$

For $l = (m-1)(n-1)+1$ $\exists m,n$:
$c(m,n,l) > 1, \rho_{c(m,n,l)} \ge \mathrm{grank}_{\mathbb{C}}(m,n,l) + 1$

Generic rank III - the real case

For $mn \leq l$ $\operatorname{grank}_{\mathbb{R}}(m,n,l) = mn$.

For $2 \leq m \leq n \leq l < mn - 1$, there exist $V_1, \ldots, V_{c(m,n,l)} \subset \mathbb{R}^{m\times n\times l}$ pairwise disjoint open connected semi-algebraic sets s.t.
Closure$(\cup_{i=1}^{c(m,n,l)} V_i) = \mathbb{R}^{m\times n\times l}$
$\operatorname{rank} \mathcal{T} = \operatorname{grank}_{\mathbb{C}}(m,n,l)$ for each $\mathcal{T} \in V_1$
$\operatorname{rank} \mathcal{T} = \rho_i$ for each $\mathcal{T} \in V_i$
$\rho_i \geq \operatorname{grank}_{\mathbb{C}}(m,n,l)$ for $i = 2, \ldots, c(m,n,l)$

For $l = (m-1)(n-1) + 1$ $\exists m, n$:
$c(m,n,l) > 1, \rho_{c(m,n,l)} \geq \operatorname{grank}_{\mathbb{C}}(m,n,l) + 1$

Examples [1]
$m = n \geq 2,\ l = (m-1)(n-1) + 1.$
$m = n = 4,\ l = 11, 12$

Generic rank III - the real case

For $mn \leq l$ $\mathrm{grank}_{\mathbb{R}}(m,n,l) = mn$.

For $2 \leq m \leq n \leq l < mn - 1$, there exist $V_1, \ldots, V_{c(m,n,l)} \subset \mathbb{R}^{m\times n\times l}$ pairwise disjoint open connected semi-algebraic sets s.t.

$\mathrm{Closure}(\cup_{i=1}^{c(m,n,l)} V_i) = \mathbb{R}^{m\times n\times l}$

$\mathrm{rank}\ \mathcal{T} = \mathrm{grank}_{\mathbb{C}}(m,n,l)$ for each $\mathcal{T} \in V_1$

$\mathrm{rank}\ \mathcal{T} = \rho_i$ for each $\mathcal{T} \in V_i$

$\rho_i \geq \mathrm{grank}_{\mathbb{C}}(m,n,l)$ for $i = 2, \ldots, c(m,n,l)$

For $l = (m-1)(n-1)+1$ $\exists m,n$:

$c(m,n,l) > 1, \rho_{c(m,n,l)} \geq \mathrm{grank}_{\mathbb{C}}(m,n,l) + 1$

Examples [1]

$m = n \geq 2,\ l = (m-1)(n-1)+1.$

$m = n = 4,\ l = 11, 12$

Problem: $\rho_i \leq \mathrm{grank}_{\mathbb{C}}(m,n,l) + 1$?

Upper bounds for generic and maximal rank

Upper bounds for generic and maximal rank

$L \cap \mathcal{R}(m,n,k,\mathbb{C}) \supsetneq \{0\}$ if codim $L < \dim \mathcal{R}(m,n,k,\mathbb{C}) = k(n+m-k)$.

Upper bounds for generic and maximal rank

$L \cap \mathcal{R}(m,n,k,\mathbb{C}) \supsetneq \{0\}$ if codim $L <$ dim $\mathcal{R}(m,n,k,\mathbb{C}) = k(n+m-k)$.

$\mathrm{grank}(n,m,m) \le \lfloor \frac{n}{2} \rfloor m + (n - 2\lfloor \frac{n}{2} \rfloor)(m - \lfloor \sqrt{n-1} \rfloor)$ if $m \ge 2\lfloor \sqrt{n-1} \rfloor$

Upper bounds for generic and maximal rank

$L \cap \mathcal{R}(m,n,k,\mathbb{C}) \supsetneq \{0\}$ if codim $L < \dim \mathcal{R}(m,n,k,\mathbb{C}) = k(n+m-k)$.

$\mathrm{grank}(n,m,m) \le \lfloor \frac{n}{2} \rfloor m + (n - 2\lfloor \frac{n}{2} \rfloor)(m - \lfloor \sqrt{n-1} \rfloor)$ if $m \ge 2\lfloor \sqrt{n-1} \rfloor$

$\mathrm{grank}(n,m,m) \le n(m - \lfloor \sqrt{n-1} \rfloor)$ if $m < 2\lfloor \sqrt{n-1} \rfloor < 2(m-1)$,

Upper bounds for generic and maximal rank

$L \cap \mathcal{R}(m,n,k,\mathbb{C}) \supsetneq \{0\}$ if codim $L < \dim \mathcal{R}(m,n,k,\mathbb{C}) = k(n+m-k)$.

$\mathrm{grank}(n,m,m) \leq \lfloor \frac{n}{2} \rfloor m + (n - 2\lfloor \frac{n}{2} \rfloor)(m - \lfloor \sqrt{n-1} \rfloor)$ if $m \geq 2\lfloor \sqrt{n-1} \rfloor$

$\mathrm{grank}(n,m,m) \leq n(m - \lfloor \sqrt{n-1} \rfloor)$ if $m < 2\lfloor \sqrt{n-1} \rfloor < 2(m-1)$,

$\mathrm{grank}(n,m,m) = \min(n, m^2)$ if $n \geq (m-1)^2 + 1$,

Upper bounds for generic and maximal rank

$L\cap \mathcal{R}(m,n,k,\mathbb{C}) \supsetneq \{0\}$ if codim $L <$ dim $\mathcal{R}(m,n,k,\mathbb{C}) = k(n+m-k)$.

$\mathrm{grank}(n,m,m) \le \lfloor \frac{n}{2}\rfloor m + (n-2\lfloor\frac{n}{2}\rfloor)(m-\lfloor\sqrt{n-1}\rfloor)$ if $m \ge 2\lfloor\sqrt{n-1}\rfloor$

$\mathrm{grank}(n,m,m) \le n(m-\lfloor\sqrt{n-1}\rfloor)$ if $m < 2\lfloor\sqrt{n-1}\rfloor < 2(m-1)$,

$\mathrm{grank}(n,m,m) = \min(n,m^2)$ if $n \ge (m-1)^2+1$,

$\mathrm{mrank}(n,m,m) \le$
$\sum_{i=1}^{\lfloor\sqrt{n-1}\rfloor}(2i-1)(m-i+1) + (m-\lfloor\sqrt{n-1}\rfloor^2)(m-\lfloor\sqrt{n-1}\rfloor)$

Rank one approximations

$\mathbb{R}^{m\times n\times l}$ IPS: $\langle \mathcal{A}, \mathcal{B}\rangle = \sum_{i=j=k}^{m,n,l} a_{i,j,k} b_{i,j,k}, \ \|\mathcal{T}\| = \sqrt{\langle \mathcal{T}, \mathcal{T}\rangle}$

$\langle \mathbf{x}\otimes\mathbf{y}\otimes\mathbf{z}, \mathbf{u}\otimes\mathbf{v}\otimes\mathbf{w}\rangle = (\mathbf{u}^\top\mathbf{x})(\mathbf{v}^\top\mathbf{y})(\mathbf{w}^\top\mathbf{z})$

Rank one approximations

$\mathbb{R}^{m\times n\times l}$ IPS: $\langle \mathcal{A}, \mathcal{B}\rangle = \sum_{i=j=k}^{m,n,l} a_{i,j,k} b_{i,j,k}, \ \|\mathcal{T}\| = \sqrt{\langle \mathcal{T}, \mathcal{T}\rangle}$
$\langle \mathbf{x}\otimes\mathbf{y}\otimes\mathbf{z}, \mathbf{u}\otimes\mathbf{v}\otimes\mathbf{w}\rangle = (\mathbf{u}^\top\mathbf{x})(\mathbf{v}^\top\mathbf{y})(\mathbf{w}^\top\mathbf{z})$

$\mathbf{X}$ subspace of $\mathbb{R}^{m\times n\times l}$, $\mathcal{X}_1, \ldots, \mathcal{X}_d$ an orthonormal basis of $\mathbf{X}$
$\mathrm{P}_{\mathbf{X}}(\mathcal{T}) = \sum_{i=1}^{d} \langle \mathcal{T}, \mathcal{X}_i\rangle \mathcal{X}_i, \quad \|\mathrm{P}_{\mathbf{X}}(\mathcal{T})\|^2 = \sum_{i=1}^{d} \langle \mathcal{T}, \mathcal{X}_i\rangle^2$

Rank one approximations

$\mathbb{R}^{m\times n\times l}$ IPS: $\langle \mathcal{A}, \mathcal{B}\rangle = \sum_{i=j=k}^{m,n,l} a_{i,j,k} b_{i,j,k}, \ \|\mathcal{T}\| = \sqrt{\langle \mathcal{T}, \mathcal{T}\rangle}$
$\langle \mathbf{x}\otimes\mathbf{y}\otimes\mathbf{z}, \mathbf{u}\otimes\mathbf{v}\otimes\mathbf{w}\rangle = (\mathbf{u}^\top\mathbf{x})(\mathbf{v}^\top\mathbf{y})(\mathbf{w}^\top\mathbf{z})$

$\mathbf{X}$ subspace of $\mathbb{R}^{m\times n\times l}$, $\mathcal{X}_1, \ldots, \mathcal{X}_d$ an orthonormal basis of $\mathbf{X}$
$\mathrm{P}_{\mathbf{X}}(\mathcal{T}) = \sum_{i=1}^{d} \langle \mathcal{T}, \mathcal{X}_i\rangle \mathcal{X}_i, \quad \|\mathrm{P}_{\mathbf{X}}(\mathcal{T})\|^2 = \sum_{i=1}^{d} \langle \mathcal{T}, \mathcal{X}_i\rangle^2$

$\|\mathcal{T}\|^2 = \|\mathrm{P}_{\mathbf{X}}(\mathcal{T})\|^2 + \|\mathcal{T} - \mathrm{P}_{\mathbf{X}}(\mathcal{T})\|^2$

Rank one approximations

$\mathbb{R}^{m\times n\times l}$ IPS: $\langle \mathcal{A}, \mathcal{B}\rangle = \sum_{i=j=k}^{m,n,l} a_{i,j,k} b_{i,j,k}, \ \|\mathcal{T}\| = \sqrt{\langle \mathcal{T}, \mathcal{T}\rangle}$
$\langle \mathbf{x}\otimes\mathbf{y}\otimes\mathbf{z}, \mathbf{u}\otimes\mathbf{v}\otimes\mathbf{w}\rangle = (\mathbf{u}^\top\mathbf{x})(\mathbf{v}^\top\mathbf{y})(\mathbf{w}^\top\mathbf{z})$

$\mathbf{X}$ subspace of $\mathbb{R}^{m\times n\times l}$, $\mathcal{X}_1,\ldots,\mathcal{X}_d$ an orthonormal basis of $\mathbf{X}$
$\mathrm{P}_{\mathbf{X}}(\mathcal{T}) = \sum_{i=1}^d \langle \mathcal{T}, \mathcal{X}_i\rangle \mathcal{X}_i, \quad \|\mathrm{P}_{\mathbf{X}}(\mathcal{T})\|^2 = \sum_{i=1}^d \langle \mathcal{T}, \mathcal{X}_i\rangle^2$

$\|\mathcal{T}\|^2 = \|\mathrm{P}_{\mathbf{X}}(\mathcal{T})\|^2 + \|\mathcal{T} - \mathrm{P}_{\mathbf{X}}(\mathcal{T})\|^2$

Best rank one approximation of $\mathcal{T}$:

Rank one approximations

$\mathbb{R}^{m\times n\times l}$ IPS: $\langle \mathcal{A}, \mathcal{B}\rangle = \sum_{i=j=k}^{m,n,l} a_{i,j,k} b_{i,j,k}, \ \|\mathcal{T}\| = \sqrt{\langle \mathcal{T}, \mathcal{T}\rangle}$
$\langle \mathbf{x}\otimes\mathbf{y}\otimes\mathbf{z}, \mathbf{u}\otimes\mathbf{v}\otimes\mathbf{w}\rangle = (\mathbf{u}^\top\mathbf{x})(\mathbf{v}^\top\mathbf{y})(\mathbf{w}^\top\mathbf{z})$

$\mathbf{X}$ subspace of $\mathbb{R}^{m\times n\times l}$, $\mathcal{X}_1,\ldots,\mathcal{X}_d$ an orthonormal basis of $\mathbf{X}$
$\mathrm{P}_{\mathbf{X}}(\mathcal{T}) = \sum_{i=1}^d \langle \mathcal{T}, \mathcal{X}_i\rangle \mathcal{X}_i, \quad \|\mathrm{P}_{\mathbf{X}}(\mathcal{T})\|^2 = \sum_{i=1}^d \langle \mathcal{T}, \mathcal{X}_i\rangle^2$

$\|\mathcal{T}\|^2 = \|\mathrm{P}_{\mathbf{X}}(\mathcal{T})\|^2 + \|\mathcal{T} - \mathrm{P}_{\mathbf{X}}(\mathcal{T})\|^2$

Best rank one approximation of $\mathcal{T}$:
$\min_{\mathbf{x},\mathbf{y},\mathbf{z}} \|\mathcal{T} - \mathbf{x}\otimes\mathbf{y}\otimes\mathbf{z}\| = \min_{\|\mathbf{x}\|=\|\mathbf{y}\|=\|\mathbf{z}\|=1,a} \|\mathcal{T} - a\,\mathbf{x}\otimes\mathbf{y}\otimes\mathbf{z}\|$

Rank one approximations

$\mathbb{R}^{m\times n\times l}$ IPS: $\langle \mathcal{A}, \mathcal{B}\rangle = \sum_{i=j=k}^{m,n,l} a_{i,j,k} b_{i,j,k}, \ \|\mathcal{T}\| = \sqrt{\langle \mathcal{T}, \mathcal{T}\rangle}$
$\langle \mathbf{x}\otimes\mathbf{y}\otimes\mathbf{z}, \mathbf{u}\otimes\mathbf{v}\otimes\mathbf{w}\rangle = (\mathbf{u}^\top\mathbf{x})(\mathbf{v}^\top\mathbf{y})(\mathbf{w}^\top\mathbf{z})$

$\mathbf{X}$ subspace of $\mathbb{R}^{m\times n\times l}$, $\mathcal{X}_1,\ldots,\mathcal{X}_d$ an orthonormal basis of $\mathbf{X}$
$\mathrm{P}_{\mathbf{X}}(\mathcal{T}) = \sum_{i=1}^d \langle \mathcal{T}, \mathcal{X}_i\rangle \mathcal{X}_i, \quad \|\mathrm{P}_{\mathbf{X}}(\mathcal{T})\|^2 = \sum_{i=1}^d \langle \mathcal{T}, \mathcal{X}_i\rangle^2$

$\|\mathcal{T}\|^2 = \|\mathrm{P}_{\mathbf{X}}(\mathcal{T})\|^2 + \|\mathcal{T} - \mathrm{P}_{\mathbf{X}}(\mathcal{T})\|^2$

Best rank one approximation of $\mathcal{T}$:
$\min_{\mathbf{x},\mathbf{y},\mathbf{z}} \|\mathcal{T} - \mathbf{x}\otimes\mathbf{y}\otimes\mathbf{z}\| = \min_{\|\mathbf{x}\|=\|\mathbf{y}\|=\|\mathbf{z}\|=1, a} \|\mathcal{T} - a\,\mathbf{x}\otimes\mathbf{y}\otimes\mathbf{z}\|$

Equivalent: $\max_{\|\mathbf{x}\|=\|\mathbf{y}\|=\|\mathbf{z}\|=1} \sum_{i=j=k}^{m,n,l} t_{i,j,k} x_i y_j z_k$

Rank one approximations

$\mathbb{R}^{m\times n\times l}$ IPS: $\langle \mathcal{A}, \mathcal{B}\rangle = \sum_{i=j=k}^{m,n,l} a_{i,j,k} b_{i,j,k}$, $\|\mathcal{T}\| = \sqrt{\langle \mathcal{T}, \mathcal{T}\rangle}$
$\langle \mathbf{x}\otimes\mathbf{y}\otimes\mathbf{z}, \mathbf{u}\otimes\mathbf{v}\otimes\mathbf{w}\rangle = (\mathbf{u}^\top\mathbf{x})(\mathbf{v}^\top\mathbf{y})(\mathbf{w}^\top\mathbf{z})$

$\mathbf{X}$ subspace of $\mathbb{R}^{m\times n\times l}$, $\mathcal{X}_1, \ldots, \mathcal{X}_d$ an orthonormal basis of $\mathbf{X}$
$\mathrm{P}_{\mathbf{X}}(\mathcal{T}) = \sum_{i=1}^d \langle \mathcal{T}, \mathcal{X}_i\rangle \mathcal{X}_i$, $\quad \|\mathrm{P}_{\mathbf{X}}(\mathcal{T})\|^2 = \sum_{i=1}^d \langle \mathcal{T}, \mathcal{X}_i\rangle^2$

$\|\mathcal{T}\|^2 = \|\mathrm{P}_{\mathbf{X}}(\mathcal{T})\|^2 + \|\mathcal{T} - \mathrm{P}_{\mathbf{X}}(\mathcal{T})\|^2$

Best rank one approximation of $\mathcal{T}$:
$\min_{\mathbf{x},\mathbf{y},\mathbf{z}} \|\mathcal{T} - \mathbf{x}\otimes\mathbf{y}\otimes\mathbf{z}\| = \min_{\|\mathbf{x}\|=\|\mathbf{y}\|=\|\mathbf{z}\|=1,a} \|\mathcal{T} - a\,\mathbf{x}\otimes\mathbf{y}\otimes\mathbf{z}\|$

Equivalent: $\max_{\|\mathbf{x}\|=\|\mathbf{y}\|=\|\mathbf{z}\|=1} \sum_{i=j=k}^{m,n,l} t_{i,j,k} x_i y_j z_k$

Lagrange multipliers: $\mathcal{T}\times\mathbf{y}\otimes\mathbf{z} := \sum_{j=k=1} t_{i,j,k} y_j z_k = \lambda\mathbf{x}$
$\mathcal{T}\times\mathbf{x}\otimes\mathbf{z} = \lambda\mathbf{y}$, $\mathcal{T}\times\mathbf{x}\otimes\mathbf{y} = \lambda\mathbf{z}$

Rank one approximations

$\mathbb{R}^{m\times n\times l}$ IPS: $\langle \mathcal{A}, \mathcal{B}\rangle = \sum_{i=j=k}^{m,n,l} a_{i,j,k} b_{i,j,k}, \ \|\mathcal{T}\| = \sqrt{\langle \mathcal{T}, \mathcal{T}\rangle}$
$\langle \mathbf{x}\otimes\mathbf{y}\otimes\mathbf{z}, \mathbf{u}\otimes\mathbf{v}\otimes\mathbf{w}\rangle = (\mathbf{u}^\top\mathbf{x})(\mathbf{v}^\top\mathbf{y})(\mathbf{w}^\top\mathbf{z})$

$\mathbf{X}$ subspace of $\mathbb{R}^{m\times n\times l}$, $\mathcal{X}_1,\ldots,\mathcal{X}_d$ an orthonormal basis of $\mathbf{X}$
$\mathrm{P}_{\mathbf{X}}(\mathcal{T}) = \sum_{i=1}^d \langle \mathcal{T}, \mathcal{X}_i\rangle \mathcal{X}_i, \quad \|\mathrm{P}_{\mathbf{X}}(\mathcal{T})\|^2 = \sum_{i=1}^d \langle \mathcal{T}, \mathcal{X}_i\rangle^2$

$\|\mathcal{T}\|^2 = \|\mathrm{P}_{\mathbf{X}}(\mathcal{T})\|^2 + \|\mathcal{T} - \mathrm{P}_{\mathbf{X}}(\mathcal{T})\|^2$

Best rank one approximation of $\mathcal{T}$:
$\min_{\mathbf{x},\mathbf{y},\mathbf{z}} \|\mathcal{T} - \mathbf{x}\otimes\mathbf{y}\otimes\mathbf{z}\| = \min_{\|\mathbf{x}\|=\|\mathbf{y}\|=\|\mathbf{z}\|=1,a} \|\mathcal{T} - a\,\mathbf{x}\otimes\mathbf{y}\otimes\mathbf{z}\|$

Equivalent: $\max_{\|\mathbf{x}\|=\|\mathbf{y}\|=\|\mathbf{z}\|=1} \sum_{i=j=k}^{m,n,l} t_{i,j,k} x_i y_j z_k$

Lagrange multipliers: $\mathcal{T}\times\mathbf{y}\otimes\mathbf{z} := \sum_{j=k=1} t_{i,j,k} y_j z_k = \lambda\mathbf{x}$
$\mathcal{T}\times\mathbf{x}\otimes\mathbf{z} = \lambda\mathbf{y}, \ \mathcal{T}\times\mathbf{x}\otimes\mathbf{y} = \lambda\mathbf{z}$
λ singular value, $\mathbf{x},\mathbf{y},\mathbf{z}$ singular vectors

Rank one approximations

$\mathbb{R}^{m\times n\times l}$ IPS: $\langle \mathcal{A}, \mathcal{B}\rangle = \sum_{i=j=k}^{m,n,l} a_{i,j,k} b_{i,j,k},\ \|\mathcal{T}\| = \sqrt{\langle \mathcal{T}, \mathcal{T}\rangle}$
$\langle \mathbf{x}\otimes\mathbf{y}\otimes\mathbf{z}, \mathbf{u}\otimes\mathbf{v}\otimes\mathbf{w}\rangle = (\mathbf{u}^\top\mathbf{x})(\mathbf{v}^\top\mathbf{y})(\mathbf{w}^\top\mathbf{z})$

$\mathbf{X}$ subspace of $\mathbb{R}^{m\times n\times l}$, $\mathcal{X}_1,\ldots,\mathcal{X}_d$ an orthonormal basis of $\mathbf{X}$
$\mathrm{P}_{\mathbf{X}}(\mathcal{T}) = \sum_{i=1}^d \langle \mathcal{T}, \mathcal{X}_i\rangle \mathcal{X}_i, \quad \|\mathrm{P}_{\mathbf{X}}(\mathcal{T})\|^2 = \sum_{i=1}^d \langle \mathcal{T}, \mathcal{X}_i\rangle^2$

$\|\mathcal{T}\|^2 = \|\mathrm{P}_{\mathbf{X}}(\mathcal{T})\|^2 + \|\mathcal{T} - \mathrm{P}_{\mathbf{X}}(\mathcal{T})\|^2$

Best rank one approximation of $\mathcal{T}$:
$\min_{\mathbf{x},\mathbf{y},\mathbf{z}} \|\mathcal{T} - \mathbf{x}\otimes\mathbf{y}\otimes\mathbf{z}\| = \min_{\|\mathbf{x}\|=\|\mathbf{y}\|=\|\mathbf{z}\|=1,a} \|\mathcal{T} - a\,\mathbf{x}\otimes\mathbf{y}\otimes\mathbf{z}\|$

Equivalent: $\max_{\|\mathbf{x}\|=\|\mathbf{y}\|=\|\mathbf{z}\|=1} \sum_{i=j=k}^{m,n,l} t_{i,j,k} x_i y_j z_k$

Lagrange multipliers: $\mathcal{T}\times\mathbf{y}\otimes\mathbf{z} := \sum_{j=k=1} t_{i,j,k} y_j z_k = \lambda\mathbf{x}$
$\mathcal{T}\times\mathbf{x}\otimes\mathbf{z} = \lambda\mathbf{y},\ \mathcal{T}\times\mathbf{x}\otimes\mathbf{y} = \lambda\mathbf{z}$
λ singular value, $\mathbf{x},\mathbf{y},\mathbf{z}$ singular vectors
How many distinct singular values are for a generic tensor?

ℓ_p maximal problem and Perron-Frobenius

ℓ_p maximal problem and Perron-Frobenius

$\|(x_1,\ldots,x_n)^\top\|_p := (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}$

Problem: $\max_{\|\mathbf{x}\|_p=\|\mathbf{y}\|_p=\|\mathbf{z}\|_p=1} \sum_{i=j=k}^{m,n,l} t_{i,j,k} x_i y_j z_k$

Lagrange multipliers: $\mathcal{T} \times \mathbf{y} \otimes \mathbf{z} := \sum_{j=k=1} t_{i,j,k} y_j z_k = \lambda \mathbf{x}^{p-1}$
$\mathcal{T} \times \mathbf{x} \otimes \mathbf{z} = \lambda \mathbf{y}^{p-1},\ \mathcal{T} \times \mathbf{x} \otimes \mathbf{y} = \lambda \mathbf{z}^{p-1}$ $(p = \frac{2t}{2s-1}, t, s \in \mathbb{N})$

ℓ_p maximal problem and Perron-Frobenius

$\|(x_1,\ldots,x_n)^\top\|_p := (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}$

Problem: $\max_{\|\mathbf{x}\|_p=\|\mathbf{y}\|_p=\|\mathbf{z}\|_p=1} \sum_{i=j=k}^{m,n,l} t_{i,j,k} x_i y_j z_k$

Lagrange multipliers: $\mathcal{T}\times\mathbf{y}\otimes\mathbf{z} := \sum_{j=k=1} t_{i,j,k} y_j z_k = \lambda \mathbf{x}^{p-1}$
$\mathcal{T}\times\mathbf{x}\otimes\mathbf{z} = \lambda\mathbf{y}^{p-1},\ \mathcal{T}\times\mathbf{x}\otimes\mathbf{y} = \lambda\mathbf{z}^{p-1}$ $(p = \frac{2t}{2s-1}, t, s \in \mathbb{N})$

$p = 3$ is most natural in view of homogeneity

ℓ_p maximal problem and Perron-Frobenius

$\|(x_1,\ldots,x_n)^\top\|_p := (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}$

Problem: $\max_{\|\mathbf{x}\|_p=\|\mathbf{y}\|_p=\|\mathbf{z}\|_p=1} \sum_{i=j=k}^{m,n,l} t_{i,j,k} x_i y_j z_k$

Lagrange multipliers: $\mathcal{T} \times \mathbf{y} \otimes \mathbf{z} := \sum_{j=k=1} t_{i,j,k} y_j z_k = \lambda \mathbf{x}^{p-1}$
$\mathcal{T} \times \mathbf{x} \otimes \mathbf{z} = \lambda \mathbf{y}^{p-1},\ \mathcal{T} \times \mathbf{x} \otimes \mathbf{y} = \lambda \mathbf{z}^{p-1}$ $(p = \frac{2t}{2s-1}, t, s \in \mathbb{N})$

$p = 3$ is most natural in view of homogeneity

Assume that $\mathcal{T} \geq 0$. Then $\mathbf{x}, \mathbf{y}, \mathbf{z} \geq 0$

For which values of p we have an analog of Perron-Frobenius theorem?

Yes, for $p \geq 3$, No, for $p < 3$,
Friedland-Gauber-Han [2]

Outline of the proof

Outline of the proof

Define: $F : \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^l$:

Outline of the proof

Define: $F : \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^l$:

$$F((\mathbf{x},\mathbf{y},\mathbf{z}))_{i,1} = \left(\|\mathbf{x}\|_p^{p-3} \sum_{j=k=1}^{n,l} f_{i,j,k} y_j z_k\right)^{\frac{1}{p-1}}, i = 1,\ldots,m$$

$$F((\mathbf{x},\mathbf{y},\mathbf{z}))_{j,2} = \left(\|\mathbf{y}\|_p^{p-3} \sum_{i=k=1}^{m,l} f_{i,j,k} x_i z_k\right)^{\frac{1}{p-1}}, j = 1,\ldots,n$$

$$F((\mathbf{x},\mathbf{y},\mathbf{z}))_{k} = \left(\|\mathbf{z}\|_p^{p-3} \sum_{i=j=1}^{m,n} f_{i,j,k} x_i y_j\right)^{\frac{1}{p-1}}, k = 1,\ldots,l$$

Outline of the proof

Define: $F:\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}^l\to\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}^l$:

$$F((\mathbf{x},\mathbf{y},\mathbf{z}))_{i,1}=\left(\|\mathbf{x}\|_p^{p-3}\sum_{j=k=1}^{n,l} f_{i,j,k}y_jz_k\right)^{\frac{1}{p-1}},\ i=1,\ldots,m$$

$$F((\mathbf{x},\mathbf{y},\mathbf{z}))_{j,2}=\left(\|\mathbf{y}\|_p^{p-3}\sum_{i=k=1}^{m,l} f_{i,j,k}x_iz_k\right)^{\frac{1}{p-1}},\ j=1,\ldots,n$$

$$F((\mathbf{x},\mathbf{y},\mathbf{z}))_{k}=\left(\|\mathbf{z}\|_p^{p-3}\sum_{i=j=1}^{m,n} f_{i,j,k}x_iy_j\right)^{\frac{1}{p-1}},\ k=1,\ldots,l$$

Assume $\sum_{j=k=1}^{n,l} f_{i,j,k}>0, i=1,\ldots,m$, $\sum_{i=k=1}^{m,l} f_{i,j,k}>0, j=1,\ldots,n$, $\sum_{i=j=1}^{m,n} f_{i,j,k}>0, k=1,\ldots,l$

Outline of the proof

Define: $F : \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^l$:

$$F((\mathbf{x},\mathbf{y},\mathbf{z}))_{i,1} = \left(\|\mathbf{x}\|_p^{p-3} \sum_{j=k=1}^{n,l} f_{i,j,k} y_j z_k\right)^{\frac{1}{p-1}}, i = 1,\dots,m$$

$$F((\mathbf{x},\mathbf{y},\mathbf{z}))_{j,2} = \left(\|\mathbf{y}\|_p^{p-3} \sum_{i=k=1}^{m,l} f_{i,j,k} x_i z_k\right)^{\frac{1}{p-1}}, j = 1,\dots,n$$

$$F((\mathbf{x},\mathbf{y},\mathbf{z}))_{k} = \left(\|\mathbf{z}\|_p^{p-3} \sum_{i=j=1}^{m,n} f_{i,j,k} x_i y_j\right)^{\frac{1}{p-1}}, k = 1,\dots,l$$

Assume $\sum_{j=k=1}^{n,l} f_{i,j,k} > 0, i = 1,\dots,m$,
$\sum_{i=k=1}^{m,l} f_{i,j,k} > 0, j = 1,\dots,n$, $\sum_{i=j=1}^{m,n} f_{i,j,k} > 0, k = 1,\dots,l$

F 1-homogeneous monotone, maps open positive cone $\mathbb{R}^m_+ \times \mathbb{R}^n_+ \times \mathbb{R}^l_+$ to itself.

$\mathcal{F} = [f_{i,j,k}]$ induces tri-partite graph on $\langle m\rangle, \langle n\rangle, \langle l\rangle$:
$i \in \langle m\rangle$ connected to $j \in \langle n\rangle$ and $k \in \langle l\rangle$ iff $f_{i,j,k} > 0$, sim. for j, k

Outline of the proof

Define: $F : \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^l$:

$$F((\mathbf{x},\mathbf{y},\mathbf{z}))_{i,1} = \left(\|\mathbf{x}\|_p^{p-3} \sum_{j=k=1}^{n,l} f_{i,j,k} y_j z_k\right)^{\frac{1}{p-1}}, i = 1,\ldots,m$$

$$F((\mathbf{x},\mathbf{y},\mathbf{z}))_{j,2} = \left(\|\mathbf{y}\|_p^{p-3} \sum_{i=k=1}^{m,l} f_{i,j,k} x_i z_k\right)^{\frac{1}{p-1}}, j = 1,\ldots,n$$

$$F((\mathbf{x},\mathbf{y},\mathbf{z}))_{k} = \left(\|\mathbf{z}\|_p^{p-3} \sum_{i=j=1}^{m,n} f_{i,j,k} x_i y_j\right)^{\frac{1}{p-1}}, k = 1,\ldots,l$$

Assume $\sum_{j=k=1}^{n,l} f_{i,j,k} > 0, i = 1,\ldots,m$,
$\sum_{i=k=1}^{m,l} f_{i,j,k} > 0, j = 1,\ldots,n$, $\sum_{i=j=1}^{m,n} f_{i,j,k} > 0, k = 1,\ldots,l$

F 1-homogeneous monotone, maps open positive cone $\mathbb{R}_+^m \times \mathbb{R}_+^n \times \mathbb{R}_+^l$ to itself.

$\mathcal{F} = [f_{i,j,k}]$ induces tri-partite graph on $\langle m\rangle, \langle n\rangle, \langle l\rangle$:
$i \in \langle m\rangle$ connected to $j \in \langle n\rangle$ and $k \in \langle l\rangle$ iff $f_{i,j,k} > 0$, sim. for j, k

If tri-partite graph is connected then F has unique positive eigenvector

Outline of the proof

Define: $F : \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^l$:

$$F((\mathbf{x},\mathbf{y},\mathbf{z}))_{i,1} = \left(\|\mathbf{x}\|_p^{p-3} \sum_{j=k=1}^{n,l} f_{i,j,k} y_j z_k\right)^{\frac{1}{p-1}}, i = 1,\ldots,m$$

$$F((\mathbf{x},\mathbf{y},\mathbf{z}))_{j,2} = \left(\|\mathbf{y}\|_p^{p-3} \sum_{i=k=1}^{m,l} f_{i,j,k} x_i z_k\right)^{\frac{1}{p-1}}, j = 1,\ldots,n$$

$$F((\mathbf{x},\mathbf{y},\mathbf{z}))_{k} = \left(\|\mathbf{z}\|_p^{p-3} \sum_{i=j=1}^{m,n} f_{i,j,k} x_i y_j\right)^{\frac{1}{p-1}}, k = 1,\ldots,l$$

Assume $\sum_{j=k=1}^{n,l} f_{i,j,k} > 0, i = 1,\ldots,m,$
$\sum_{i=k=1}^{m,l} f_{i,j,k} > 0, j = 1,\ldots,n, \sum_{i=j=1}^{m,n} f_{i,j,k} > 0, k = 1,\ldots,l$

F 1-homogeneous monotone, maps open positive cone $\mathbb{R}_+^m \times \mathbb{R}_+^n \times \mathbb{R}_+^l$ to itself.

$\mathcal{F} = [f_{i,j,k}]$ induces tri-partite graph on $\langle m\rangle, \langle n\rangle, \langle l\rangle$:
$i \in \langle m\rangle$ connected to $j \in \langle n\rangle$ and $k \in \langle l\rangle$ iff $f_{i,j,k} > 0$, sim. for j, k

If tri-partite graph is connected then F has unique positive eigenvector

If F completely irreducible, i.e. F^N maps nonzero nonnegative vectors to positive, nonnegative eigenvector is unique and positive

Outline of the proof

Define: $F : \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^l$:

$$F((\mathbf{x},\mathbf{y},\mathbf{z}))_{i,1} = \left(\|\mathbf{x}\|_p^{p-3} \sum_{j=k=1}^{n,l} f_{i,j,k} y_j z_k\right)^{\frac{1}{p-1}}, i = 1,\dots,m$$

$$F((\mathbf{x},\mathbf{y},\mathbf{z}))_{j,2} = \left(\|\mathbf{y}\|_p^{p-3} \sum_{i=k=1}^{m,l} f_{i,j,k} x_i z_k\right)^{\frac{1}{p-1}}, j = 1,\dots,n$$

$$F((\mathbf{x},\mathbf{y},\mathbf{z}))_k = \left(\|\mathbf{z}\|_p^{p-3} \sum_{i=j=1}^{m,n} f_{i,j,k} x_i y_j\right)^{\frac{1}{p-1}}, k = 1,\dots,l$$

Assume $\sum_{j=k=1}^{n,l} f_{i,j,k} > 0, i = 1,\dots,m,$
$\sum_{i=k=1}^{m,l} f_{i,j,k} > 0, j = 1,\dots,n, \sum_{i=j=1}^{m,n} f_{i,j,k} > 0, k = 1,\dots,l$

F 1-homogeneous monotone, maps open positive cone $\mathbb{R}^m_+ \times \mathbb{R}^n_+ \times \mathbb{R}^l_+$ to itself.

$\mathcal{F} = [f_{i,j,k}]$ induces tri-partite graph on $\langle m\rangle, \langle n\rangle, \langle l\rangle$:
$i \in \langle m\rangle$ connected to $j \in \langle n\rangle$ and $k \in \langle l\rangle$ iff $f_{i,j,k} > 0$, sim. for j, k

If tri-partite graph is connected then F has unique positive eigenvector

If F completely irreducible, i.e. F^N maps nonzero nonnegative vectors to positive, nonnegative eigenvector is unique and positive

$p < 3$ numerical counterexamples $m = n = l = 2$

Analogs of SVD decomposition: I

Analogs of SVD decomposition: I

Unfolding tensor $\mathcal{T} = [t_{i,j,k}] \in \mathbb{F}^{m\times n\times l}$ in 1-index: viewing as a matrix $A_1 \in \mathbb{F}^{m\times (nl)}$.

Analogs of SVD decomposition: I

Unfolding tensor $\mathcal{T} = [t_{i,j,k}] \in \mathbb{F}^{m\times n\times l}$ in 1-index: viewing as a matrix $A_1 \in \mathbb{F}^{m\times(nl)}$.

For a cubic tensor $\mathcal{T} \in \mathbb{R}^{n\times n\times n}$

do orthonormal change of coordinates in each three components $\mathbb{R}^n$:

$\mathcal{T}_1 = \mathcal{T} \times_1 Q_1 \times_2 Q_2 \times_3 Q_3$

Analogs of SVD decomposition: I

Unfolding tensor $\mathcal{T} = [t_{i,j,k}] \in \mathbb{F}^{m\times n\times l}$ in 1-index: viewing as a matrix $A_1 \in \mathbb{F}^{m\times (nl)}$.
For a cubic tensor $\mathcal{T} \in \mathbb{R}^{n\times n\times n}$
do orthonormal change of coordinates in each three components $\mathbb{R}^n$:
$\mathcal{T}_1 = \mathcal{T} \times_1 Q_1 \times_2 Q_2 \times_3 Q_3$

One can have at most $\frac{3n(n-1)}{2}$ zero entries in $\mathcal{T}_1$: $t_{j,i,i} = t_{i,j,i} = t_{i,i,j}$ for $i < j$.

Analogs of SVD decomposition: I

Unfolding tensor $\mathcal{T} = [t_{i,j,k}] \in \mathbb{F}^{m\times n\times l}$ in 1-index: viewing as a matrix $A_1 \in \mathbb{F}^{m\times(nl)}$.
For a cubic tensor $\mathcal{T} \in \mathbb{R}^{n\times n\times n}$
do orthonormal change of coordinates in each three components $\mathbb{R}^n$:
$\mathcal{T}_1 = \mathcal{T} \times_1 Q_1 \times_2 Q_2 \times_3 Q_3$

One can have at most $\frac{3n(n-1)}{2}$ zero entries in $\mathcal{T}_1$: $t_{j,i,i} = t_{i,j,i} = t_{i,i,j}$ for $i < j$.

Apply QR algorithm to the n columns of the unfolded matrix in mode k indexed by $(j,j), j = 1,\ldots,n$ for $k = 1,2,3,1,2,3,...$

Analogs of SVD decomposition: I

Unfolding tensor $\mathcal{T} = [t_{i,j,k}] \in \mathbb{F}^{m\times n\times l}$ in 1-index: viewing as a matrix $A_1 \in \mathbb{F}^{m\times(nl)}$.
For a cubic tensor $\mathcal{T} \in \mathbb{R}^{n\times n\times n}$
do orthonormal change of coordinates in each three components $\mathbb{R}^n$:
$\mathcal{T}_1 = \mathcal{T} \times_1 Q_1 \times_2 Q_2 \times_3 Q_3$

One can have at most $\frac{3n(n-1)}{2}$ zero entries in $\mathcal{T}_1$: $t_{j,i,i} = t_{i,j,i} = t_{i,i,j}$ for $i < j$.

Apply QR algorithm to the n columns of the unfolded matrix in mode k indexed by $(j,j), j = 1,\ldots,n$ for $k = 1,2,3,1,2,3,...$

Example $2\times 2\times 2$: $\begin{bmatrix} t_{1,1,1} & t_{1,2,2} \\ t_{2,1,1} & t_{2,2,2} \end{bmatrix}$; $\begin{bmatrix} t_{1,1,1} & t_{2,1,2} \\ t_{1,2,1} & t_{2,2,2} \end{bmatrix}$; $\begin{bmatrix} t_{1,1,1} & t_{2,2,1} \\ t_{1,1,2} & t_{2,2,2} \end{bmatrix}$

Analogs of SVD decomposition: I

Unfolding tensor $\mathcal{T} = [t_{i,j,k}] \in \mathbb{F}^{m\times n\times l}$ in 1-index: viewing as a matrix $A_1 \in \mathbb{F}^{m\times (nl)}$.
For a cubic tensor $\mathcal{T} \in \mathbb{R}^{n\times n\times n}$
do orthonormal change of coordinates in each three components $\mathbb{R}^n$:
$\mathcal{T}_1 = \mathcal{T} \times_1 Q_1 \times_2 Q_2 \times_3 Q_3$

One can have at most $\frac{3n(n-1)}{2}$ zero entries in $\mathcal{T}_1$: $t_{j,i,i} = t_{i,j,i} = t_{i,i,j}$ for $i < j$.

Apply QR algorithm to the n columns of the unfolded matrix in mode k indexed by $(j, j), j = 1, \ldots, n$ for $k = 1, 2, 3, 1, 2, 3, ...$

Example $2 \times 2 \times 2$: $\begin{bmatrix} t_{1,1,1} & t_{1,2,2} \\ t_{2,1,1} & t_{2,2,2} \end{bmatrix}; \begin{bmatrix} t_{1,1,1} & t_{2,1,2} \\ t_{1,2,1} & t_{2,2,2} \end{bmatrix}; \begin{bmatrix} t_{1,1,1} & t_{2,2,1} \\ t_{1,1,2} & t_{2,2,2} \end{bmatrix}$

Do QR on each two columns successively to obtain:
$t_{2,1,1} = t_{1,2,1} = t_{1,1,2} = 0.$

CUR approximation of matrices

CUR approximation of matrices

For given $A \in \mathbb{R}^{m\times n}, F \in \mathbb{R}^{m\times p}, E \in \mathbb{R}^{q\times n}$
$\min_{U\in\mathbb{C}^{p\times q}} \|A - EUF\|_F$ achieved for $U = E^\dagger A F^\dagger$

CUR approximation of matrices

For given $A \in \mathbb{R}^{m \times n}, F \in \mathbb{R}^{m \times p}, E \in \mathbb{R}^{q \times n}$
$\min_{U \in \mathbb{C}^{p \times q}} \|A - EUF\|_F$ achieved for $U = E^{\dagger} A F^{\dagger}$

CUR approximation $C \in \mathbb{R}^{m \times p}, R \in \mathbb{R}^{q \times n}$ some submatrices of A.

CUR approximation of matrices

For given $A \in \mathbb{R}^{m\times n}, F \in \mathbb{R}^{m\times p}, E \in \mathbb{R}^{q\times n}$
$\min_{U\in\mathbb{C}^{p\times q}} \|A - EUF\|_F$ achieved for $U = E^\dagger A F^\dagger$

CUR approximation $C \in \mathbb{R}^{m\times p}, R \in \mathbb{R}^{q\times n}$ some submatrices of A.

$C(C^\dagger A R^\dagger)R$ best rank $\leq min(p, q)$ approximation matrix based on C, R submatrices of A.

CUR approximation of matrices

For given $A \in \mathbb{R}^{m\times n}, F \in \mathbb{R}^{m\times p}, E \in \mathbb{R}^{q\times n}$
$\min_{U\in\mathbb{C}^{p\times q}} \|A - EUF\|_F$ achieved for $U = E^\dagger A F^\dagger$

CUR approximation $C \in \mathbb{R}^{m\times p}, R \in \mathbb{R}^{q\times n}$ some submatrices of A.

$C(C^\dagger A R^\dagger)R$ best rank $\le min(p,q)$ approximation matrix based on C, R submatrices of A.

Goreinov-Tyrtyshnikov-Zmarashkin 95: for $p = q$ choose $U = A[I,J]^{-1}$

CUR approximation of matrices

For given $A \in \mathbb{R}^{m\times n}, F \in \mathbb{R}^{m\times p}, E \in \mathbb{R}^{q\times n}$
$\min_{U\in\mathbb{C}^{p\times q}} \|A - EUF\|_F$ achieved for $U = E^\dagger A F^\dagger$

CUR approximation $C \in \mathbb{R}^{m\times p}, R \in \mathbb{R}^{q\times n}$ some submatrices of A.

$C(C^\dagger A R^\dagger)R$ best rank $\le min(p,q)$ approximation matrix based on C, R submatrices of A.

Goreinov-Tyrtyshnikov-Zmarashkin 95: for $p = q$ choose $U = A[I,J]^{-1}$ (corresponds to best CUR approximation on the entries read)

CUR approximation of matrices

For given $A \in \mathbb{R}^{m\times n}, F \in \mathbb{R}^{m\times p}, E \in \mathbb{R}^{q\times n}$
$\min_{U\in\mathbb{C}^{p\times q}} \|A - EUF\|_F$ achieved for $U = E^\dagger A F^\dagger$

CUR approximation $C \in \mathbb{R}^{m\times p}, R \in \mathbb{R}^{q\times n}$ some submatrices of A.

$C(C^\dagger A R^\dagger)R$ best rank $\le min(p,q)$ approximation matrix based on C, R submatrices of A.

Goreinov-Tyrtyshnikov-Zmarashkin 95: for $p = q$ choose $U = A[I,J]^{-1}$
(corresponds to best CUR approximation on the entries read)
good approximation when the corresponding $\det A[I,J]$ is maximal

CUR approximation of matrices

For given $A \in \mathbb{R}^{m\times n}, F \in \mathbb{R}^{m\times p}, E \in \mathbb{R}^{q\times n}$
$\min_{U\in\mathbb{C}^{p\times q}} \|A - EUF\|_F$ achieved for $U = E^\dagger A F^\dagger$

CUR approximation $C \in \mathbb{R}^{m\times p}, R \in \mathbb{R}^{q\times n}$ some submatrices of A.

$C(C^\dagger A R^\dagger)R$ best rank $\le min(p, q)$ approximation matrix based on C, R submatrices of A.

Goreinov-Tyrtyshnikov-Zmarashkin 95: for $p = q$ choose $U = A[I, J]^{-1}$
(corresponds to best CUR approximation on the entries read)
good approximation when the corresponding $\det A[I, J]$ is maximal

Friedland-Mehrmann-Miedlar-Nkengla 08 choose several random choices of I, J set of rows and columns of A such that $A[I, J]$ has maximal product of significant singular values

Extension to 3-tensors I:

Extension to 3-tensors I:

For given $\mathcal{A} \in \mathbb{R}^{m\times n\times l}, F \in \mathbb{R}^{m\times p}, E \in \mathbb{R}^{n\times q}, G \in \mathbb{R}^{l\times r}$,
where $\langle p\rangle \subset \langle n\rangle \times \langle l\rangle, \langle q\rangle \subset \langle m\rangle \times \langle l\rangle, \langle r\rangle \subset \langle m\rangle \times \langle l\rangle$

$\min_{\mathcal{U}\in\mathbb{C}^{p\times q\times r}} \|\mathcal{A} - \mathcal{U} \times F \times E \times G\|_F$ achieved for $\mathcal{U} = \mathcal{A} \times E^\dagger \times F^\dagger \times G^\dagger$

Extension to 3-tensors I:

For given $\mathcal{A} \in \mathbb{R}^{m \times n \times l}, F \in \mathbb{R}^{m \times p}, E \in \mathbb{R}^{n \times q}, G \in \mathbb{R}^{l \times r}$,
where $\langle p \rangle \subset \langle n \rangle \times \langle l \rangle, \langle q \rangle \subset \langle m \rangle \times \langle l \rangle, \langle r \rangle \subset \langle m \rangle \times \langle l \rangle$

$\min_{\mathcal{U} \in \mathbb{C}^{p \times q \times r}} \|\mathcal{A} - \mathcal{U} \times F \times E \times G\|_F$ achieved for $\mathcal{U} = \mathcal{A} \times E^\dagger \times F^\dagger \times G^\dagger$

CUR approximation of $\mathcal{A}$ obtained by choosing E, F, G submatrices of unfolded $\mathcal{A}$ in the mode $1, 2, 3$.

Extensions to 3-tensors: II

$\mathcal{A} = [a_{i,j,k}] \in \mathbb{R}^{m \times n \times \ell}$ - 3-tensor
given $I \subset \langle m \rangle, J \subset \langle n \rangle, K \subset \langle \ell \rangle$ define
$R := \mathcal{A}_{\langle m \rangle, J, K} = [a_{i,j,k}]_{\langle m \rangle, J, K} \in \mathbb{R}^{m \times (\# J \cdot \# K)}$,
$C := \mathcal{A}_{I, \langle n \rangle, K} \in \mathbb{R}^{\langle n \rangle \times (\# I \cdot \# K)}$,
$D := \mathcal{A}_{I, J, \langle \ell \rangle} \in \mathbb{R}^{l \times (\# I \cdot \# J)}$
Problem: Find 3-tensor $\mathcal{U} = \in \mathbb{R}^{(\# J \cdot \# K) \times (\# I \cdot \# K) \times (\# I \cdot \# J)}$
such that $\mathcal{A}$ is approximated by the Tucker tensor
$\mathcal{V} = \mathcal{U} \times_1 C \times_2 R \times_3 D$
where $\mathcal{U}$ is the least squares solution

$$\mathcal{U}_{\mathrm{opt}} \in \arg \min_{\mathcal{U} \in \mathbb{R}^{\text{three tensor}}} \sum_{(i,j,k) \in \mathcal{S}} \left(a_{i,j,k} - (\mathcal{U} \times_1 C \times_2 R \times_3 D)_{i,j,k} \right)^2$$

$\mathcal{S} = (\langle m \rangle \times J \times K) \cup (I \times \langle n \rangle \times K) \cup (I \times J \times \langle \ell \rangle)$

Extension to 3-tensors: III

For $\#I = \#J = p,\ \#K = p^2,\ I \subset \langle m \rangle, J \subset \langle n \rangle, K \subset \langle \ell \rangle$
generically there is an exact solution to $\mathcal{U}_{\rm opt} \in \mathbb{R}^{p^3 \times p^3 \times p^2}$
obtained by unfolding in third direction
View $\mathcal{A}$ as $A \in \mathbb{R}^{(mn) \times \ell}$ by identifying
$\langle m \rangle \times \langle n \rangle \equiv \langle mn \rangle,\ I_1 = I \times J,\ J_1 = K$ and apply CUR again.

More generally, given $\#I = p,\ \#J = q,\ \#K = r$.
For $L = I \times J$ approximate $\mathcal{A}$ by $\mathcal{A}_{\langle m \rangle, \langle n \rangle, K} E^{\dagger}_{L,K} \mathcal{A}_{I,J,\langle \ell \rangle}$
Then for each $k \in K$ approximate each matrix $\mathcal{A}_{\langle m \rangle, \langle n \rangle, \{k\}}$ by
$\mathcal{A}_{\langle m \rangle, J, \{k\}} E^{\dagger}_{I,J,\{k\}} \mathcal{A}_{I, \langle n \rangle, \{k\}}$

Scaling of nonnegative tensors to balanced tensors

$0 \le \mathcal{T} = [t_{i,j,k}] \in \mathbb{R}^{m \times n \times l}$ balanced if each unfolding has fixed row sum:
$\sum_{j,k} t_{i,j,k} = \alpha > 0, \ \sum_{i,k} t_{i,j,k} = \beta > 0, \sum_{i,j} t_{i,j,k} = \gamma > 0$

Scaling of nonnegative tensors to balanced tensors

$0 \le \mathcal{T} = [t_{i,j,k}] \in \mathbb{R}^{m \times n \times l}$ balanced if each unfolding has fixed row sum:
$\sum_{j,k} t_{i,j,k} = \alpha > 0, \ \sum_{i,k} t_{i,j,k} = \beta > 0, \sum_{i,j} t_{i,j,k} = \gamma > 0$

Find nec. and suf. conditions for scaling:
$\mathcal{T}' = [x_i y_j z_k t_{i,j,k}], \mathbf{x}, \mathbf{y}, \mathbf{z} > \mathbf{0}$ such that $\mathcal{T}'$ balanced

Scaling of nonnegative tensors to balanced tensors

$0 \le \mathcal{T} = [t_{i,j,k}] \in \mathbb{R}^{m\times n\times l}$ balanced if each unfolding has fixed row sum:
$\sum_{j,k} t_{i,j,k} = \alpha > 0, \ \sum_{i,k} t_{i,j,k} = \beta > 0, \sum_{i,j} t_{i,j,k} = \gamma > 0$

Find nec. and suf. conditions for scaling:
$\mathcal{T}' = [x_i y_j z_k t_{i,j,k}], \mathbf{x}, \mathbf{y}, \mathbf{z} > \mathbf{0}$ such that $\mathcal{T}'$ balanced
Solution: Linear Programming

Scaling of nonnegative tensors to balanced tensors

$0 \le \mathcal{T} = [t_{i,j,k}] \in \mathbb{R}^{m\times n\times l}$ balanced if each unfolding has fixed row sum:
$\sum_{j,k} t_{i,j,k} = \alpha > 0, \ \sum_{i,k} t_{i,j,k} = \beta > 0, \sum_{i,j} t_{i,j,k} = \gamma > 0$

Find nec. and suf. conditions for scaling:
$\mathcal{T}' = [x_i y_j z_k t_{i,j,k}], \mathbf{x}, \mathbf{y}, \mathbf{z} > \mathbf{0}$ such that $\mathcal{T}'$ balanced
Solution: Linear Programming

THM (SC): $\mathcal{T} \in \mathbb{R}_+^{m\times n\times l}, 1 < m \le n \le l$, each $m \times m$ submatrix of the unfolded tensor $A \in \mathbb{R}^{m\times nl}$ in the first mode has positive permanent, and there exists one $m \times m$ fully indecomposable submatrix. Then there exists a "unique" scaling of $\mathcal{T}'$ of $\mathcal{T}$ such that $A' \in \mathbb{R}^{m\times nl}$ is stochastic.

Scaling of nonnegative tensors to balanced tensors

$0 \le \mathcal{T} = [t_{i,j,k}] \in \mathbb{R}^{m\times n\times l}$ balanced if each unfolding has fixed row sum:
$\sum_{j,k} t_{i,j,k} = \alpha > 0,\ \sum_{i,k} t_{i,j,k} = \beta > 0, \sum_{i,j} t_{i,j,k} = \gamma > 0$

Find nec. and suf. conditions for scaling:
$\mathcal{T}' = [x_i y_j z_k t_{i,j,k}], \mathbf{x}, \mathbf{y}, \mathbf{z} > \mathbf{0}$ such that $\mathcal{T}'$ balanced
Solution: Linear Programming

THM (SC): $\mathcal{T} \in \mathbb{R}_+^{m\times n\times l}, 1 < m \le n \le l$, each $m \times m$ submatrix of the unfolded tensor $A \in \mathbb{R}^{m\times nl}$ in the first mode has positive permanent, and there exists one $m \times m$ fully indecomposable submatrix. Then there exists a "unique" scaling of $\mathcal{T}'$ of $\mathcal{T}$ such that $A' \in \mathbb{R}^{m\times nl}$ is stochastic.

Fact: Sinkhorn scaling algorithm works

Scaling of nonnegative tensors to balanced tensors

$0 \le \mathcal{T} = [t_{i,j,k}] \in \mathbb{R}^{m\times n\times l}$ balanced if each unfolding has fixed row sum:
$\sum_{j,k} t_{i,j,k} = \alpha > 0, \ \sum_{i,k} t_{i,j,k} = \beta > 0, \sum_{i,j} t_{i,j,k} = \gamma > 0$

Find nec. and suf. conditions for scaling:
$\mathcal{T}' = [x_i y_j z_k t_{i,j,k}], \mathbf{x}, \mathbf{y}, \mathbf{z} > \mathbf{0}$ such that $\mathcal{T}'$ balanced
Solution: Linear Programming

THM (SC): $\mathcal{T} \in \mathbb{R}_+^{m\times n\times l}, 1 < m \le n \le l$, each $m \times m$ submatrix of the unfolded tensor $A \in \mathbb{R}^{m\times nl}$ in the first mode has positive permanent, and there exists one $m \times m$ fully indecomposable submatrix. Then there exists a "unique" scaling of $\mathcal{T}'$ of $\mathcal{T}$ such that $A' \in \mathbb{R}^{m\times nl}$ is stochastic.

Fact: Sinkhorn scaling algorithm works

Newton method works, since the scaling problem can be reformulated to unique minimum of strict convex function

References I

- S. Friedland, On the generic rank of 3-tensors, arXiv: 0805.3777v2.
- S. Friedland, S. Gauber and L. Han, Perron-Frobenius theorem for nonnegative multilinear forms, arXiv:0905.1626.
- S. Friedland, V. Mehrmann, A. Miedlar, and M. Nkengla, Fast low rank approximations of matrices and tensors, submitted, www.matheon.de/preprints/4903.
- S.A. Goreinov, E.E. Tyrtyshnikov, N.L. Zmarashkin, Pseudo-skeleton approximations of matrices, *Reports of the Russian Academy of Sciences* 343(2) (1995), 151-152.
- S.A. Goreinov, E.E. Tyrtyshnikov, N.L. Zmarashkin, A theory of pseudo-skeleton approximations of matrices, *Linear Algebra Appl.* 261 (1997), 1-21.

References II

- L.H. Lim, Singular values and eigenvalues of tensors: a variational approach, *CAMSAP* 05, 1 (2005), 129-132.
- M.W. Mahoney and P. Drineas, CUR matrix decompositions for improved data analysis, *PNAS* 106, (2009), 697-702.