Exact conditions for countable inclusion-exclusion identity and extensions

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Abstract

We give simple necessary and sufficient conditions for the inclusion-exclusion identity to hold for an infinite countable number of sets. In terms of a random variable, whose range are nonnegative integers, this condition is equivalent to the convergence to zero of binomial moments. Some standard extensions of the countable inclusion-exclusion identity are also given.

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1 Introduction

The method of inclusion and exclusion and the accompanying Bonferroni-type inequalities are very useful and versatile tools in probability and combinatorics. See for example the relative recent book of Galambos and Simonelli [1] and references therein.

Let $(\Omega, \Sigma, \Pr)$ be a probability space consisting of the sample space Ω, a σ-algebra Σ and a probability measure $\Pr$ on Σ. Let $A_1, \ldots, A_n \in \Sigma$. Then the *Inclusion-Exclusion Identity* states:

$$\Pr(\cup_{i=1}^n A_i) = \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \le i_1 < \ldots < i_k \le n} \Pr(A_{i_1} \cap \ldots \cap A_{i_k}). \tag{1.1}$$

The main purpose of this paper is to give simple necessary and sufficient conditions for the *Countable Inclusion-Exclusion Identity*

$$\Pr(\cup_{i \in \mathbb{N}} A_i) = \sum_{k \in \mathbb{N}} (-1)^{k-1} \sum_{1 \le i_1 < \ldots < i_k} \Pr(A_{i_1} \cap \ldots \cap A_{i_k}), \ \textit{where } A_i \in \Sigma \text{ for } i \in \mathbb{N}. \tag{1.2}$$

(Here $\mathbb{N}, \mathbb{Z}_+$ is the set of positive and nonnegative integers respectively.) As in [1], let

$$S_k := \sum_{1 \le i_1 < \ldots < i_k} \Pr(A_{i_1} \cap \ldots \cap A_{i_k}) \in [0, \infty] \text{ for } k \in \mathbb{N}. \tag{1.3}$$

Clearly, for (1.2) to hold we must assume that each S_k is finite, and the sequence $S_k, k \in \mathbb{N}$ converges to 0. The result of Takacs [2] claims that (1.2) holds if $S_k, k \in \mathbb{N}$ is a sequence of nonnegative numbers that converge exponentially to zero:

$$S_k \in [0, \infty) \text{ for } k \in \mathbb{N} \text{ and } \limsup_{k \to \infty} S_k^{\frac{1}{k}} < 1. \tag{1.4}$$

Moreover, it follows from [2, p. 111, (39)], that the above conditions yield the following generalization of (1.2)

$$\Pr(\cup_{1\le i_1<\ldots<i_k}(A_{i_1}\cap\ldots\cap A_{i_k}))=\sum_{j\in\mathbb{Z}_+}(-1)^j\binom{j+k-1}{k-1}S_{j+k} \text{ for all } k\in\mathbb{N}. \tag{1.5}$$

In this note we show

Theorem 1.1 *Let* $(\Omega,\Sigma,\Pr)$ *be probability space and assume that* $A_i\in\Omega$ *for* $i\in\mathbb{N}$. *Then (1.5) holds for some* $k\in\mathbb{N}$ *if and only if* $S_l, l\in\mathbb{N}$ *is a sequence of nonnegative numbers such that* $\lim_{l\to\infty} l^{k-1}S_l=0$.

We now list briefly the contents of this paper. In §2 we prove Theorem 1.1. In §3 we discuss a partition of Ω induced by $A_i, i\in\mathbb{N}$ and discuss a few applications. In §4 we give an analog of Theorem 1.1 for the random variable $X:\Omega\to\mathbb{Z}_+$.

2 Proof of the main theorem

Let the assumptions of Theorem 1.1 hold. Denote

$$S_{k,n}:=\sum_{1\le i_1<\ldots<i_k\le n}\Pr(A_{i_1}\cap\ldots\cap A_{i_k}) \text{ for } k\le n \text{ and } S_{k,n}:=0 \text{ for } k>n. \tag{2.1}$$

Clearly, $0\le S_{k,n}\le\binom{n}{k}$ for each $k,n\in\mathbb{N}$.

Lemma 2.1 *Let the assumptions of Theorem 1.1 hold. Then for each* $k\in\mathbb{N}$ *the sequence* $S_{k,n}, n\in\mathbb{N}$ *is a nondecreasing sequence that converges in the generalized sense to* $S_k\in[0,\infty]$. *That is,* $S_k<\infty$ *if and only* $S_{k,n}, n\in\mathbb{N}$ *is a bounded sequence converging to* S_k, *and* $S_k=\infty$ *if and only if* $S_{k,n}, n\in\mathbb{N}$ *is an unbounded sequence.*

The proof of this lemma is standard and is left to the reader.

Proof of Theorem 1.1. Fix $k\in\mathbb{N}$. Assume first that (1.5) holds. First, $S_j, j\in\mathbb{N}$ is a sequence of nonnegative numbers. Second, the convergence the series in (1.5) implies that $\lim_{l\to\infty}\binom{l-1}{k-1}S_l=0$, which is equivalent to $\lim_{j\to\infty} l^{k-1}S_l=0$.

Assume first that $S_l, l\in\mathbb{N}$ is a sequence of nonnegative numbers. Recall the Bonferroni inequalities [1, Ineq. I.2]. Let $1\le k\le n$ and assume that $d,r\in\mathbb{Z}_+$. Then

$$\sum_{j=0}^{2d+1}(-1)^j\binom{j+k-1}{k-1}S_{j+k,n}\le\Pr(\cup_{i_1<\ldots<i_k\le n}(A_{i_1}\cap\ldots\cap A_{i_k}))\le \sum_{j=0}^{2r}(-1)^j\binom{j+k-1}{k-1}S_{j+k,n}. \tag{2.2}$$

Let $n\to\infty$. Clearly, $\Pr(\cup_{i_1<\ldots<i_k\le n}(A_{i_1}\cap\ldots\cap A_{i_k}))\nearrow\Pr(\cup_{i_1<\ldots<i_k}(A_{i_1}\cap\ldots\cap A_{i_k}))$. Use Lemma 2.1 to deduce the Bonferroni type inequality

$$\sum_{j=0}^{2d+1}(-1)^j\binom{j+k-1}{k-1}S_{j+k}\le\Pr(\cup_{i_1<\ldots<i_k}(A_{i_1}\cap\ldots\cap A_{i_k}))\le \sum_{j=0}^{2r}(-1)^j\binom{j+k-1}{k-1}S_{j+k}. \tag{2.3}$$

Let $a_d\le c_k\le b_r$ be the left-hand side, the middle part and the right-hand side of the above inequalities. Then $c_k\in[a_d,b_d]$ for any $d\in\mathbb{Z}_+$. Assume second that $\lim_{l\to\infty}\binom{l}{k-1}S_l=0$. Hence $b_d-a_d=\binom{2d+k}{k-1}S_{2d+k+1}\to 0$. Therefore the left-hand side and the right-hand side of (2.3) converge to c_k. □

Corollary 2.2 *Let* $(\Omega, \Sigma, \Pr)$ *be a probability space, assume that* $S_l \in [0, \infty), l \in \mathbb{N}$ *and (1.5) holds for some* $k = m > 1$. *Then (1.5) holds for* $k = 1, \ldots, m-1$.

3 A decomposition of a countable sets

Definition 3.1 *Let* Ω *be an infinite set and* $A_i \subseteq \Omega, i \in \mathbb{N}$. *Then*

- *Let* $\mathcal{F} \subset 2^\Omega$ *be the set of all nonempty finite subsets of* Ω, *let* $\tilde{\mathcal{F}} := \mathcal{F} \cup \{\emptyset\} \subset 2^\Omega$ *and for each* $j \in \mathbb{Z}_+$ *let* $\mathcal{F}_j \subset \tilde{\mathcal{F}}$ *be the set all finite subsets of* $\mathbb{N}$ *of cardinality* j.
- $B_\emptyset := \Omega \backslash \cup_{i \in \mathbb{N}} A_i$.
- $B_\infty := \limsup A_i = \cap_{i \in \mathbb{N}} \cup_{j \geq i} A_j$ *the set of points that belong to an infinite number of* $A_i, i \in \mathbb{N}$.
- *For each* $U \in \mathcal{F}$ *denote by* $A_U := \cap_{i \in U} A_i$, *and by* $B_U := A_U \backslash \cup_{i \in \mathbb{N} \backslash U} A_{U \cup \{i\}}$ *the set of points belonging only to* $A_i, i \in U$. *Let* $A'_U := A_U \backslash B_\infty$.

Proposition 3.2 *Let* Ω *be an infinite set and* $A_i \subseteq \Omega, i \in \mathbb{N}$. *Then*

1. B_∞ *and* $B_U, U \in \tilde{\mathcal{F}}$ *form a countable partition of* Ω.
2. $\cup_{i \in \mathbb{N}} A_i = B_\infty \cup (\cup_{U \in \mathcal{F}} B_U)$.
3. $A_U = A'_U \cup (A_U \cap B_\infty)$.

Assume in addition that $(\Omega, \Sigma, \Pr)$ *is a probability space and* $A_i \in \Sigma, i \in \mathbb{N}$. *If* S_k *is finite for some* $k \in \mathbb{N}$ *then* $\Pr(B_\infty) = 0$. *Let*

$$T_j := \sum_{U \in \mathcal{F}_j} \Pr(B_U), \quad \text{for each } j \in \mathbb{Z}_+. \tag{3.1}$$

Then $\sum_{j \in \mathbb{Z}_+} T_j \leq 1$ *and equality holds if and only if* $\Pr(B_\infty) = 0$. *Assume that* $\Pr(B_\infty) = 0$. *Then*

$$S_k = \sum_{j \in \mathbb{Z}_+} \binom{j+k}{k} T_{j+k} \in [0, \infty], \quad \text{for each } k \in \mathbb{N}. \tag{3.2}$$

Let $l \in \mathbb{N}$ *and assume that* $S_l \in [0, \infty)$. *Then for each positive integer* $k < l$ $S_k \in [0, \infty)$. *Suppose furthermore that* $2d + k + 1, 2r + k \in [1, l]$ *for some* $d, r \in \mathbb{Z}_+, k \in \mathbb{N}$. *Then (2.3) holds.*

Proof. Claims *1, 2, 3* of the proposition are straightforward. Assume that $(\Omega, \Sigma, \Pr)$ is a probability space and $A_i \in \Sigma, i \in \mathbb{N}$. Since $\mathcal{F}$ is countable, $A_U, A'_U, B_\infty \in \Sigma$ for each $U \in \tilde{\mathcal{F}}$.

Assume that $S_k = \sum_{U \in \mathcal{F}_k} \Pr(A_U) < \infty$. Let $B_{k,\infty}$ be the set of elements in Ω which belong to an infinite number of A_U, where $U \in \mathcal{F}_k$. Use the Borel-Cantelli Lemma to deduce that $\Pr(B_{k,\infty}) = 0$. It is straightforward to show that $B_{k,\infty} = B_\infty$. Hence $\Pr(B_\infty) = 0$.

From *1* it follows that $\Pr(B_\infty) + \sum_{j \in \mathbb{Z}_+} T_j = 1$. Hence $\sum_{j \in \mathbb{Z}_+} T_j \leq 1$ and equality holds if and only if $\Pr(B_\infty) = 0$.

Assume that $\Pr(B_\infty) = 0$. Then $\Pr(A_U) = Pr(A'_U)$ for $U \in \mathcal{F}$ and $S_k = \sum_{U \in \mathcal{F}_k} \Pr(A'_U)$. Let $V \in \mathcal{F}_l$, where $l \geq k$. Then $B_V \subset A'_U$ for each $U \subset V$. Suppose that $\#U = k$. Then V has exactly $\binom{l}{k}$ distinct k-elements subsets U. Thus B_V is a subset of exactly $\binom{l}{k}$ sets $A'_U, U \in \mathcal{F}_k$, and all other subsets $A'_U, U \in \mathcal{F}_k$ are disjoint from B_V. Hence (3.2) holds in the generalized sense, i.e. $S_k = \infty$ if and only if the right-hand side of (3.2) diverges.

Suppose that $S_l < \infty$. Then $\Pr(B_\infty) = 0$. Furthermore the series (3.2) converges for $k = l$. Let $l > k \in \mathbb{N}$. As $\binom{p}{l} > \binom{p}{k}$ for $p \geq 2l - 1$ it follows that

$$S_l \geq \sum_{p \geq 2l-1} \binom{p}{l} T_l \geq \sum_{p \geq 2l-1} \binom{p}{k} T_l = S_k - \sum_{k \leq p \leq 2l-2} \binom{p}{l} T_l.$$

Hence $S_k < \infty$.

Suppose furthermore that $2d + k + 1, 2r + k \in [1, l]$ for some $d, r \in \mathbb{Z}_+, k \in \mathbb{N}$. Then the proof of Theorem 1.1 yields (2.3). □

4 Random variables with values in $\mathbb{Z}_+$

As in [1], Theorem 1.1 or its variation can be reformulate in terms of binomial moments of a random variable $X : \Omega \to \mathbb{Z}_+$. Assume that $\Pr(X = j) = T_j$ for each $j \in \mathbb{Z}_+$. Hence $\sum_{j\in\mathbb{Z}_+} T_j = 1$. Let $S_j := \mathrm{E}(\binom{X}{j})$ for $j \in \mathbb{Z}_+$. Then $S_0 = 1$ and S_k is given by (3.2) for $k \in \mathbb{N}$.

Theorem 4.1 *Let* $(\Omega, \Sigma, \Pr)$ *be probability space and assume that* $X : \Omega \to \mathbb{Z}_+$ *is a random variable. Let* $S_j := \mathrm{E}(\binom{X}{j}) \in [0, \infty]$ *for* $j \in \mathbb{Z}_+$. *Suppose that* $S_l < \infty$. *Then* $S_k < \infty$ *for each* $l > k \in \mathbb{N}$. *Suppose furthermore that* $2d + k + 1, 2r + k \in [1, l]$ *for some* $d, r \in \mathbb{Z}_+, k \in \mathbb{N}$. *Then*

$$\sum_{j=0}^{2d+1} (-1)^j \binom{j+k-1}{k-1} S_{j+k} \le \Pr(X \ge k) \le \sum_{j=0}^{2r} (-1)^j \binom{j+k-1}{k-1} S_{j+k}, \tag{4.1}$$

$$\sum_{j=0}^{2d+1} (-1)^j \binom{j+k-1}{k-1} S_{j+k-1} \le \Pr(X = k-1) \le \sum_{j=0}^{2r} (-1)^j \binom{j+k-1}{k-1} S_{j+k-1}. \tag{4.2}$$

Furthermore for $k \in \mathbb{N}$ *one has the equalities*

$$\Pr(X \ge k) = \sum_{j\in\mathbb{Z}_+} (-1)^j \binom{j+k-1}{k-1} S_{j+k}, \tag{4.3}$$

$$\Pr(X = k-1) = \sum_{j\in\mathbb{Z}_+} (-1)^j \binom{j+k-1}{k-1} S_{j+k-1}, \tag{4.4}$$

if and only if $S_l, l \in \mathbb{N}$ *is a sequence of nonnegative numbers such that* $\lim_{l\to\infty} l^{k-1} S_l = 0$.

Proof. Assume that $n_0 \in \mathbb{Z}_+$ is the first nonnegative integer such that $\Pr(X \le n_0) > 0$. For $\mathbb{N} \ni n \ge n_0$ let $X_n : \Omega \to [0, n] \cap \mathbb{Z}_+$ be the random variable whose distribution is given by $\Pr(X_n = k) = \frac{\Pr(X=k)}{\Pr(X\le n)}$ for $k = 0, \ldots, n$. Let $S_{j,n} := \mathrm{E}(\binom{X_n}{j})$ for $\in \mathbb{Z}_+$. Then

$$\lim_{n\to\infty} S_{j,n} = S_j, \ \lim_{n\to\infty} \Pr(X_n \ge j) = \Pr(X \ge j), \ \lim_{n\to\infty} \Pr(X_n = j) = \Pr(X = j), \text{ for all } j \in \mathbb{N}.$$

Use the two type of Bonferroni inequalities given in [1, Ineq. I.2] and the arguments of the proof of Theorem 1.1 and Proposition 3.2 to deduce the theorem. □

If $S_k, k \in \mathbb{N}$ is a sequence of nonnegative numbers such that $\limsup S_k^{\frac{1}{k}} < 1$ then the equalities (4.3-4.4) are due to Takacs [2].

References

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